



Minimum eigenvalue inequalities for \mathbf{Z} -transformations on proper and symmetric cones

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ABSTRACT

A linear transformation L defined on a finite dimensional real Hilbert space is said to be a \mathbf{Z} -transformation on a proper cone K if

$$x \in K, y \in K^*, \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0,$$

where K^* is the dual of K . Examples of such transformations include \mathbf{Z} -matrices on R_+^n , Lyapunov and Stein transformations on the semi-definite cone. For a \mathbf{Z} -transformation L ,

$$\tau(L) := \min\{\operatorname{Re}(\lambda) : \lambda \in \sigma(L)\}$$

is an eigenvalue of L with a corresponding eigenvector in K . In this article, when K is a product cone, we relate the \mathbf{Z} -property/positive stable property/minimum real eigenvalue of L with those of a sub-transformation of L and its Schur complement.

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1. Introduction

Consider a finite dimensional real inner product space H , a proper cone K in H , and a linear transformation L on H . We say that L has the \mathbf{Z} -property on K , or that it is a \mathbf{Z} -transformation on K , and write $L \in \mathbf{Z}(K)$ if

$$x \in K, y \in K^*, \text{ and } \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0,$$

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where K^* denotes the dual of K . If both L and $-L$ are in $\mathbf{Z}(K)$, we say that L is a *Lyapunov-like transformation*. \mathbf{Z} -transformations – the negatives of which are called cross-positive matrices in [18] and exponentially K -nonnegative matrices in [2] – are generalizations of \mathbf{Z} -matrices. Such transformations have appeared in numerous studies, particularly in connection with dynamical systems [21,22,2,13], complementarity problems [7,10], optimization [8], and Lie algebras of cone automorphism groups [11]. They also have properties similar to those of \mathbf{Z} -matrices [21,7,1]. Here are some equivalent properties:

Theorem 1. Suppose L is a \mathbf{Z} -transformation on K . Then the following are equivalent:

- (1) There exists a $d \in K^\circ$ such that $L(d) \in K^\circ$, where K° denotes the interior of K .
- (2) L is invertible with $L^{-1}(K) \subseteq K$.
- (3) L is positive stable, that is, the real part of any eigenvalue of L is positive.
- (4) The dynamical system $\frac{dx}{dt} + L(x) = 0$ is asymptotically stable, that is, every trajectory of the system from any starting point converges to the origin as $t \rightarrow \infty$.
- (5) $L + tI$ is invertible for all $t \in [0, \infty)$.
- (6) All real eigenvalues of L are positive.
- (7) There is an $e \in (K^*)^\circ$ such that $L^T(e) \in (K^*)^\circ$.
- (8) The implication $[x \in K, -L(x) \in K] \Rightarrow x = 0$ holds.
- (9) For any $q \in H$, the (cone) linear complementarity problem $\text{LCP}(L, K, q)$ has a solution.

Moreover, when $H = R^n$ and $K = R_+^n$, the above properties (for a \mathbf{Z} -matrix) are further equivalent to

- (10) L is a \mathbf{P} -matrix, that is, $x * L(x) \leq 0 \Rightarrow x = 0$, where $x * y$ refers to the componentwise product of two vectors.
- (11) All principal minors of L are positive.
- (12) L is a nonsingular \mathbf{M} -matrix, that is, L is of the form $L = rI - B$, where B is a nonnegative matrix whose spectral radius is less than r .
- (13) For any $q \in R^n$, the linear complementarity problem $\text{LCP}(L, R_+^n, q)$ has a unique solution.

Given $L \in \mathbf{Z}(K)$, we let

$$\tau(L) := \min\{\text{Re}(\lambda) : \lambda \in \sigma(L)\},$$

where $\sigma(L)$ denotes the spectrum of L . Schneider and Vidyasagar [18] have shown that $\tau(L)$ is an eigenvalue of L with a corresponding eigenvector in K . As in matrix theory ([12], page 129), we call $\tau(L)$, the ‘minimum eigenvalue of L ’.

In the context of nonsingular \mathbf{M} -matrices, various inequalities have been described for the minimum eigenvalue [15,16,20,25]. Many of these inequalities are derived by exploiting properties (10)–(13) given in the above theorem. Unlike \mathbf{Z} -matrices which are always of the form

$$A = rI - B \quad \text{with } r \in R \quad \text{and } B(R_+^n) \subseteq R_+^n,$$

it turns out that on a general cone, not every \mathbf{Z} -transformation (even if it satisfies any of the properties (1) – (9)) can be written in the form $L = rI - S$, with $S(K) \subseteq K$ and $r \in R$. Also, the concept of \mathbf{P} -property of a linear transformation is so far limited to symmetric cones in Euclidean Jordan algebras, and even in that setting, the equivalence of \mathbf{P} -property with properties (1) – (9) is yet to be established [10]. Thus, it becomes interesting and useful to derive inequalities for the minimum eigenvalue in the general (proper cone) setting, and for symmetric cones in particular. We remark that symmetric cones in Euclidean Jordan algebras form an important class of proper cones, and, in the last decade such cones have become highly relevant in (conic) optimization [14] and complementarity problems [6,10]. To motivate and demonstrate the importance of \mathbf{Z} and Lyapunov-like transformations that go beyond \mathbf{Z} -matrices and \mathbf{M} -matrices, we cite two examples (more can be found in [7]).

Example 1. In the Hilbert space \mathcal{S}^n of all real symmetric $n \times n$ matrices with the trace inner product, let, for any $A \in \mathbb{R}^{n \times n}$,

$$L_A(X) := AX + XA^T \quad (X \in \mathcal{S}^n). \quad (1)$$

L_A is the *Lyapunov transformation* corresponding to A and its importance is well-known in the study of linear continuous dynamical systems. L_A is a Lyapunov-like transformation on the symmetric cone \mathcal{S}_+^n of all positive semidefinite matrices within \mathcal{S}^n [7]. Its complementarity properties have been well documented in the literature, see e.g., [5]. It turns out that unless A is a multiple of the Identity matrix, L_A can never be written in the form $L = rI - S$, where $S(\mathcal{S}_+^n) \subseteq \mathcal{S}_+^n$ and $r \in \mathbb{R}$ (see Section 3).

Example 2. In the Hilbert space \mathcal{S}^n , let, for any $A \in \mathbb{R}^{n \times n}$,

$$S_A(X) := X - AXA^T \quad (X \in \mathcal{S}^n). \quad (2)$$

S_A is the *Stein transformation* corresponding to A and it appears in the study of linear discrete dynamical systems. It is a **Z**-transformation on the symmetric cone \mathcal{S}_+^n . For some of its complementarity properties, see [4]. Note that $S_A = I - S$, where $S(\mathcal{S}_+^n) \subseteq \mathcal{S}_+^n$.

Our main contributions are the following: Assuming the block form

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

on a product space/cone, we show that under certain conditions, the **Z** and positive stability properties of L are inherited by A and the Schur complement $D - CA^{-1}B$ and, additionally, relate the minimum eigenvalues of these three transformations. We then specialize these results to Euclidean Jordan algebras. Motivation for our work comes, partly, from the results in [24], where the inheritance of various **P**-properties are discussed.

Here is a brief outline of the paper. In Section 2, we collect necessary background material. In Section 3, we prove the existence of (positive stable) **Z**-transformations that are not of the form $L = rI - S$ with $S(K) \subseteq K$ and $r \in \mathbb{R}$. Section 4 shows that under certain conditions, the **Z** and $\mathbf{Z} \cap \mathbf{S}$ properties are inherited by subtransformations and Schur complements. Section 5 deals with the computation of the minimum eigenvalue in some specific instances. In Section 6, we prove minimum eigenvalue inequalities. Finally, in Section 7, we specialize the previous results to Euclidean Jordan algebras.

2. Preliminaries

Throughout, we assume that H denotes a finite dimensional real Hilbert space and K denotes a proper cone, that is, K is a closed convex pointed cone with nonempty interior. The interior of K is denoted by K° and the dual of K is given by

$$K^* := \{y \in H : \langle y, x \rangle \geq 0 \ \forall x \in K\}.$$

We use the notation $x \geq 0$ ($x > 0$) to mean $x \in K$ (respectively, $x \in K^\circ$). Furthermore, we write $x \perp y$ when $\langle x, y \rangle = 0$.

For a linear transformation L defined on H , we say that L has the **S**-property on K or that

*L is an **S** – transformation on K if there exists $d > 0$ such that $L(d) > 0$.*

Similar to $\mathbf{Z}(K)$, the notation $\mathbf{S}(K)$ has an obvious meaning; we simply use the symbols **Z** and **S** when the context is clear. In view of Theorem 1, for an $L \in \mathbf{Z}(K)$,

L is positive stable if and only if $L \in \mathbf{S}(K)$.

We use the notation $L \succeq 0$ if $L(K) \subseteq K$, and for two transformations L_1 and L_2 , $L_1 \succeq L_2$ or $L_2 \preceq L_1$ if $L_1 - L_2 \succeq 0$. The following is immediate:

$$L_1 \succeq L_2 \quad \text{and} \quad L_2 \in \mathbf{S} \Rightarrow L_1 \in \mathbf{S}. \quad (3)$$

Now, let H_1 and H_2 be two finite dimensional real inner product spaces with proper cones $K_i \subset H_i$, $i = 1, 2$. On the product space $H_1 \times H_2$, we assume that the inner product is the sum of the inner products of the components. Then, $K_1 \times K_2$ is proper in $H_1 \times H_2$ and $(K_1 \times K_2)^* = K_1^* \times K_2^*$. Given a linear transformation L on $H_1 \times H_2$, we decompose L in the block form by writing

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (4)$$

where any element in $H_1 \times H_2$ is regarded as a column pair, $A : W_1 \rightarrow W_1$, $B : W_2 \rightarrow W_1$, $C : W_1 \rightarrow W_2$, and $D : W_2 \rightarrow W_2$ are linear transformations. For $i = 1, 2$, we denote the Identity transformation on H_i by I_i .

If A is invertible, we define the *Schur complement of L with respect to A* by

$$L/A := D - CA^{-1}B$$

and the *principal pivotal transform of L with respect to A* by

$$L^\diamond := \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix}.$$

It is easy to see that

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Leftrightarrow L^\diamond \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}.$$

We note that on $K_1 \times K_2$, L has the **S**-property if and only if L^\diamond has the **S**-property.

All of our results are applicable to symmetric cones in Euclidean Jordan algebras. While [3] is our main source, brief introductions to Euclidean Jordan algebras and symmetric cones can be found in [6] and [19].

3. Transformations that are not of the form $\alpha I - S$ with $S(K) \subseteq K$

In [18], it is shown that $L \in \mathbf{Z}(K)$ if and only if

$$e^{-tL}(K) \subseteq K \quad \forall t \geq 0 \text{ in } \mathbb{R}.$$

Thus, $L = \lim_{t \downarrow 0} \frac{I - e^{-tL}}{t} = \lim_{k \rightarrow \infty} (\alpha_k I - S_k)$, where $\alpha_k > 0$ and S_k is linear with $S_k(K) \subseteq K$ for all $k = 1, 2, \dots$. Example 6 given in [7] shows that a Lyapunov transformation L_A on S_+^n need not be of the form $\alpha I - S$ with $\alpha \in \mathbb{R}$ and $S(S_+^n) \subseteq S_+^n$. (See Exercise 4.6, page 20 in [2] for a similar example on the Jordan spin algebra). The following result explains why it is possible to construct examples of this type.

Proposition 1 ([8], Theorem 4). Suppose that H is a simple Euclidean Jordan algebra. If L is a Lyapunov-like transformation on the symmetric cone K with $L(K) \subseteq K$, then L is a (nonnegative) multiple of the Identity transformation.

Now, if H is a simple Euclidean Jordan algebra (the algebra \mathcal{S}^n is one such) and L is Lyapunov-like on its symmetric cone with $L = \alpha I - S$ and $S(K) \subseteq K$, then applying the above result to the Lyapunov-like transformation $S = \alpha I - L$, we see that S , and hence, L must be a multiple of the Identity. In particular, on a simple Euclidean Jordan algebra, Lyapunov-like transformations which are not multiples of the Identity can never be written in the form $L = \alpha I - S$ with $S(K) \subseteq K$. This is the case when $H = \mathcal{S}^n$ and $L = L_A$ (see Example 1), where $A \in \mathbb{R}^{n \times n}$ ($n > 2$) is not a multiple of the Identity matrix.

4. Inheritance of the \mathbf{Z} -property by subtransformations and Schur complements

From now on, we assume that $H = H_1 \times H_2$ and $K = K_1 \times K_2$ as given in Section 2. We assume that L is given by (4) and consider its Schur complement/pivotal transform when A is invertible.

Proposition 2. Suppose L is a \mathbf{Z} -transformation on $K_1 \times K_2$. Then the following hold:

- (i) $-B(K_2) \subseteq K_1$ and $-C(K_1) \subseteq K_2$.
- (ii) A is a \mathbf{Z} -transformation on K_1 and D is a \mathbf{Z} -transformation on K_2 .
- (iii) L/A has the \mathbf{Z} -property on K_2 when A has the \mathbf{S} -property.

Proof. (i) Consider any $x \in K_2$ and $y \in K_1^*$. Then

$$\bar{x} = \begin{bmatrix} 0 \\ x \end{bmatrix} \in K_1 \times K_2 \quad \text{and} \quad \bar{y} = \begin{bmatrix} y \\ 0 \end{bmatrix} \in (K_1 \times K_2)^*.$$

Moreover, $\langle \bar{x}, \bar{y} \rangle = 0$. As L has the \mathbf{Z} -property, $\langle L(\bar{x}), \bar{y} \rangle \leq 0$, which, upon simplification gives $\langle Bx, y \rangle \leq 0$. As x is arbitrary in K_2 and y is arbitrary in K_1^* , we see that $-B(K_2) \subseteq K_1$. Similarly, we have $-C(K_1) \subseteq K_2$.

- (ii) Take $u \in K_1$ and $v \in K_1^*$ with $u \perp v$. Then considering $\bar{u} = \begin{bmatrix} u \\ 0 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$ in $H_1 \times H_2$ and applying the \mathbf{Z} -property of L , we get $\langle L(\bar{u}), \bar{v} \rangle \leq 0$. This leads to $\langle Au, v \rangle \leq 0$, proving the \mathbf{Z} -property of A . The proof that D is a \mathbf{Z} -transformation on K_2 is similar.

- (iii) Here we assume that A has the \mathbf{S} -property. Since A has the \mathbf{Z} -property by (ii), A is invertible and $A^{-1}(K_1) \subseteq K_1$ (see Theorem 1). From (i), $CA^{-1}B(K_2) \subseteq K_2$. Now take $u \in K_2$, $v \in K_2^*$ with $u \perp v$. Then, by (ii), $\langle Du, v \rangle \leq 0$ and, using (i), $\langle CA^{-1}Bu, v \rangle \geq 0$. Thus,

$$\langle (L/A)u, v \rangle = \langle (D - CA^{-1}B)u, v \rangle = \langle Du, v \rangle - \langle CA^{-1}Bu, v \rangle \leq 0.$$

Hence, L/A has the \mathbf{Z} -property on K_2 . \square

Remarks. Consider the situation where L is assumed to be Lyapunov-like on the product cone $K_1 \times K_2$. In this case, in the proof of Item (i) above, $\langle Bx, y \rangle = 0$ for all $x \in K_1$ and $y \in K_2$. As the cones involved are generating cones, this leads to $B = 0$. Similarly, $C = 0$. Thus, when L is Lyapunov-like on $K_1 \times K_2$, L will have only diagonal blocks which are Lyapunov-like on their respective cones.

Theorem 2. Suppose that $L \in \mathbf{Z}$. Then $L \in \mathbf{S}$ if and only if $A \in \mathbf{Z} \cap \mathbf{S}$ and $L/A \in \mathbf{Z} \cap \mathbf{S}$.

Proof. Suppose that $L \in \mathbf{S}$. Then there exist $0 < u \in H_1$ and $0 < v \in H_2$ such that $L \begin{bmatrix} u \\ v \end{bmatrix} > 0$. This

implies that $Au + Bv > 0$ in H_1 . As $-Bv \in K_1$ (from Item (i) in the previous result), we have $Au > -Bv \geq 0$ in H_1 . This shows that A has the **S**-property and (along with the **Z**-property) $A^{-1}(K_1) \subseteq K_1$. Now, from Item (iii) of the previous result, L/A has the **Z**-property. To show that L/A has the **S**-property, we proceed as follows. The **S**-property of L implies the **S**-property of the principal pivotal transform

L^\diamond . Thus, there exist $0 < u \in H_1$ and $0 < v \in H_2$ such that $L^\diamond \begin{bmatrix} u \\ v \end{bmatrix} > 0$. This implies that

$CA^{-1}u + (L/A)v > 0$, which further implies, $(L/A)v > -CA^{-1}u \geq 0$ as $A^{-1}(K_1) \subseteq K_1$ and $-C(K_1) \subseteq K_2$. Thus, L/A has the **S**-property.

Now, for the ‘if’ part. Assume that $A, L/A \in \mathbf{Z} \cap \mathbf{S}$. Then there exist $0 < u \in H_1$ and $0 < v \in H_2$ such that $Au > 0$ and $(L/A)v > 0$. As $A^{-1}(K_1) \subseteq K_1$, we must have $A^{-1}(K_1^\circ) \subseteq K_1^\circ$ and so $A^{-1}u > 0$. Now, pick a large positive number λ such that $CA^{-1}u + \lambda(L/A)v > 0$. As $-B(v) \in K_1$ and $A^{-1}(K_1) \subseteq K_1$, we have $A^{-1}u - \lambda A^{-1}Bv > 0$. Then

$$L^\diamond \begin{bmatrix} u \\ \lambda v \end{bmatrix} = \begin{bmatrix} A^{-1}u - \lambda A^{-1}Bv \\ CA^{-1}u + \lambda(L/A)v \end{bmatrix} > 0.$$

Thus, L^\diamond has the **S**-property, and consequently, L has the **S**-property. \square

Remarks. When $H = \mathbb{R}^n$ and $K = \mathbb{R}_+^n$, for a **Z**-matrix, **S** and **P** properties are equivalent (see Theorem 1). In this setting, the above result reads: A **Z**-matrix L is a **P**-matrix if and only if A and L/A are **P**-matrices. Easy examples can be constructed to show that this is false without the **Z**-property. On a Euclidean Jordan algebra, one defines the **P**-property via the following implication [6]:

$$x \text{ and } L(x) \text{ operator commute, } x \circ L(x) \leq 0 \Rightarrow x = 0.$$

While the **P**-property always implies the **S**-property, the following two problems remain open:

- For a **Z**-transformation, does **S** imply the **P**-property?
- Suppose L (given in the block form) has the **Z**-property on a product Euclidean Jordan algebra. If both A and L/A have the **P**-property, can we conclude that L has the **P**-property?

These questions have positive answers when L is Lyapunov-like, see [10] and the previous remark.

An interpretation. Consider the setting of the previous theorem. When L , given by (4), has the **Z**-property on $K = K_1 \times K_2$, all trajectories of the dynamical system

$$\frac{dz}{dt} + L(z) = 0, \quad z(0) = z_0 \in K$$

which originate in K stay in K . (This is because, the **Z**-property is equivalent to the inclusion $e^{-tL}(K) \subseteq K$ for all $t \geq 0$.) If A is invertible, we get two related systems

$$\frac{dx}{dt} + Ax = 0 \quad \text{and} \quad \frac{dy}{dt} + (L/A)(y) = 0.$$

Now, the above theorem says that all the trajectories of the system $\frac{dz}{dt} + L(z) = 0$, $z(0) = z_0 \in K$ which originate in $K_1 \times K_2$ stay in $K_1 \times K_2$ and converge to zero if and only if all trajectories of $\frac{dx}{dt} + Ax = 0$

which originate in K_1 stay in K_1 and converge to zero and all trajectories of $\frac{dy}{dt} + (D - CA^{-1}B)y = 0$ which originate in K_2 stay in K_2 and converge to zero. Putting it in a different way:

When $L \in \mathbf{Z}$, the system $\frac{dz}{dt} + L(z) = 0$ is asymptotically stable if and only if the systems $\frac{dx}{dt} + Ax = 0$ and $\frac{dy}{dt} + (D - CA^{-1}B)y = 0$ are asymptotically stable.

5. Minimum eigenvalue computations

As we noted earlier, on \mathcal{R}_+^n , every \mathbf{Z} -transformation is of the form $L = rI - B$, where B is a nonnegative matrix. In this case, $\tau(L) = r - \rho(B)$. We depart from this classical case, and provide the following examples.

Example 3. Let $H = \mathcal{H}^n$ denote the set of all $n \times n$ complex Hermitian matrices with inner product given by $\langle X, Y \rangle = \text{trace}(XY)$, where the trace of a matrix is the sum of all its diagonal elements (or equivalently, the sum of all its eigenvalues). Let $K = \mathcal{H}_+^n$ denote the set of all positive semidefinite matrices in \mathcal{H}^n . For any $A \in \mathcal{C}^{n \times n}$, consider the Lyapunov transformation L_A defined by

$$L_A(X) := AX + XA^*,$$

where A^* denotes the conjugate transpose of A . It is well known that L_A is Lyapunov-like on \mathcal{H}_+^n and L_A is positive stable if and only if A is positive stable. (In view of the equivalence (1) \Leftrightarrow (3) in Theorem 1, this is Lyapunov's Theorem.) Writing $\tau(A) := \min \operatorname{Re} \sigma(A)$, we see that for any real number r ,

$$\tau(A) > r \Leftrightarrow \tau(A - rI) > 0 \Leftrightarrow \tau(L_{A-rI}) > 0 \Leftrightarrow \tau(L_A) > 2r.$$

Thus,

$$\tau(L_A) = 2\tau(A). \quad (5)$$

A similar statement holds for L_A defined on \mathcal{S}^n , with $A \in \mathcal{R}^{n \times n}$.

Example 4. Let K be a proper cone in any finite dimensional real Hilbert space. For any real number r and a linear transformation S on H with $S(K) \subseteq K$, consider $L = rI - S$. Then L is a \mathbf{Z} -transformation on H . By Krein–Rutman theorem, there is nonzero $u \in K$ such that $S(u) = \rho(S)u$, see [1], page 6. Thus $\rho(S) = \max \operatorname{Re} \sigma(S)$ and consequently,

$$\tau(L) = r - \rho(S).$$

Example 5. Let $H = \mathcal{H}^n$ and $K = \mathcal{H}_+^n$ as in Example 3. For any $A \in \mathcal{C}^{n \times n}$, consider the Stein transformation defined by $S_A(X) := X - AXA^*$. Since the eigenvalues of the transformation $S : X \mapsto AXA^*$, are of the form $\lambda\bar{\mu}$, where $\lambda, \mu \in \sigma(A)$ ([17], page 14), we see that $\rho(S) = \rho(A)^2$. Thus,

$$\tau(S_A) = 1 - \rho(A)^2.$$

A similar statement holds for S_A defined on \mathcal{S}^n , with $A \in \mathcal{R}^{n \times n}$.

Example 6. Let $(H, \circ, \langle \cdot, \cdot \rangle)$ be a Euclidean Jordan algebra with $x \circ y$ and $\langle x, y \rangle$ denoting, respectively, the Jordan product and the inner product in H . Let K denote the corresponding symmetric cone in H . It is known, see [10], that every Lyapunov-like transformation L on K is of the form $L = L_a + D$, where $a \in H$ and L_a is defined by $L_a(x) := a \circ x$ and D is a derivation, i.e., it satisfies the condition

$D(x \circ y) = D(x) \circ y + x \circ D(y)$. As D is skew-symmetric, that is, $\langle D(x), x \rangle = 0$ for all x , we see that L_a is the symmetric (=self-adjoint) part of L and D is the skew-symmetric part of L .

Since $\sigma(L_a) \subseteq \left\{ \frac{\lambda + \mu}{2} : \lambda, \mu \in \sigma(a) \right\}$ (with equality when H is simple, see [9]), we see that

$$\tau(L_a) = \min \sigma(a),$$

where $\sigma(a)$ is the spectrum of a within the algebra H .

Since D is Lyapunov-like, we have from ([18], Theorem 6), $D(u) = \tau(D)u$ for some nonzero $u \in K$. Then $\langle D(u), u \rangle = 0$ implies that

$$\tau(D) = 0.$$

Example 7. Let K be a (general) proper cone and $L \in \mathbf{Z} \cap \mathbf{S}$ with respect to K . Then

$$\tau(L) = (\rho(L^{-1}))^{-1}.$$

To see this, let $\rho := \rho(L^{-1})$ and $\tau := \tau(L)$. Since $L \in \mathbf{Z} \cap \mathbf{S}$, L is positive stable (by Theorem 1). In this case,

$$\tau = \min\{\operatorname{Re}(\lambda) : \lambda \in \sigma(L)\} \leq \min\{|\lambda| : \lambda \in \sigma(L)\} \leq \tau,$$

where the last inequality comes from the fact that τ is a positive eigenvalue of L . Also,

$$\rho = \max\{|\mu| : \mu \in \sigma(L^{-1})\} = \max\left\{\left|\frac{1}{\lambda}\right| : \lambda \in \sigma(L)\right\}.$$

Since this max is the reciprocal of $\min\{|\lambda| : \lambda \in \sigma(L)\}$, we see that $\rho = \frac{1}{\tau}$. This proves the required equality.

6. Minimum eigenvalue inequalities

In this section, we present our minimum eigenvalue inequalities. We begin with a simple observation.

Proposition 3. If $L \in \mathbf{Z}$ and $\alpha = \tau(L)$, then $L - \alpha I + \varepsilon I \in \mathbf{Z} \cap \mathbf{S}$ for all $\varepsilon > 0$.

Proof. With the given conditions on L , $L - \alpha I + \varepsilon I$ has the \mathbf{Z} -property and $\tau(L - \alpha I + \varepsilon I) = \varepsilon > 0$. Thus, $L - \alpha I + \varepsilon I$ is positive stable, and by Theorem 1, has the \mathbf{S} -property. \square

Proposition 4. Suppose L given by (4) has the \mathbf{Z} -property. Then $\tau(A) \geq \tau(L)$.

Proof. Let $\alpha = \tau(L)$. Then, by Proposition 3, $L - \alpha I + \varepsilon I$ has the \mathbf{S} -property for any $\varepsilon > 0$. By Theorem 2, $A - \alpha I_1 + \varepsilon I_1$ has the \mathbf{Z} -property and the \mathbf{S} -property; hence it is positive stable. This implies that $\tau(A) - \alpha + \varepsilon = \tau(A - \alpha I_1 + \varepsilon I_1) > 0$ for all $\varepsilon > 0$. Thus, $\tau(A) \geq \alpha$ proving the result. \square

Our next objective is to relate the minimum eigenvalues of L and its Schur complement. In preparation for this, we prove the following.

Proposition 5. Let $L_1, L_2 \in \mathbf{Z}$ such that $L_1 \succeq L_2$. Then the following statements hold:

$$(a) \quad \tau(L_1) \geq \tau(L_2).$$

If, in addition, $L_2 \in \mathbf{S}$, then

$$(b) \ L_2^{-1} \succeq L_1^{-1}.$$

Proof. (a) Let $\theta = \tau(L_2)$. Then by Proposition 3, $L_2 - \theta I + \varepsilon I$ has the \mathbf{S} -property. Since $L_1 - \theta I + \varepsilon I \succeq L_2 - \theta I + \varepsilon I$, by (3), $L_1 - \theta I + \varepsilon I$ will have the \mathbf{S} -property. Thus, $L_1 - \theta I + \varepsilon I$ is positive stable and $\tau(L_1) - \theta + \varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, it follows that $\tau(L_1) \geq \theta = \tau(L_2)$.

(b) Since L_2 has the \mathbf{S} -property and $L_1 \succeq L_2$, L_1 has the \mathbf{S} -property by (3). Since L_1 and L_2 are in $\mathbf{Z} \cap \mathbf{S}$, they are both invertible and, moreover, $L_1^{-1}(K) \subseteq K$ and $L_2^{-1}(K) \subseteq K$. Thus, for any $x \in K$, $L_2^{-1}(x) - L_1^{-1}(x) = L_1^{-1}(L_1 - L_2)L_2^{-1}(x) \geq 0$. Hence, $L_2^{-1} \succeq L_1^{-1}$. \square

Corollary 1. Consider two linear transformations on $H_1 \times H_2$ given in the block form:

$$L_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

Suppose $L_1, L_2 \in \mathbf{Z}$ and $L_1 \succeq L_2$. Then $A_1 \succeq A_2$ and $D_1 \succeq D_2$. If, in addition, $A_2 \in \mathbf{S}$, then $L_1/A_1 \succeq L_2/A_2$.

Proof. From $L_1 \succeq L_2$, it is easy to see that $A_1 \succeq A_2$ and $D_1 \succeq D_2$. Now assume that A_2 is in \mathbf{S} . Then $A_1 \in \mathbf{S}$, $A_2^{-1} \succeq A_1^{-1}$, and by Proposition 2, $C_2 A_2^{-1} B_2 \succeq C_1 A_1^{-1} B_1$. Thus, $L_1/A_1 - L_2/A_2 = (D_1 - D_2) + (C_2 A_2^{-1} B_2 - C_1 A_1^{-1} B_1) \geq 0$. \square

We now compare the minimum eigenvalues of L and its Schur complement. Our proof is similar to the one given by Smith [20] for matrices. We recall that I denotes the Identity transformation on $H_1 \times H_2$, while I_i denotes the Identity transformation on H_i for $i = 1, 2$.

Theorem 3. Suppose L given by (4) has the \mathbf{Z} -property and $A \in \mathbf{S}$. Then the following hold:

- (i) If $L \in \mathbf{S}$, then $\tau(L/A) \geq \tau(L)$.
- (ii) If $\tau(L/A) > \tau(L)$, then $L \in \mathbf{S}$.

Proof. By Proposition 2, L/A has the \mathbf{Z} -property.

- (i) Assume that $L \in \mathbf{S}$ and let $\alpha = \tau(L)$. Then $\alpha > 0$ and $A, L/A \in \mathbf{Z} \cap \mathbf{S}$ by Theorem 2. Since $\tau(A) \geq \tau(L)$ by Proposition 4, $A - \alpha I_1 + \varepsilon I_1$ is positive stable for $\varepsilon > 0$. Now, $L - \alpha I + \varepsilon I$ is positive stable (by Proposition 3) and has the \mathbf{Z} -property; thus, $(L - \alpha I + \varepsilon I)/(A - \alpha I_1 + \varepsilon I_1)$ is positive stable and has the \mathbf{Z} -property.

Since $A - \alpha I_1 + \varepsilon I_1 \preceq A$ for all small $\varepsilon > 0$, $(A - \alpha I_1 + \varepsilon I_1)^{-1} \succeq A^{-1}$ by Proposition 5. Then for all small $\varepsilon > 0$,

$$\begin{aligned} (L - \alpha I + \varepsilon I)/(A - \alpha I_1 + \varepsilon I_1) &= D - \alpha I_2 + \varepsilon I_2 - C(A - \alpha I_1 + \varepsilon I_1)^{-1}B \\ &\preceq D - \alpha I_2 + \varepsilon I_2 - CA^{-1}B \\ &= (L/A) - \alpha I_2 + \varepsilon I_2. \end{aligned}$$

This implies that $(L/A) - \alpha I_2 + \varepsilon I_2$ has the \mathbf{S} -property, and hence positive stable. Thus, $\tau(L/A) \geq \alpha - \varepsilon$ for all small $\varepsilon > 0$, and $\tau(L/A) \geq \alpha$.

- (ii) Suppose, if possible, $\tau(L/A) > \tau(L)$ and L does not have \mathbf{S} -property. Since $A \in \mathbf{Z} \cap \mathbf{S}$, we must have $\tau(L/A) \leq 0$. Else, $\tau(L/A) > 0$ implies that L/A has the \mathbf{S} -property, and by Theorem 2 L

would have the **S**-property. Now, Let $r = \tau(L)$. To get a contradiction, we show that r is not an eigenvalue of L . As $r < \tau(L/A) \leq 0$, $L/A - rI_2$ is positive stable, hence has the **S**-property. Now, since A has the **S**-property and $A - rI_1 \succeq A$, $A - rI_1$ has the **S**-property and $(A - rI_1)^{-1} \preceq A^{-1}$ by Proposition 5. Since $L - rI$ has the **Z**-property, $(L - rI)/(A - rI_1)$ has the **Z**-property, and

$$(L - rI)/(A - rI_1) = D - rI_2 - C(A - rI_1)^{-1}B \succeq D - rI_2 - CA^{-1}B = L/A - rI_2.$$

Note that Proposition 2 is used in proving the last inequality. Since $L/A - rI_2$ has the **S**-property, $(L - rI)/(A - rI_1)$ has the **S**-property, i.e., it is positive stable. So $0 < \det(A - rI_1) \det((L - rI)/(A - rI_1)) = \det(L - rI)$. This implies that r is not an eigenvalue of L leading to a contradiction. Thus, the implication (ii) holds. \square

Remarks. Suppose L , given by (4), has the **Z**-property and $A \in \mathbf{S}$. When $\tau(L/A) = \tau(L)$, L may or may not belong to \mathbf{S} . To see this, consider the following examples of 2×2 **Z**-matrices (relative to R_+^2): First, let L be the 2×2 Identity matrix. In this case, $A = [1]$, $L/A = [1]$, $L \in \mathbf{S}$ with $\tau(L) = \tau(L/A)$. Next, let

$$L = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

Here $A = [1]$, $L/A = [0]$, and $\tau(L) = \tau(L/A)$. Yet $L \notin \mathbf{S}$.

Remarks. In the above theorem, the condition that A has the **S**-property cannot be dropped. For example, consider the **Z**-matrix $L = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$. Then $A = [-1]$ and L do not have the **S**-property and $\tau(L) = \frac{-1-\sqrt{5}}{2} < 1 = \tau(L/A)$.

7. Z-transformations on Euclidean Jordan algebras

In the previous sections, we dealt with the **Z** and **S** properties of linear transformations defined over product spaces/cones. In this section, we specialize our results to Euclidean Jordan algebras/symmetric cones.

Let V be any Euclidean Jordan algebra of rank r , K be the corresponding symmetric cone and e denote the unit element in V . Given any nonzero idempotent c in V , we consider the subalgebras

$$V_1 := V(c, 1) := \{x \in V : x \circ c = x\} \quad \text{and} \quad V_0 := V(c, 0) := \{x \in V : x \circ c = 0\}.$$

We also let

$$V_{\frac{1}{2}} := V\left(c, \frac{1}{2}\right) := \left\{x \in V : x \circ c = \frac{1}{2}x\right\}.$$

Then the following (orthogonal) Peirce decomposition holds [3]:

$$V = V_1 + V_0 + V_{\frac{1}{2}}.$$

Now, given any linear transformation L on V , using the above decomposition, we write L in the block form

$$L = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}, \quad (6)$$

where the blocks are linear transformations acting on appropriate spaces. (Note that when $c = e$, all blocks except A will be vacuous.) Corresponding to the algebras V_1 and V_0 , we let K_1 and K_0 denote their respective symmetric cones. Let P_c denote the quadratic representation corresponding to c , that is, $P_c(x) := 2c \circ (c \circ x) - c^2 \circ x$. Then, $L_{V(c,1)} := P_c L : V(c,1) \rightarrow V(c,1)$ is, by definition, a *principal subtransformation* of L . We note that P_c is nothing but the (orthogonal) projection of V onto $V(c,1)$ and $L_{V(c,1)}$ is the transformation A defined in the block form of L above.

Now suppose that L has the \mathbf{Z} -property on the symmetric cone of V . By abuse of language, we say that L has the \mathbf{Z} -property on V . Since the symmetric cones of V_1 and V_0 are subsets of K (actually they are faces of K), it follows that

If L has the \mathbf{Z} -property on V , then $L_{V(c,1)}$ has the \mathbf{Z} -property on $V(c,1)$.

A natural question is the following: Do principal subtransformations (defined above) inherit the $\mathbf{Z} \cap \mathbf{S}$ -property? When $V = R^n$, the answer is affirmative as \mathbf{S} and \mathbf{P} properties are equivalent for a \mathbf{Z} -matrix [1] and these three properties are inherited by principal submatrices. However, this is not true on a general Euclidean Jordan algebra as the following example shows. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For the above A , consider the Lyapunov transformation L_A (defined by (1)) on $V = S^2$. As A is positive stable, L_A is also positive stable; hence $L_A \in \mathbf{Z} \cap \mathbf{S}$ on S^2_+ . Now, the principal subtransformation T of L_A defined on $V(E_1,1) = RE_1$ is the zero transformation as it takes λE_1 to $(L_A(\lambda E_1))_{11} = 0$. Hence, this principal subtransformation T cannot have the \mathbf{S} -property, proving that the \mathbf{S} -property need not be inherited by principal subtransformations. (Note that $\tau(T) = 0$, while $\tau(L) > 0$ in contrast to Proposition 4 which deals with the \mathbf{Z} -property on a product cone.) However, as we see below, *some* principal subtransformation inherits the \mathbf{S} -property.

Theorem 4. Suppose L has the $\mathbf{Z} \cap \mathbf{S}$ -property on V . Then there exists a Jordan frame $\{e_1, e_2, \dots, e_r\}$ such that for any $1 \leq k \leq r$ with $c = e_1 + e_2 + \dots + e_k$, the block decomposition of L given by (6) has the following property: The transformation

$$M := \begin{bmatrix} A & B \\ D & E \end{bmatrix} : V_1 \times V_0 \rightarrow V_1 \times V_0$$

has the $\mathbf{Z} \cap \mathbf{S}$ -property on $K_1 \times K_0$. Consequently, A has the $\mathbf{Z} \cap \mathbf{S}$ -property on K_1 and $E - DA^{-1}B$ has the $\mathbf{Z} \cap \mathbf{S}$ -property on K_0 .

Proof. Let $d \in V$ with $d > 0$ and $L(d) > 0$. Corresponding to d , there exists a Jordan frame $\{e_1, e_2, \dots, e_r\}$ such that

$$d = d_1 e_1 + d_2 e_2 + \dots + d_r e_r$$

is the spectral decomposition of d . Note that $d_i > 0$ for all i . Fix k such that $1 \leq k \leq r$ and let $c = e_1 + e_2 + \cdots + e_k$. Corresponding to c , we define spaces V_1 , V_0 , and $V_{\frac{1}{2}}$ and the block form of L given by (6). Clearly,

$$M := \begin{bmatrix} A & B \\ D & E \end{bmatrix} : V_1 \times V_0 \rightarrow V_1 \times V_0.$$

We now proceed to show that M has the specified properties. When $k = r$, $V_1 = V$ and all blocks of L except A become vacuous. In this case, $A = M = L$ and the result is clearly true. We assume that $1 \leq k < r$. Let $0 \leq x \perp z \geq 0$ in V_1 and $0 \leq y \perp w \geq 0$ in V_0 . Then $0 \leq [x, 0, 0]^T \perp [z, 0, 0]^T \geq 0$ in V . Using the **Z**-property of L , we get $\langle Ax, z \rangle \leq 0$. Also, from $0 \leq [x, 0, 0]^T \perp [0, w, 0]^T \geq 0$ in V , we get $\langle Dx, w \rangle \leq 0$. Similarly, we get $\langle By, z \rangle \leq 0$ and $\langle Ey, w \rangle \leq 0$. Combining these, we see that

$$0 \leq \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \perp \begin{bmatrix} z \\ w \\ 0 \end{bmatrix} \geq 0 \Rightarrow \left\langle \begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right\rangle \leq 0.$$

Thus, M has the **Z**-property on $K_1 \times K_0$. Now let $u := d_1 e_1 + d_2 e_2 + \cdots + d_k e_k > 0$ in K_1 and $v := d_{k+1} e_{k+1} + \cdots + d_r e_r > 0$ in K_0 . Then $d = u + v + 0$ in $V = V_1 + V_0 + V_{\frac{1}{2}}$. This yields,

$$0 < L(d) = [Au + Bv, Du + Ev, Gu + Hv]^T.$$

It follows that $Au + Bv > 0$ in K_1 and $Du + Ev > 0$ in K_0 . Thus,

$$\begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} > 0$$

in $K_1 \times K_0$. This proves the **S**-property of M on $K_1 \times K_0$. Now we can apply our earlier results to conclude that $A \in \mathbf{Z} \cap \mathbf{S}$ on K_1 and $E - DA^{-1}B$ has the **Z** \cap **S**-property on K_0 . \square

Given $L \in \mathbf{Z}(K)$ and a natural number k , $1 \leq k \leq r$, we let

$$\tau_k(L) := \sup_{\{c \in V: c^2=c, \text{rank}(c)=k\}} \tau(L_{V(c,1)}).$$

Theorem 5. Let V be an Euclidean Jordan algebra of rank r and $L \in \mathbf{Z}(K)$. Then

$$\tau(L) = \tau_r(L) \leq \tau_{r-1}(L) \leq \cdots \leq \tau_2(L) \leq \tau_1(L) \leq \|L\|.$$

Proof. Fix a natural number k with $1 \leq k \leq r$. We first show that

$$\tau(L) \leq \tau_k(L). \quad (7)$$

Let $\theta := \tau(L)$ so that for any $\varepsilon > 0$, $L_\varepsilon := L - \theta I + \varepsilon I \in \mathbf{Z} \cap \mathbf{S}$. Then, by the above theorem, there exists an idempotent c_ε of rank k such that $A_\varepsilon := (L_\varepsilon)_{V(c_\varepsilon,1)}$ has the **Z** \cap **S**-property on $V(c_\varepsilon, 1)$. Thus, $\tau(A_\varepsilon) > 0$. Now, let $A_\varepsilon^0 := L_{V(c_\varepsilon,1)}$. Using the definition that $L_{V(c,1)} := P_c L$ for any idempotent c , we easily see that $A_\varepsilon = A_\varepsilon^0 - \theta I_1 + \varepsilon I_1$, where I_1 is the Identity transformation on $V(c_\varepsilon, 1)$. Since $\tau(A_\varepsilon) > 0$, we have $\tau(A_\varepsilon^0) - \theta + \varepsilon > 0$. This gives,

$$\tau_k(L) \geq \tau(A_\varepsilon^0) \geq \theta - \varepsilon = \tau(L) - \varepsilon$$

for all $\varepsilon > 0$. Since ε is arbitrary, we get (7).

We now proceed to prove the chain of inequalities.

- (i) Let $k = r$. Then any idempotent c with rank r must be the unit element in V . In this case, $V(c, 1) = V$ and $L_{V(c,1)} = L$. Hence $\tau_r(L) = \tau(L)$.
- (ii) Let $k = 1$. In this case, any idempotent of rank k must be primitive and $V(c, 1) = Rc$ and $L_{V(c,1)}(x) = \frac{\langle L(c), c \rangle}{\|c\|^2} x$ for all $x \in V(c, 1)$. Hence, by Cauchy–Schwarz inequality, $\tau(L_{V(c,1)}) = \frac{\langle L(c), c \rangle}{\|c\|^2} \leq \|L\|$, where $\|L\|$ denotes the operator norm of L on V . This gives, $\tau_1(L) \leq \|L\|$.
- (iii) Now let $1 \leq l \leq k \leq r$. Consider an idempotent c of rank k in V . For simplicity, let

$$V_1 := V(c, 1) \quad \text{and} \quad T := L_{V(c,1)}.$$

Applying (7) to T , we get

$$\tau(T) \leq \sup_{\{u \in V_1 : u^2 = u, \text{rank}(u) = l\}} \tau(T_{V_1(u,1)}).$$

We note that for $u \in V_1$ with $u^2 = u$, we have (see [23], Lemma 1 and the corresponding Remark)

$$V_1(u, 1) = V(u, 1).$$

Since c is the unit element in $V(c, 1)$, c and u operator commute, and $u \circ c = u$. Hence, on $V(u, 1)$, $T_{V_1(u,1)} = P_u T = P_u P_c L = P_{u \circ c} L = P_u L = L_{V(u,1)}$, and so,

$$\tau(T) \leq \sup_{\{u \in V_1 : u^2 = u, \text{rank}(u) = l\}} \tau(T_{V_1(u,1)}) \leq \sup_{\{u \in V : u^2 = u, \text{rank}(u) = l\}} \tau(L_{V(u,1)}) = \tau_l(L).$$

This gives $\tau(L_{V(c,1)}) \leq \tau_l(L)$. Now taking the supremum over all idempotents c in V of rank k , we get $\tau_k(L) \leq \tau_l(L)$, proving the final part of the chain of inequalities. \square

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