On the inheritance of some complementarity properties by Schur complements

Roman Sznajder
Department of Mathematics
Bowie State University
Bowie, MD 20715
rsznajder@bowiestate.edu
*
M. Seetharama Gowda
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21250
gowda@math.umbc.edu
*
Jiyuan Tao
Department of Mathematics and Statistics
Loyola University Maryland
Baltimore, Maryland 21210
jtao@loyola.edu

July 17, 2012
Revised: February 10, 2013

Abstract

In this paper, we consider the Schur complement of a subtransformation of a linear transformation defined on the product of two finite dimensional real Hilbert spaces, and in particular, on two Euclidean Jordan algebras. We study complementarity properties of linear transformations that are inherited by principal subtransformations, principal pivot transformations, and Schur complements.

Keywords: Inheritance property, Principal pivotal transformation, Schur complement, Principal subtransformation, Complementarity problem, Symmetric cone, Euclidean Jordan algebra

AMS subject classifications: 90C33, 17C55, 17C20

Abbreviated title: Inheritance properties

1 Introduction

Given a matrix $M \in \mathcal{R}^{n\times n}$ and a vector $q \in \mathcal{R}^n$, the *(standard) linear complementarity problem* LCP(M, q) [5] is to find a vector $x \in \mathcal{R}^n$ such that

$$0 \leq x \perp Mx + q \geq 0,$$

where $x \geq 0$ means that x belongs to the nonnegative orthant in $\mathcal{R}^n$. In the complementarity literature, numerous classes such as $\mathbf{Q}, \mathbf{GUS}, \mathbf{R}$, etc., have been introduced specifically to study the existence, uniqueness, stability, and computational issues. When M is written in the block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with A invertible, the Schur complement of A in M is given by

$$M/A = D - CA^{-1}B.$$

The Schur complement enjoys numerous properties and appears in various applications, see [4], [13], [18] and the references therein. In particular, it is well known that if M is positive definite, then so are A and M/A. In the setting of linear complementarity problems, it is known, see [5], that if M is a **P**-matrix (which means that all principal minors of M are positive, or equivalently, LCP(M, q) has a unique solution for all q), then A and M/A are also **P**-matrices. Recently, Chua et al. [3] introduced the concept of *uniform nonsingularity property* (**UNS**-property for short) of a (linear) transformation on a Euclidean Jordan algebra as a generalization of this **P**-matrix property. The motivation for our paper comes from the question whether the **UNS**-property is inherited by subtransformations and Schur complements. To elaborate, consider two finite dimensional real Hilbert spaces V_1 and V_2 with corresponding proper cones K_1 and K_2; in particular, these could be Euclidean Jordan algebras with their symmetric cones. Given a linear transformation L on the product space $V_1 \times V_2$, we write L in the block form similar to the matrix case (given above), where the blocks are now linear transformations. With A invertible, we define the Schur complement L/A as in the matrix case. The question raised in this paper is the following: *What complementarity properties of L are inherited by A and L/A?* In particular, is the **UNS**-property one such property when both V_1 and V_2 are Euclidean Jordan algebras? We answer these by proving the inheritance of various generalizations of the **P**-property, namely, the Cartesian **P**-property [6], [1], **GUS**-property [10], and the **UNS**-property [3]. Some of these results are proved via the so-called principal pivotal transform of L defined as in the matrix case.

The framework of the product spaces appears in various instances particularly in connection with complementarity problems on product symmetric cones [1], [12], [17]. Here is an example from conic optimization. Assume $K_1 \subseteq \mathcal{R}^n$ and $K_2 \subseteq \mathcal{R}^m$ are proper cones and consider the quadratic problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^TQx + \langle c, x\rangle \\ \text{subject to} & Ax \underset{K_2}{\geq} b \\ & x \underset{K_1}{\geq} 0 \end{array}$$

where $Q \in \mathcal{R}^{n\times n}$ is a symmetric matrix, $c \in \mathcal{R}^n$, $A \in \mathcal{R}^{m\times n}$ and $b \in \mathcal{R}^m$. Here, $z \underset{K}{\geq} 0$ means that $z \in K$. Under certain constraint qualifications, if x is a solution of the above problem, then there exists $y \in K_2$ (see [5], [6]) such that

$$\begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

with

$$\begin{bmatrix} x \\ y \end{bmatrix} \in K_1 \times K_2, \begin{bmatrix} u \\ v \end{bmatrix} \in K_1^* \times K_2^*, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} \perp \begin{bmatrix} u \\ v \end{bmatrix}.$$

The linear transformation $L = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$ acts on the cone $K_1 \times K_2$. In case of invertible matrix Q, $L/Q = AQ^{-1}A^T$.

The organization of the paper is as follows. In the next section, we cover some basic material including the definitions of Schur complement, principal pivotal transform, and all LCP concepts. In Section 3, we consider some complementarity properties that are invariant under principal pivotal transforms. Section 4 deals with the subtransformation inheritance properties, and finally in Section 5, we deal with those complementarity properties that are inherited by Schur complements.

2 Preliminaries

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Hilbert space and K denote a *proper convex cone*, i.e., K is a *closed convex pointed cone with a nonempty interior*. We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (respectively, $x \in K^\circ$, the interior of K). Also, K^* denotes the dual cone of K given by $K^* := \{x \in V : \langle x, y\rangle \geq 0 \text{ for all } y \in K\}$.

Specializing, we (also) let $(V, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra of rank r and $K := \{x^2 : x \in V\}$ be its symmetric cone of squares [7], [10]. It is well known that any Euclidean Jordan algebra is a product of simple Euclidean Jordan algebras and every simple algebra is isomorphic to the Jordan spin algebra $\mathcal{L}^n$ or to the algebra of all $n \times n$ real/complex/quaternion Hermitian matrices, or to the algebra of all 3×3 octonion Hermitian matrices. Given any $a \in V$, we let L_a denote the corresponding Lyapunov transformation on V:

$$L_a(x) := a \circ x.$$

We say that objects a and b in V *operator commute* if $L_aL_b = L_bL_a$. It is known that a and b operator commute if and only if they have their spectral decompositions with respect to a common Jordan frame. One sufficient condition for operator commutativity is: $0 \leq a \perp b \geq 0$, see [10], Proposition 6.

2.1 Linear transformations on product spaces

In what follows, we will focus on linear transformations defined on a product of two finite dimensional real Hilbert spaces, and in particular, on the product of two Euclidean Jordan algebras. Let

$V = V_1 \times V_2$ be the product of two such spaces and consider a linear transformation $L : V \to V$. Then, L can be uniquely presented in a block form as

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

where entries of this "matrix" are linear operators acting in the following way:

$$A : V_1 \to V_1,\ B : V_2 \to V_1,\ C : V_1 \to V_2,\ \text{and } D : V_2 \to V_2. \tag{2}$$

When A is invertible, we define the Schur complement of A in L by

$$L/A := D - CA^{-1}B \tag{3}$$

and the *principal pivotal transform* of L with respect to A by

$$L^{\diamond} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & L/A \end{bmatrix}. \tag{4}$$

Note that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = L^{\diamond} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = L \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -Aq_1 \\ q_2 - Cq_1 \end{bmatrix}. \tag{5}$$

In case when L and L/A are invertible, we obtain the known formula ([13]):

$$L^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(L/A)^{-1}CA^{-1} & -A^{-1}B(L/A)^{-1} \\ -(L/A)^{-1}CA^{-1} & (L/A)^{-1} \end{bmatrix}. \tag{6}$$

More information on principal pivotal transformations and a historical account can be found in [16].

We assume that each of the spaces V_i $(i = 1, 2)$ is equipped with a proper cone K_i. This way, we have a natural proper (product) cone $K = K_1 \times K_2$ in V. Thus, the ordering in V by the cone K is determined by the ordering of factor spaces V_i by the cones K_i, in the sense that

$$\begin{bmatrix} x \\ y \end{bmatrix} \underset{K}{\geq} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{if and only if} \quad x \underset{K_1}{\geq} 0 \quad \text{and} \quad y \underset{K_2}{\geq} 0\,.$$

When the context is clear, we will drop the explicit mentioning of the cone.

2.2 LCP Concepts

For a linear transformation $L : V \to V$ and a vector $q \in V$, the *linear complementarity problem* $\mathrm{LCP}(L, K, q)$ is to find a vector $x \in K$ such that $y := L(x) + q \in K^*$ and $\langle x, y \rangle = 0$.

Given the space $V = V_1 \times V_2$ and the cone $K = K_1 \times K_2$, for a linear transformation $L : V \to V$ of the form (1) and a vector $[q_1, q_2]^T$ (which is the column vector with components $q_1 \in V_1$

and $q_2 \in V_2$), the linear complementarity problem $\mathrm{LCP}(L, K_1 \times K_2, [q_1, q_2]^T)$ is to find a vector $[x_1, x_2]^T \in V$ such that

$$0 \underset{K_1 \times K_2}{\leq} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \perp \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \underset{K_1^* \times K_2^*}{\geq} 0\,, \tag{7}$$

where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

The solution set of $\mathrm{LCP}(L, K, q)$ is denoted by $\mathrm{SOL}(L, K, q)$, or (when L and K are fixed and the context is clear) by $\mathrm{SOL}(q)$.

We remark that the complementarity problems of L and $L^\diamond$ are defined on different cones, namely L on $K_1 \times K_2$ and $L^\diamond$ on $K_1^* \times K_2$. While dealing with L and the corresponding cone $K_1 \times K_2$, we abbreviate $\mathrm{SOL}(L, K_1 \times K_2, q)$ by $\mathrm{SOL}(L, q)$. Correspondingly, while dealing with $L^\diamond$ and the related cone $K_1^* \times K_2$, we abbreviate $\mathrm{SOL}(L^\diamond, K_1^* \times K_2, p)$ by $\mathrm{SOL}(L^\diamond, p)$.

Given a proper cone K in V, we say that a linear transformation L on V has the

(*i*) **Q**-*property* if for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a solution;

(*ii*) **GUS**-*property* if for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a unique solution;

(*iii*) *Lipschitzian property* if for all $p, q \in V$, such that $\mathrm{SOL}(p) \neq \emptyset$ and $\mathrm{SOL}(q) \neq \emptyset$, it holds that

$$\mathrm{SOL}(p) \subseteq \mathrm{SOL}(q) + c\|p - q\|\mathbb{B},$$

where $\mathbb{B}$ is the unit ball in V and $c > 0$ is a constant independent of p and q;

(*iv*) *strict monotonicity* property if $\langle L(x), x\rangle > 0$ for all $x \neq 0$;

(*v*) *Cartesian* **P**-*property* if $V = E_1 \times E_2 \times \cdots \times E_l$ and

$$\max_{1 \leq i \leq l} \langle L_i(x), x_i\rangle > 0 \ \ \forall\, x \neq 0,$$

where $L_i(x) := (L(x))_i$ for all i.

Now suppose that V is a Euclidean Jordan algebra and K denotes its symmetric cone. We say that a linear transformation L defined on V has the

(*a*) *cross-commutative* property if for any $q \in V$ and any two solutions x_1 and x_2 of $\mathrm{LCP}(L, K, q)$, x_1 operator commutes with y_2 and x_2 operator commutes with y_1, where $y_i = L(x_i) + q$ $(i = 1, 2)$.

(*b*) **P**-*property* if x and $L(x)$ operator commute and $x \circ L(x) \leq 0 \Rightarrow x = 0$;

(*c*) *Uniform nonsingularity property* (**UNS**) if there exists $\Delta > 0$ such that for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V, any $x \in V$ with its Peirce decomposition $x = \sum_{i\le j} x_{ij}$ (with respect to the given Jordan frame), and any $r \times r$ nonnegative, symmetric matrix $D = [d_{ij}]$,

$$\|L(x) + \sum_{i\le j} d_{ij} x_{ij}\| \ge \Delta \|x\|. \tag{8}$$

The **UNS**-property was introduced in [2], see also [6], [1], and [3], where the following implications were proved:

$$\textit{strict monotonicity} \Rightarrow \textit{Cartesian } \mathbf{P} \Rightarrow \mathbf{UNS} \Rightarrow \mathbf{GUS}.$$

We use the notation $L \in \mathbf{T}(K)$ or $L \in \mathbf{T}$ to say that L has the $\mathbf{T}$-property with respect to K.

Proposition 1 *([10], [11]) We have:*
(a) strict monotonicity $\Rightarrow$ Lipschitz $\cap$ $\mathbf{GUS} \Rightarrow \mathbf{GUS} \Rightarrow \mathbf{Q}$,
(b) In the context of a Euclidean Jordan algebra, $\mathbf{GUS} \Leftrightarrow$ cross-commutative $\cap$ $\mathbf{P}$, $\mathbf{P} \Rightarrow \mathbf{Q}$.

3 Invariance under principal pivotal transformations

It is easy to see from classical matrix theory results that (strict) monotonicity is inherited by principal pivotal transforms, principal subtransformations, and Schur complements. In this section, we show that many of the complementarity properties are similarly inherited by principal pivotal transforms.

Theorem 1 *When $\mathbf{T} \in \{\mathbf{Q}$, $\mathbf{GUS}$, Cartesian $\mathbf{P}$, Lipschitzian, $\mathbf{P}$, cross-commutative$\}$ and A is invertible, L has the $\mathbf{T}$-property if and only if $L^\diamond$ has the $\mathbf{T}$-property.*

Proof. Since $(L^\diamond)^\diamond = L$, it is enough to show that $L \in \mathbf{T} \Rightarrow L^\diamond \in \mathbf{T}$.
(1) Let $\mathbf{T} \in \{\mathbf{Q}, \mathbf{GUS}\}$. Given $[q_1, q_2]^T \in V$, we have from (5),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{SOL}\left(L^\diamond, K_1^* \times K_2, \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) \iff \begin{bmatrix} y_1 \\ x_2 \end{bmatrix} \in \mathrm{SOL}\left(L, K_1 \times K_2, \begin{bmatrix} -Aq_1 \\ q_2 - Cq_1 \end{bmatrix}\right), \tag{9}$$

where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = L^\diamond \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

We easily verify that if $\mathbf{T}$ has either the $\mathbf{Q}$ or $\mathbf{GUS}$ property, then $L^\diamond$ will also have the same property.

(2) Let $\mathbf{T}$ = Cartesian $\mathbf{P}$. For $1 \le m < l$, we write $V = E_1 \times \cdots \times E_l$, with $V_1 = E_1 \times \cdots \times E_m$ and $V_2 = E_{m+1} \times \cdots \times E_l$. Let

$$0 \ne \begin{bmatrix} x \\ y \end{bmatrix} \in V_1 \times V_2 \text{ and } \mathrm{L} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

We know that $\max_{1\le i\le m}\langle u_i, x_i\rangle > 0$ or $\max_{m+1\le j\le l}\langle v_j, y_j\rangle > 0$. Since $L^\diamond[u,y]^T = [x,v]^T$, $L^\diamond$ has the Cartesian $\mathbf{P}$-property on $V = E_1\times\cdots\times E_m\times E_{m+1}\times\cdots\times E_l$.

(3) Let $\mathbf{T}$ = Lipschitzian. Fix $p=[p_1,p_2]^T$ and $q=[q_1,q_2]^T$ with $\mathrm{SOL}(L^\diamond,p)$ and $\mathrm{SOL}(L^\diamond,q)$ nonempty. Let

$$\begin{bmatrix} u_1\\ u_2\end{bmatrix}\in \mathrm{SOL}(L^\diamond,p) \text{ and } \mathrm{L}^\diamond\begin{bmatrix} u_1\\ u_2\end{bmatrix}+\begin{bmatrix} p_1\\ p_2\end{bmatrix}=\begin{bmatrix} x_1\\ x_2\end{bmatrix}.$$

Then by (9),

$$\begin{bmatrix} x_1\\ u_2\end{bmatrix}\in \mathrm{SOL}\left(L,\begin{bmatrix} -Ap_1\\ p_2-Cp_1\end{bmatrix}\right).$$

From the Lipschitzian property of L, there exists a constant c and

$$\begin{bmatrix} \bar{x}_1\\ \bar{u}_2\end{bmatrix}\in \mathrm{SOL}\left(L,\begin{bmatrix} -Aq_1\\ q_2-Cq_1\end{bmatrix}\right)$$

such that

$$\left\|\begin{bmatrix} x_1\\ u_2\end{bmatrix}-\begin{bmatrix} \bar{x}_1\\ \bar{u}_2\end{bmatrix}\right\|\le c\left\|\begin{bmatrix} -Ap_1\\ p_2-Cp_1\end{bmatrix}-\begin{bmatrix} -Aq_1\\ q_2-Cq_1\end{bmatrix}\right\|\le \bar{c}\|p-q\|,$$

where $\bar{c}$ depends on c and L.

Hence, $\|x_1-\bar{x}_1\|\le\bar{c}\|p-q\|$ and $\|u_2-\bar{u}_2\|\le\bar{c}\|p-q\|$. By letting

$$\begin{bmatrix} \bar{u}_1\\ \bar{x}_2\end{bmatrix}=L\begin{bmatrix} \bar{x}_1\\ \bar{u}_2\end{bmatrix}+\begin{bmatrix} -Aq_1\\ q_2-Cq_1\end{bmatrix},$$

and using (5) and (9), we have

$$L^\diamond\begin{bmatrix} \bar{u}_1\\ \bar{u}_2\end{bmatrix}+\begin{bmatrix} q_1\\ q_2\end{bmatrix}=\begin{bmatrix} \bar{x}_1\\ \bar{x}_2\end{bmatrix}, \text{ and } \begin{bmatrix} \bar{u}_1\\ \bar{u}_2\end{bmatrix}\in\mathrm{SOL}(\mathrm{L}^\diamond,\mathrm{q}).$$

Now,

$$\left\|\begin{bmatrix} u_1\\ u_2\end{bmatrix}-\begin{bmatrix} \bar{u}_1\\ \bar{u}_2\end{bmatrix}\right\|=\left\|\begin{bmatrix} A(x_1-\bar{x}_1)+B(u_2-\bar{u}_2)+A(q_1-p_1)\\ u_2-\bar{u}_2\end{bmatrix}\right\|\le c'\,\|q-p\|,$$

where c' is a positive constant, that depends on $\bar{c}$ and the blocks of L.
In the rest of the proof, we deal with Euclidean Jordan algebras.

(4) Let $\mathbf{T}=\mathbf{P}$. Let $x=[x_1,x_2]^T$ and $L^\diamond(x)=y=[y_1,y_2]^T$ operator commute and $x\circ y\le 0$. Then x_i and y_i operator commute for $i=1,\,2$. Thus, using (5) with $q=0$, $[x_1,y_2]^T=L[y_1,x_2]^T$ operator commutes with $[y_1,x_2]^T$. Since $L\in\mathbf{P}$, it follows that $x=0$, proving the $\mathbf{P}$-property of $L^\diamond$.

(5) Let $\mathbf{T}$ = cross-commutative. Fix any $[q_1,q_2]^T\in V$.

Suppose that $[x_1, x_2]^T$ and $[u_1, u_2]^T$ are two solutions to $\mathrm{LCP}\big(L^\diamond, [q_1, q_2]^T\big)$, and let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = L^\diamond \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathrm{L}^\diamond \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Now, using (9), we obtain

$$\begin{bmatrix} y_1 \\ x_2 \end{bmatrix} \in \mathrm{SOL}\left(L, \begin{bmatrix} -Aq_1 \\ q_2 - Cq_1 \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \in \mathrm{SOL}\left(\mathrm{L}, \begin{bmatrix} -Aq_1 \\ q_2 - Cq_1 \end{bmatrix}\right),$$

which together with (5) and the cross-commutative property of L imply that

$$\langle u_1, y_1\rangle = 0,\ \langle v_2, x_2\rangle = 0,\ \langle x_1, v_1\rangle = 0,\ \langle y_2, u_2\rangle = 0.$$

From [10], Proposition 6, these give the operator commutativity of various vectors and leads to the verification of the cross-commutative property of $L^\diamond$. □

4 Inheritance of complementarity properties by principal subtransformations

A property $\mathbf{T}$ is called a *principal subtransformation inheritance property*, if for any linear transformation L given by (1), we have $L \in \mathbf{T} \Rightarrow A \in \mathbf{T}$ and $D \in \mathbf{T}$.

Theorem 2 *On a proper cone $K_1 \times K_2$,* **GUS** *and Cartesian* **P** *are principal subtransformation inheritance properties.*

Proof. (1) Let $\mathbf{T} = \mathbf{GUS}$. For any $q \in V_1$, suppose that $\mathrm{LCP}(A, q)$ has two solutions x_1 and x_2.

Now, since K_2 is proper, we can choose $p \in V_2$ such that $p + Cx_1 \geq 0$ and $p + Cx_2 \geq 0$. Then

$$L \begin{bmatrix} x_i \\ 0 \end{bmatrix} + \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} Ax_i + q \\ Cx_i + p \end{bmatrix} \geq 0$$

for $i = 1, 2$. Hence, $[x_1, 0]^T$ and $[x_2, 0]^T$ are solutions of $\mathrm{LCP}\big(L, [q, p]^T\big)$. Since $L \in \mathbf{GUS}$, $x_1 = x_2$. Thus, $A \in \mathbf{GUS}$.

(2) Let $\mathbf{T} =$ Cartesian $\mathbf{P}$. We assume that $V = E_1 \times \cdots \times E_l$ and $\max_{1\leq i\leq l}\langle L_i(x), x_i\rangle > 0\ \forall\, x \neq 0$. For $1 \leq m < l$, let $V_1 = E_1 \times \cdots \times E_m$ and $V_2 = E_{m+1} \times \cdots \times E_l$. We will show that A defined on V_1 has the Cartesian $\mathbf{P}$-property.

Let $u \neq 0$ in V_1 and $x := [u, 0]^T \neq 0$ in V. Since $L \in$ Cartesian $\mathbf{P}$, there is an index $i \in \{1, 2, \ldots, l\}$ such that

$$\langle L_i(x), x_i\rangle > 0.$$

As this index must be in $\{1, 2, \ldots, m\}$, we see that $\langle A_i(u), u_i\rangle > 0$. This completes the proof.

Similarly, we can show that $D \in \mathbf{T}$. $\square$

In the above result we showed that $L \in \mathbf{GUS}$ implies $A, D \in \mathbf{GUS}$. While the converse is not true in general, see the example given in the next section, the converse does hold when $C = 0$. To see this, let $L = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ with $A, D \in \mathbf{GUS}$. We verify the **GUS**-property for L. Let $[p, q]^T \in V_1 \times V_2$ be any element and, for $i = 1, 2$, $[x_i, y_i]^T \in \mathrm{SOL}\left(L, K_1 \times K_2, [p, q]^T\right)$. Thus, we get

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} \in K_1 \times K_2, \begin{bmatrix} u_i \\ v_i \end{bmatrix} := L \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \in K_1^* \times K_2^*, \text{ and } \begin{bmatrix} x_i \\ y_i \end{bmatrix} \perp \begin{bmatrix} u_i \\ v_i \end{bmatrix}. \qquad (10)$$

From the second item in (10), we get

$$\begin{aligned} Ax_i + By_i + p &= u_i \\ Dy_i + q &= v_i. \end{aligned}$$

As $y_i \in K_2, v_i = Dy_i + q \in K_2^*$, and $y_i \perp v_i$, we see that $y_1, y_2 \in \mathrm{SOL}(D, K_2, q)$. Hence, $y_1 = y_2$, as $D \in \mathbf{GUS}$. Now, we put $\bar{y} := y_1 = y_2$. Then for $i = 1, 2$,

$$Ax_i + (B\bar{y} + p) = u_i \in K_1^*, \text{ and } x_i \perp u_i,$$

so $x_1, x_2 \in \mathrm{SOL}(A, K_1, B\bar{y} + p)$. As $A \in \mathbf{GUS}$, $x_1 = x_2$. Finally, $[x_1, y_1]^T = [x_2, y_2]^T$; hence $L \in \mathbf{GUS}$.

Theorem 3 *On a symmetric cone of the form $K_1 \times K_2$, $\mathbf{P}$, cross-commutative, and $\mathbf{UNS}$ properties are inherited by principal subtransformations.*

Proof. (a) Let $\mathbf{T} = \mathbf{P}$: Suppose that Ax_1 and x_1 operator commute and $x_1 \circ Ax_1 \leq 0$. Then

$$L \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \circ Ax_1 \\ 0 \end{bmatrix} \leq 0.$$

Since $L \in \mathbf{P}$, and $[Ax_1, Cx_1]^T$ and $[x_1, 0]^T$ operator commute, we must have $x_1 = 0$.
(b) Let $\mathbf{T} =$ *Cross-commutative*: For any $q \in V_1$, suppose that $\mathrm{LCP}(A, q)$ has two solutions x_1 and x_2. Put $y_1 := Ax_1 + q$ and $y_2 := Ax_2 + q$. Now choose $p \in V_2$ such that $Cx_1 + p \geq 0$ and $Cx_2 + p \geq 0$, so that for $i = 1, 2$,

$$L \begin{bmatrix} x_i \\ 0 \end{bmatrix} + \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} Ax_i + q \\ Cx_i + p \end{bmatrix} = \begin{bmatrix} y_i \\ Cx_i + p \end{bmatrix} \geq 0.$$

Thus, $[x_1, 0]^T$ and $[x_2, 0]^T$ are solutions of $\mathrm{LCP}\left([q, p]^T\right)$. Since L has the cross-commutative property, for $i = 1, 2$, $[x_i, 0]^T$ operator commutes with $[y_{3-i}, Cx_{3-i} + p]^T$, which implies the operator commutativity of x_1 and y_2, and of x_2 and y_1. This shows the cross-commutative property for A.

(c) Let $\mathbf{T} = \mathbf{UNS}$. It follows from Theorem 3.1 in [14] that every principal subtransformation of L has the **UNS**-property. Since A can be treated as a principal subtransformation of L (see [14] for definition and details on principal subtransformations) we get our claim. See the proof of Theorem 4 below for an alternate proof. □

Remarks. The following example shows that the Lipschitzian property is *not* inherited by principal subtransformations. Let $M = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$, $K_1 = \mathcal{R}_+$, $K_2 = \mathcal{R}_+$. Then the set of all q's, for which $\text{LCP}(M, \mathcal{R}^2_+, q)$ is solvable, is $\mathcal{R}^2_+$. Moreover, for $q = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$,

$$\text{SOL}(q) = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if } \beta = 0, \\ \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right\} & \text{if } \alpha > \beta > 0, \\ \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \beta - \alpha \\ \alpha \end{bmatrix} \right\} & \text{if } \beta \geq \alpha > 0, \\ \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, 0 \leq x \leq \beta \right\} & \text{if } \alpha = 0\,. \end{cases}$$

One easily checks, using the 1-norm, that M is a Lipschitzian transformation with $c = 2$ (for the Euclidean norm, we may put $c = 2\sqrt{2}$). Consider now $N = [0]$, a linear subtransformation of M. We have $\text{SOL}(N, \mathcal{R}_+, 0) = \mathcal{R}_+$, while $\text{SOL}(N, \mathcal{R}_+, 1) = \{0\}$. Observe that the set of all q's for which $\text{SOL}(N, \mathcal{R}_+, q) \neq \emptyset$ is $\mathcal{R}_+$. Clearly the inclusion $\text{SOL}(N, \mathcal{R}_+, 0) \subseteq \text{SOL}(N, \mathcal{R}_+, 1) + c|1-0|\mathbb{B}$ cannot hold as the right-hand side set is bounded, while $\text{SOL}(N, \mathcal{R}_+, 0)$ is unbounded. This contradiction shows that N is not Lipschitzian.

5 Inheritance of complementarity properties by Schur complements

The Schur complement L/A of a block linear transformation L is intimately related to its principal pivot transformation $L^\diamond$. In this section, we study the preservation of certain complementarity properties by Schur complements.

Theorem 4 *Let* $\mathbf{T} \in \{\mathbf{GUS}$*, Cartesian* $\mathbf{P}$*, cross-commutative,* $\mathbf{P}$*,* $\mathbf{UNS}\}$*. If* L *has the* $\mathbf{T}$*-property and* A *is invertible, then* L/A *has the* $\mathbf{T}$*-property.*

Proof.
(1) Suppose that $L \in \mathbf{GUS}$. Then $L^\diamond \in \mathbf{GUS}$ by Theorem 1. Hence, $L/A \in \mathbf{GUS}$ follows from Theorem 2.

(2) Suppose that $L \in$ Cartesian $\mathbf{P}$ on $V = E_1 \times \cdots \times E_l$. Again, for $1 \leq m < l$, let $V_1 = E_1 \times \cdots \times E_m$ and $V_2 = E_{m+1} \times \cdots \times E_l$. We claim that L/A has the Cartesian $\mathbf{P}$-property on $E_{m+1} \times \cdots \times E_l$. Observe that the Cartesian $\mathbf{P}$-property of L implies that L is invertible and $L^{-1} \in$ Cartesian $\mathbf{P}$. By (6),

$$L^{-1} = \begin{bmatrix} * & * \\ * & (L/A)^{-1} \end{bmatrix}.$$

As $L^{-1} \in$ Cartesian $\mathbf{P}$, by Theorem 2, $(L/A)^{-1} \in$ Cartesian $\mathbf{P}$, hence $L/A \in$ Cartesian $\mathbf{P}$ on $E_{m+1} \times \cdots \times E_l$. This implies that L/A has the Cartesian $\mathbf{P}$-property.

In the rest of the proof, we deal with Euclidean Jordan algebras.

(3) If L has the cross-commutativity property, then by Theorem 1, $L^{\diamond}$ has the cross-commutativity property. By Theorem 3, L/A has the cross-commutativity property.

(4) Suppose that $L \in \mathbf{P}$. Then $L^{\diamond} \in \mathbf{P}$ by Theorem 1. Hence, $L/A \in \mathbf{P}$ by Theorem 3.

(5) Let L have the **UNS**-property on $V_1 \times V_2$. Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V_1 and $\{f_1, f_2, \ldots, f_s\}$ be a Jordan frame in V_2. Then

$$\left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ f_1 \end{bmatrix}, \begin{bmatrix} 0 \\ f_2 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ f_s \end{bmatrix} \right\}$$

is a Jordan frame in $V_1 \times V_2$. For $u \in V_1$ and $v \in V_2$, write their Peirce decompositions $u = \sum u_{ij}$ and $v = \sum v_{ij}$. Since L has the **UNS**-property,

$$\left\| \begin{bmatrix} Au + Bv + \sum \alpha_{ij} u_{ij} \\ Cu + Dv + \sum \beta_{ij} v_{ij} \end{bmatrix} \right\| \geq \Delta \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| \quad \forall \alpha_{ij},\, \beta_{ij} \geq 0. \tag{11}$$

In (11), we let $u = -A^{-1}Bv$ and $\alpha_{ij} = 0$ for all i, j. Then

$$||(D - CA^{-1}B)v + \sum \beta_{ij} v_{ij}|| \geq \Delta \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| \geq \Delta ||v||.$$

So L/A has the **UNS**-property. □

As a by-product of the proof given in part (5) above, we can show that the subtransformation A of an **UNS** transformation L also has the **UNS**-property. Here is the argument:
In (11), we fix $u = \sum u_{ij}$ and let $w := -Cu = \sum w_{ij}$, $\beta_{ij} := k$ (natural number) for all i, j, and $v := \frac{1}{k} w$. Then (11) becomes

$$\left\| \begin{bmatrix} Au + \frac{1}{k} Bw + \sum \alpha_{ij} u_{ij} \\ Cu + \frac{1}{k} Dw + w \end{bmatrix} \right\| \geq \Delta \left\| \begin{bmatrix} u \\ \frac{1}{k} w \end{bmatrix} \right\| \quad \forall\, k = 1, 2, \ldots.$$

Letting $k \to \infty$ and using $Cu + w = 0$, we get $||Au + \sum \alpha_{ij} u_{ij}|| \geq \Delta ||u||$. Thus, *$A$ has the* **UNS***-property.*

The example given below shows that the converse relations in the above theorem need not hold.

Example Consider on R^2,

$$L = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

Here, $A = [1] \in \mathbf{P}$ and $L/A \in \mathbf{P}$, but $L \notin \mathbf{P}$. For symmetric matrices, however, the conditions $A \in \mathbf{P}$ and $L/A \in \mathbf{P}$ imply $L \in \mathbf{P}$ (see [9]).

Remarks. In this paper, we considered some complementarity properties that are generalizations of the **P**-property of a matrix. Another such property is the so-called *Jordan* **P**-property, defined on a Euclidean Jordan algebra by the implication $x \circ L(x) \leq 0 \Rightarrow x = 0$, see [10]. That this property is inherited by principal pivotal transformations, principal subtransformations, and Schur complements can be easily seen by modifying the proofs given for the **P**-property. Below, we consider several complementarity properties and investigate whether they remain the same under principal pivotal transformations and are inherited by principal subtransformations and Schur complements. We follow the notation of Section 2.2.

(1) We say that L has the $\mathbf{R}_0$-property on K if $\mathrm{SOL}(L, K, 0) = \{0\}$. When $K = K_1 \times K_2$, it follows from (9) that the $\mathbf{R}_0$-property is inherited by principal pivotal transforms. However, the $\mathbf{R}_0$-property need not be inherited by principal subtransformations and Schur complements. To see this, consider the following two examples in the setting of $K = R_+ \times R_+$:

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Clearly, the first matrix has the $\mathbf{R}_0$-property on R^2_+, but has a principal submatrix which is not $\mathbf{R}_0$; the second matrix also has the $\mathbf{R}_0$-property on R^2_+, but has zero Schur complement.

(2) We say that L has the $\mathbf{R}$-property on K if it has the $\mathbf{R}_0$-property and there is a $d \in (K^*)^\circ$ such that $\mathrm{SOL}(L, K, d) = \{0\}$. The following example shows that this property need not be inherited by principal pivotal transforms, principal subtransformations, and Schur complements. Let

$$L = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad \mathrm{L}^\diamond = \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}.$$

It can be easily shown that on R^2_+, L has the $\mathbf{R}$-property with respect to $d = [1 \quad 1]^T$. However, the principal submatrix $[-1]$ and its Schur complement, namely, $[-2]$ do not have the $\mathbf{R}$-property with respect to any $p > 0$ in R. In addition, for any $d = [p \quad q]^T > 0$ in R^2_+, $[p \quad 0]^T$ is a nonzero solution of $\mathrm{LCP}(L^\diamond, R^2_+, d)$ implying that $L^\diamond$ does not have the $\mathbf{R}$-property. We remark that the construction of $L^\diamond$ was inspired by Theorem 6.6.4, [5] on the so-called **N**-matrices.

(3) We say that L has the $\mathbf{S}$-property on K if there exists a $d \in K^\circ$ with $L(d) \in K^\circ$. Now suppose that $K = K_1 \times K_2$ with $K_1^* = K_1$. (Note that this condition holds when K is a

symmetric cone, or, more generally, a self-dual cone.) In this setting, if L has the **S**-property on $K_1 \times K_2$, then also $L^\diamond$ has the **S**-property on $K_1 \times K_2$. This follows from (5) with $q_1 = 0, q_2 = 0, x_1, y_1 \in (K_1)^\circ$ and $x_2, y_2 \in (K_2)^\circ$.

It is easy to see that even in the standard LCP case, the **S**-property is *not* inherited by principal subtransformations and Schur complements. For example, consider the following two **S**-matrices on R^2_+:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In the first matrix, the **S**-property is not inherited by the principal submatrix [0]. In the second matrix, the Schur complement of [1], namely, [0], does not have the **S**-property.

(4) The transformation L is said to be copositive on K if $\langle L(x), x\rangle \geq 0$ for all $x \in K$. It is easy to see that this property is inherited by principal subtransformations. However, the following example shows that even in the standard LCP setting, principal pivotal transforms and Schur complements do not inherit the copositivity property. Let

$$L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{L}^\diamond = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Clearly, L is copositive on R^2_+, but $L^\diamond$ and the Schur complement of [1] are not copositive.

6 Inheritance of the UNS-property in simple Euclidean Jordan algebras

In the previous sections, we dealt with the inheritance of some complementarity properties of linear transformations defined over product spaces/algebras. What happens in the case of a simple Euclidean Jordan algebra? Since it is known that **P** and **GUS** properties are not inherited by principal subtransformations in simple algebras, see [8], we consider only the **UNS** property. Let V be any simple Euclidean Jordan algebra of rank r. Given any idempotent c in V with $0 \neq c \neq e$, where e is the unit element in V, we consider the subalgebras

$$V_1 := \{x \in V : x \circ c = x\} \quad \text{and} \quad V_0 = \{x \in V : x \circ c = 0\}.$$

We also let

$$V_{\frac{1}{2}} := \{x \in V : x \circ c = \frac{1}{2}x\}.$$

Then the following (orthogonal) Peirce decomposition holds [7]:

$$V = V_1 + V_0 + V_{\frac{1}{2}}.$$

Now, given any linear transformation L on V, using the above decomposition, we write L in the block form

$$L = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}, \tag{12}$$

where the blocks are linear transformations acting on appropriate spaces.

Theorem 5 *Suppose that L has the* **UNS***-property on V. Given any idempotent c with $0 \neq c \neq e$, consider the block decomposition of L given by (12). Then the transformation*

$$\begin{bmatrix} A & B \\ D & E \end{bmatrix} : V_1 \times V_0 \to V_1 \times V_0$$

also has the **UNS***-property. Consequently, A has the* **UNS***-property on V_1 and the $E - DA^{-1}B$ has the* **UNS***-property on V_0.*

Proof. Let $\{e_1, e_2, \dots, e_k\}$ be a Jordan frame in V_1 and $\{f_{k+1}, f_{k+2}, \dots, f_r\}$ be a Jordan frame in V_0, so that $\{e_1, e_2, \dots, e_k, f_{k+1}, \dots, f_r\}$ is a Jordan frame in V, $e_1 + e_2 + \cdots + e_k = c$ and $f_{k+1} + f_{k+2} + \cdots + f_r = e - c$. From the **UNS**-property of L, we may write

$$||L(z) + \sum_{i \leq j} d_{ij} z_{ij}|| \geq \Delta ||z||,$$

for all $z \in V$ and $d_{ij} \geq 0$, where $\sum_{i \leq j} z_{ij}$ is the Peirce decomposition of $z \in V$ with respect to $\{e_1, e_2, \dots, e_k, f_{k+1}, \dots, f_r\}$.
Now, we fix $u \in V_1$ with its Peirce decomposition $\sum_{1 \leq i \leq j \leq k} u_{ij}$ relative to $\{e_1, e_2, \dots, e_k\}$ and $v \in V_0$ with its Peirce decomposition $\sum_{k+1 \leq i \leq j \leq r} v_{ij}$ relative to $\{f_{k+1}, f_{k+2}, \dots, f_r\}$. Assume also $\alpha_{ij} \geq 0, \beta_{ij} \geq 0$ for all i, j. Let $w := -Gu - Hv \in V_{\frac{1}{2}}$ and $\gamma_{ij} = k$, for all i, j with k an arbitrary natural number. Note that we have $w' = \frac{1}{k} w \in V_{\frac{1}{2}}$. Now, $\sum u_{ij} + \sum v_{ij} + \sum w_{ij}$ and $\sum u_{ij} + \sum v_{ij} + \sum \frac{1}{k} w_{ij}$ are, respectively, Peirce decompositions of $u + v + w$ and $u + v + w'$ in V with respect to $\{e_1, e_2, \dots, e_k, f_{k+1}, \dots, f_r\}$. Now we have the inequality

$$\left\| L(u + v + w') + \sum \alpha_{ij} u_{ij} + \sum \beta_{ij} v_{ij} + \sum k \frac{w_{ij}}{k} \right\| \geq \Delta ||u + v + \frac{1}{k} w||,$$

for all $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$, i, j, and $k = 1, 2, \dots$. Rewriting this in the "matrix" form, we get

$$\left\| \begin{bmatrix} Au + \sum \alpha_{ij} u_{ij} + Bv + \frac{1}{k} Cw \\ Du + Ev + \sum \beta_{ij} v_{ij} + \frac{1}{k} Fw \\ Gu + Hv + \frac{1}{k} Jw + w \end{bmatrix} \right\| \geq \Delta \left\| \begin{bmatrix} u \\ v \\ \frac{1}{k} w \end{bmatrix} \right\|.$$

Letting $k \to \infty$ and using $w = -Gu - Hv$, we get

$$\left\| \begin{bmatrix} Au + \sum \alpha_{ij} u_{ij} + Bv \\ Du + Ev + \sum \beta_{ij} v_{ij} \end{bmatrix} \right\| \geq \Delta \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|.$$

This gives the stated assertion about the specified principal subblock of L and, in particular, via Theorem 4, gives the **UNS**-property of A and $E - DA^{-1}B$. This completes the proof. $\square$

Remarks. Suppose that the conditions of the above theorem are in place and let

$$M = \begin{bmatrix} A & B \\ D & E \end{bmatrix}.$$

Then, as in the proof of Theorem 4, one can show that the Schur complement of M in L has the following property:

$$||(L/M)w + \sum \gamma_{ij} w_{ij}|| \geq \Delta ||w|| \quad \forall w = \sum w_{ij} \in V_{\frac{1}{2}},\, \gamma_{ij} \geq 0,$$

so, L/M has some sort of the **UNS**-property on $V_{\frac{1}{2}}$.

Concluding remarks and open problems. In this paper, we have described some complementarity properties of linear transformations on product (symmetric) cones that are inherited by principal pivotal transforms, principal subtransformations, and Schur complements. We end this paper with a short list of open problems: Suppose that L has the **UNS**-property.

- Does L^{-1} have the **UNS**-property?
- Does $L^{\diamond}$ have the **UNS**-property?
- Should L have the Lipschitzian property?

Acknowledgements. We thank two anonymous referees for their very constructive comments and suggestions.

References

[1] X. Chen and H. Qi, Cartesian **P**–property and its applications to the semidefinite linear complementarity problems, *Math. Program. Ser. A* 106 (2006) 177–201.

[2] C.B. Chua and P. Yi, A continuation method for nonlinear complementarity problems over symmetric cones, *SIAM J. Optim.* 20 (2010) 2560–2583.

[3] C.B. Chua, H. Lin and P. Yi, Uniform nonsingularity and complementarity problems over symmetric cones. Research Report, School of Physical & Mathematical Sciences, Nanyang Technological University, Singapore (2009)

[4] R.W. Cottle, On manifestations of the Schur complement, *Rend. Sem. Mat. Fis. Milano* 45 (1975) 31–40.

[5] R.W. Cottle, J.–S. Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[6] F. Facchinei and J.–S. Pang, *Finite–dimensional Variational Inequalities and Complementarity Problems Vol. I & II*, Springer–Verlag, New York, 2003.

[7] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[8] M.S. Gowda, Y. Song and G. Ravindran, On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems, *Linear Algebra Appl.* 370 (2003) 355–368.

[9] M.S. Gowda and R. Sznajder, Schur complements, Schur determinantal and Haynsworth inertia formulas in Euclidean Jordan algebras, *Linear Algebra Appl.* 432 (2010) 1553–1559.

[10] M.S. Gowda, R. Sznajder and J. Tao, Some **P**–properties for linear transformations on Euclidean Jordan algebras, *Linear Algebra Appl.* 393 (2004) 203–232.

[11] M.S. Gowda and J. Tao, Z–transformations on proper and symmetric cones, *Math. Program. Ser. B 117* (2009) 195–221.

[12] Y. Kanno, J.A.C. Martins and A. Pinto da Costa, Three–dimensional quasi–static frictional contact by using second–order cone linear complementarity problem, *Internat. J. Numer. Methods Engrg.* 65 (2006) 62–83.

[13] D.V. Ouellette, Schur complements and statistics, *Linear Algebra Appl.* 36 (1981) 187–295.

[14] R. Sznajder, M.S. Gowda and J. Tao, On the uniform nonsingularity property for linear transformations on Euclidean Jordan algebras, *J. Optim. Theory Appl.* 153 (2012) 306–319.

[15] J. Tao and M.S. Gowda, Some **P**–properties for nonlinear transformations on Euclidean Jordan algebras, *Math. Oper. Res.* 30 (2005) 985–1004.

[16] M.J. Tsatsomeros, Principal pivot transforms: properties and applications, *Linear Algebra Appl.* 307 (2000) 151–165.

[17] A. Yoshise, Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones, *SIAM J. Optim.* 16 (2006) 1129–1153.

[18] F. Zhang, *The Schur Complement and its Applications*, Springer, New York, 2005.