International Game Theory Review
Vol. 15, No. 4 (2013) 1340034 (11 pages)

DOI: 10.1142/S0219198913400343

A REPRESENTATION THEOREM FOR LYAPUNOV-LIKE TRANSFORMATIONS ON EUCLIDEAN JORDAN ALGEBRAS

JIYUAN TAO
Department of Mathematics and Statistics
Loyola University Maryland
Baltimore, Maryland 21210, USA
jtao@loyola.edu

M. SEETHARAMA GOWDA
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, Maryland 21250, USA
gowda@math.umbc.edu

Received 15 February 2012
Revised 19 January 2013
Accepted 28 May 2013
Published 21 August 2013

A Lyapunov-like (linear) transformation L on a Euclidean Jordan algebra V is defined by the condition

$$x \in K,\ \ y \in K^*, \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0,$$

where K is the symmetric cone of V. In this paper, we give an elementary proof (avoiding Lie algebraic ideas and results) of the fact that Lyapunov-like transformations on V are of the form $L_a + D$, where $a \in V$, D is a derivation, and $L_a(x) = a \circ x$ for all $x \in V$.

Keywords: Euclidean Jordan algebra; symmetric cone; **Z** and Lyapunov-like transformations.

Subject Classification: 90C33, 17C55, 15A48

1. Introduction

Given a proper cone K in a finite dimensional real Hilbert space H, a linear transformation L on H is said to be *Lyapunov-like* on K if

$$x \in K,\ \ y \in K^*, \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0,$$

where K^* denotes the dual of K in H. Such transformations appear in complementarity theory, dynamical systems, and optimization, see Gowda and Sznajder [2007], Moldovan and Gowda [2010], Gowda and Tao [2011], and Gowda *et al.* [2013].

Our primary example (and the name) of such a transformation comes from taking $H = \mathcal{S}^n$ (the space of all $n \times n$ real symmetric matrices) and $K = \mathcal{S}^n_+$ (the semidefinite cone) and considering, for any matrix $A \in R^{n\times n}$, the Lyapunov transformation L_A defined by

$$L_A(X) = AX + XA^{\mathrm{T}} \quad (X \in \mathcal{S}^n).$$

On $H = R^n$ with $K = R^n_+$, Lyapunov-like matrices are nothing but diagonal matrices.

In Damm [2004], it is shown that every Lyapunov-like transformation on $\mathcal{S}^n_+$ is of the form L_A for some $A \in R^{n\times n}$. This raises the problem of describing/characterizing Lyapunov-like transformations on other proper cones. A recent paper by Gowda *et al.* [2013] studies this problem for completely positive cones. In the case of a symmetric cone in a Euclidean Jordan algebra, there is a neat answer: Every Lyapunov-like transformation is of the form $L_a + D$, where L_a is the Lyapunov transformation corresponding to an element a in the algebra and D is a derivation. A proof of this (as given in Gowda *et al.* [2012]) depends on using a result of Schneider and Vidyasagar [1970] relating Lyapunov-like transformations with their exponentials (which belong to the automorphism group of the cone) and then using Lie algebraic ideas. The main objective here is to derive this result by (only) Jordan algebraic means thus avoiding Lie algebraic ideas and results. In addition to having pedagogical advantage, our unified presentation/derivation yields a number of results: a characterization of self-adjoint Lyapunov-like transformations on any Euclidean Jordan algebra (Proposition 2) and characterizations of Lyapunov-like transformations on $\mathcal{S}^n_+$ (thus giving an alternate proof of a result of Damm) and on $\mathcal{L}^n_+$ (Theorem 5). We show that a result similar to that of Damm holds for quaternions (Theorem 2) but fails for octonions (Theorem 3).

2. Preliminaries

Let V denote a Euclidean Jordan algebra of rank r, see Faraut and Korányi [1994], where the inner product and Jordan product of two elements x and y are given, respectively, by $\langle x, y\rangle$ and $x \circ y$. The symmetric cone of V (which is self-dual) is denoted by K. The unit element of V is denoted by e. We use the notation $x \geq 0$ when $x \in K$ and write $x \perp y$ to mean $\langle x, y\rangle = 0$.

A linear transformation D on V is said to be a *derivation* if for all $x, y \in V$,

$$D(x \circ y) = D(x) \circ y + x \circ D(y).$$

Recalling that for an element $a \in V$, the corresponding Lyapunov transformation L_a is defined by

$$L_a(x) = a \circ x,$$

any finite sum of commutators of the form

$$[L_a, L_b] := L_aL_b - L_bL_a$$

is a derivation, called the *inner derivation*. It is known that on a Euclidean Jordan algebra, every derivation is inner, see, Propositon VI.1.2 in Faraut and Korányi [1994] or Theorem 8 in Koecher [1999].

We say that a linear transformation L on V is a **Z**-transformation on K (or on V) if

$$0 \leq x \perp y \geq 0 \Rightarrow \langle L(x), y\rangle \leq 0$$

and is a Lyapunov-like transformation on K if both L and $-L$ are **Z**-transformations, that is,

$$0 \leq x \perp y \geq 0 \Rightarrow \langle L(x), y\rangle = 0.$$

It has been observed in Gowda *et al.* [2012] that the **Z** and Lyapunov-like properties remain the same if we replace the given inner product by the canonical inner product $\langle x, y\rangle_{\mathrm{tr}} := \mathrm{trace}(x \circ y)$.

In the rest of the paper, we assume that V denotes a Euclidean Jordan algebra with symmetric cone K and which carries the canonical inner product. We also freely use results from Faraut and Korányi [1994].

3. Lyapunov-Like Transformations

We recall the following result.

Proposition 1 (Theorem 4, Gowda *et al.* [2012]). *The following are equivalent*:

(1) *L is Lyapunov-like on K.*
(2) *For any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V,*

$$\langle L(e_i), e_j\rangle = 0 \quad \forall\, i \neq j.$$

Here is our representation theorem.

Theorem 1. *A linear transformation L on a Euclidean Jordan algebra V is Lyapunov-like if and only if it is of the form*

$$L = L_a + D,$$

where $a \in V$ and D is a (inner) derivation. In this situation, L_a is the symmetric part of L and D is the skew-symmetric part of L.

Before giving the proof, we consider some special cases.

Proposition 2. *Suppose that L is self-adjoint on V. Then L is Lyapunov-like if and only if it is of the form L_a for some $a \in V$.*

Proof. Suppose $L = L_a$ for some $a \in V$. If $0 \leq x \perp y \geq 0$, then $x \circ y = 0$, see Proposition 6 in Gowda *et al.* [2004]. Hence

$$\langle L_a(x), y\rangle = \langle a \circ x, y\rangle = \langle a, x \circ y\rangle = 0$$

proving the Lyapunov-like property of L_a. Now, in addition to being self-adjoint, suppose that L is Lyapunov-like. Define $a := L(e)$. Then for any $x \in V$, we have

$$\langle L(x), e\rangle = \langle x, L(e)\rangle = \langle x, a\rangle = \langle x \circ a, e\rangle = \langle L_a(x), e\rangle$$

and hence $\langle (L - L_a)(x), e\rangle = 0$. Let $M := L - L_a$. Then $M(e) = L(e) - L_a(e) = a - a = 0$. Now, for any x in V, there exists a Jordan frame $\{e_1, \dots, e_r\}$ such that $x = \sum x_i e_i$. Since M is a Lyapunov-like, $\langle M(e_i), e_j\rangle = 0$ for all $i \neq j$, and

$$0 = M(e) = \sum M(e_i) \Rightarrow \langle M(e_i), e_i\rangle = 0 \quad \forall\, i.$$

Thus, $\langle M(x), x\rangle = 0$ for all x. Replacing x by $x + ty$, where x and y are arbitrary and t is real, we see that $\langle M(x), y\rangle = 0$. This implies that $M(x) = 0$ for all x; hence $M = 0$, i.e., $L = L_a$. □

In what follows, we write L^{T} for the transpose/adjoint of L which is defined by the condition $\langle L^{\mathrm{T}}(x), y\rangle = \langle x, L(y)\rangle$ for all $x, y \in V$.

Proposition 3. *The following are equivalent for a linear transformation L on V:*

(1) *L is a derivation.*
(2) *L is Lyapunov-like and $L(e) = 0$.*
(3) *L is Lyapunov-like and skew-symmetric (i.e., $L + L^{\mathrm{T}} = 0$).*

Proof. (1) ⇒ (2): Let L be a derivation, $\{e_1, e_2, \dots, e_r\}$ be a Jordan frame in V, and $i \neq j$. Using $e_i \circ e_j = 0$ and writing $0 = L(e_i \circ e_j) = L(e_i) \circ e_j + e_i \circ L(e_j)$, we get, upon taking the inner product with e_j,

$$0 = \langle L(e_i) \circ e_j + e_i \circ L(e_j), e_j\rangle = \langle L(e_i), e_j \circ e_j\rangle + \langle L(e_j), e_i \circ e_j\rangle = \langle L(e_i), e_j\rangle,$$

where we used the properties $e_j \circ e_j = e_j$ and $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ in V. This proves that for all $i \neq j$,

$$\langle L(e_i), e_j\rangle = 0,$$

that is, L is Lyapunov-like.

Now, for any i, $e_i \circ e_i = e_i$. Since L is a derivation, we get $2L(e_i) \circ e_i = L(e_i)$ and $2\langle L(e_i) \circ e_i, e_i\rangle = \langle L(e_i), e_i\rangle$. This implies that $\langle L(e_i), e_i\rangle = 0$, $\forall\, i = 1, 2, \dots, r$. So we have proved that when L is a derivation, for any Jordan frame $\{e_1, e_2, \dots, e_r\}$,

$$\langle L(e_i), e_j\rangle = 0 \quad \forall\, i, j = 1, 2, \dots, r.$$

Fixing j and summing over i, we get $\langle L(e), e_j\rangle = 0$ for all j. As the Jordan frame is arbitrary, writing the spectral decomposition of any x as $x = \sum x_j e_j$, we get $\langle L(e), x\rangle = 0$. As x is arbitrary, this gives $L(e) = 0$. Thus we have proved that L is Lyapunov-like and $L(e) = 0$.

(2) ⇒ (3): Let $\{e_1, e_2, \dots, e_r\}$ be any Jordan frame. Then $\langle L(e_i), e_j\rangle = 0$, $\forall\, i \neq j$. The condition $L(e) = 0$ implies that $\sum L(e_i) = 0$ and hence $\langle L(e_i), e_j\rangle = 0$ even

when $i = j$. Now for any $x \in V$, we have the spectral expansion $x = \sum_1^r x_i e_i$ for some Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and eigenvalues $x_1, x_2, \ldots, x_r$. Then

$$\langle L(x), x\rangle = \sum_{i,j} x_i x_j \langle L(e_i), e_j\rangle = 0.$$

This implies that $L + L^{\mathrm{T}} = 0$, i.e., L is skew-symmetric.

(3) $\Rightarrow$ (2): Assume that L is Lyapunov-like and skew-symmetric. Then for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we have $\langle L(e_i), e_j\rangle = 0$ for all i and j. As in the last part of the proof of (1) $\Rightarrow$ (2), we get $L(e) = 0$.

(3) $\Rightarrow$ (1): Let $D = L$ be Lyapunov-like and skew-symmetric.

Claim (i): For any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, and for all i and $k \neq l$,

$$2e_i \circ D(e_i) = D(e_i) \quad \text{and} \quad e_k \circ D(e_l) + e_l \circ D(e_k) = 0. \tag{1}$$

To see this, fix an index k, $1 \leq k \leq r$. Write the Peirce decomposition of $D(e_k)$ as $D(e_k) = \sum x_i e_i + \sum_{i<j} x_{ij}$. Since D is skew symmetric, $\langle D(e_i), e_i\rangle = 0$ for $i = 1, \ldots, r$. Since D is a Lyapunov-like, $\langle D(e_k), e_j\rangle = 0$ for $k \neq j$. Thus, $x_i = 0$ for $i = 1, \ldots, r$. Hence

$$D(e_k) = \sum_{i<j} x_{ij}.$$

Now, if a certain $x_{ij} = 0$, then $\langle D(e_k), x_{ij}\rangle = 0$. If $x_{ij} \neq 0$, then $i \neq j$ and $\lambda(e_i + e_j) + x_{ij} \geq 0$, where $\lambda = \frac{\|x_{ij}\|}{\sqrt{2}}$ (see Lemma 6 in Gowda *et al.* [2012]). From $\langle e_k, \lambda(e_i + e_j) + x_{ij}\rangle = 0$ for $k \neq i, j$ we have $\langle D(e_k), \lambda(e_i + e_j) + x_{ij}\rangle = 0$. This implies $\langle D(e_k), x_{ij}\rangle = 0$.

Thus,

$$D(e_k) = \sum_{i=1}^{k-1} x_{ik} + \sum_{j=k+1}^{r} x_{kj}.$$

Multiplying both sides of this equality by e_k and using $x_{ik} \circ e_k = \frac{1}{2} x_{ik}$, etc., we get $e_k \circ D(e_k) = \frac{1}{2} D(e_k)$. This proves the first part of (1). Now for the second part. Based on the discussion above, we write the Peirce decomposition of $D(e_k)$ for $k = 1, 2, \ldots, r$:

$$\begin{aligned}
D(e_1) &= x_{12} + x_{13} + \cdots + x_{1r},\\
D(e_2) &= y_{12} + y_{23} + \cdots + y_{2r},\\
D(e_3) &= z_{13} + z_{23} + \cdots + z_{3r},\\
\vdots\ \ &= \qquad\quad \vdots\\
D(e_k) &= \sum_{i=1}^{k-1} p_{ik} + \sum_{j=k+1}^{r} p_{kj},\\
\vdots\ \ &= \qquad\quad \vdots
\end{aligned}$$

$$
\begin{aligned}
D(e_l) &= \sum_{i=1}^{l-1} q_{il} + \sum_{j=l+1}^{r} q_{lj},\\
\vdots\ &= \qquad \vdots\\
D(e_r) &= w_{1r} + w_{2r} + \cdots + w_{r-1r}.
\end{aligned}
$$

Now, adding these equations, we get

$$D(e) = (x_{12} + y_{12}) + (x_{13} + z_{13}) + \cdots + (p_{lk} + q_{lk}) + \cdots.$$

Let $l \neq k$. Since the grouped terms in $D(e)$ belong to orthogonal Peirce spaces and $D(e) = 0$, we have $p_{lk} + q_{lk} = 0$. Since $e_k \circ D(e_l) = \frac{1}{2}q_{lk}$ and $e_l \circ D(e_k) = \frac{1}{2}p_{lk}$, we have $e_k \circ D(e_l) + e_l \circ D(e_k) = \frac{1}{2}(p_{lk} + q_{lk}) = 0$. This proves the second part of (1).

Claim (ii): For any $x \in V$,

$$D(x^2) = 2x \circ D(x). \tag{2}$$

To see this, let $x \in V$ with spectral decomposition $x = \sum x_i e_i$. Then

$$
\begin{aligned}
2x \circ D(x) &= 2\Big(\sum x_i e_i\Big) \circ \Big(\sum x_j D(e_j)\Big)\\
&= 2\left(\sum_i x_i^2 e_i \circ D(e_i)\right) + 2\sum_{i<j} x_i x_j (e_i \circ D(e_j) + e_j \circ D(e_i))\\
&= 2\left(\sum_i x_i^2 e_i \circ D(e_i)\right)\\
&= \sum_i x_i^2 D(e_i)\\
&= D\left(\sum_i x_i^2 e_i\right) = D(x^2).
\end{aligned}
$$

Now, replacing x by $x + \lambda y$ in (2) and comparing coefficients of λ, we get $D(x \circ y) = x \circ D(y) + y \circ D(x)$. Thus, D is a derivation. □

Proof of Theorem 1. If L is Lyapunov-like, then so are $\frac{L+L^{\mathrm{T}}}{2}$ and $\frac{L-L^{\mathrm{T}}}{2}$. Note that $\frac{L+L^{\mathrm{T}}}{2}$ is self-adjoint and $\frac{L-L^{\mathrm{T}}}{2}$ is skew-symmetric. By the above propositions, we can let $\frac{L+L^{\mathrm{T}}}{2} = L_a$ and $\frac{L-L^{\mathrm{T}}}{2} = D$, where $a \in V$ and D is a derivation. Then $L = L_a + D$. The converse statement in the theorem follows from the above propositions. □

3.1. *Lyapunov-like transformations on matrix algebras*

Let $\mathcal{F}$ denote any one of the following: the set of all real numbers $\mathcal{R}$, the set of all complex numbers $\mathcal{C}$, the set of all quaternions $\mathcal{H}$, the set of all octonions $\mathcal{O}$.

Given any element $p \in \mathcal{F}$, we write $\mathrm{Re}(p)$ for its real part and $\overline{p}$ for its conjugate. We note that quaternions are noncommutative but associative, while octonions are noncommutative and nonassociative. Still, for any three elements a, b and c in $\mathcal{F}$, we have, see Dray and Manogue [1998],

$$\mathrm{Re}(a) = \mathrm{Re}(\overline{a}), \quad \mathrm{Re}(ab) = \mathrm{Re}(ba), \quad \text{and} \quad \mathrm{Re}[a(bc)] = \mathrm{Re}[(ab)c]. \tag{3}$$

For any $A \in \mathcal{F}^{n\times n}$, let $\mathrm{tr}(A)$ denote the sum of the diagonal elements of A. Then, for any three matrices A, B and C in $\mathcal{F}^{n\times n}$, see Proposition V.2.1, Faraut and Korányi [1994],

$$\begin{gathered} \mathrm{Re}\,\mathrm{tr}(A) = \mathrm{Re}\,\mathrm{tr}(A^*), \quad \mathrm{Re}\,\mathrm{tr}(AB) = \mathrm{Re}\,\mathrm{tr}(BA), \\ \mathrm{Re}\,\mathrm{tr}(A(BC)) = \mathrm{Re}\,\mathrm{tr}((AB)C), \end{gathered} \tag{4}$$

where A^* is the conjugate transpose of A.

Let $\mathrm{Herm}(\mathcal{F}^{n\times n})$ denote the space of all Hermitian $n \times n$ matrices with entries from $\mathcal{F}$. For any given $A \in \mathcal{F}^{n\times n}$, we define the Lyapunov transformation L_A on $\mathrm{Herm}(\mathcal{F}^{n\times n})$ by

$$L_A(X) = AX + XA^*.$$

The following extends a result of Damm [2004] and at the same time gives an alternate proof.

Theorem 2. *Let $\mathcal{F}$ denote real numbers, complex numbers, or quaternions. A linear transformation L on the Euclidean Jordan algebra $\mathrm{Herm}(\mathcal{F}^{n\times n})$ is Lyapunov-like if and only if there exists an $A \in \mathcal{F}^{n\times n}$ such that $L = L_A$.*

Proof. Suppose that $A \in \mathcal{F}^{n\times n}$ and consider $X, Y \in \mathrm{Herm}(\mathcal{F}^{n\times n})$ such that

$$X \geq 0, \quad Y \geq 0, \quad \textit{and} \quad \langle X, Y\rangle = 0,$$

where $X \geq 0$ means that X belongs to the symmetric cone of $\mathrm{Herm}(\mathcal{F}^{n\times n})$. Then $XY = YX = 0$. (This is well known for $\mathcal{F} = \mathcal{R}$ or $\mathcal{C}$; see Remark 3 in Moldovan and Gowda [2009] for $\mathcal{F} = \mathcal{H}$.) Now, relying on the associativity in $\mathcal{F}$, and using (4),

$$\begin{aligned} \langle L_A(X), Y\rangle = \mathrm{Re}\,\mathrm{tr}(L_A(X)Y) = \mathrm{Re}\,\mathrm{tr}(AXY + XA^*Y) &= 2\mathrm{Re}\,\mathrm{tr}(XA^*Y) \\ &= 2\mathrm{Re}\,\mathrm{tr}(A^*YX) = 0. \end{aligned}$$

This proves the Lyapunov-like property of L_A on $\mathrm{Herm}(\mathcal{F}^{n\times n})$. Now for the converse. Suppose L is Lyapunov-like on $\mathrm{Herm}(\mathcal{F}^{n\times n})$. By the previous theorem, $L = L_A + D$, where $A \in \mathrm{Herm}(\mathcal{F}^{n\times n})$ and D is a derivation. As D is inner, we can write

$$D = \sum_1^m [L_{A_i}, L_{B_i}],$$

where $A_i, B_i \in \mathrm{Herm}(\mathcal{F}^{n\times n})$, $i = 1, 2, \ldots, m$. Using the associativity of the ordinary matrix product of matrices in $\mathcal{F}^{n\times n}$,

$$[L_{A_i}, L_{B_i}] = L_{[A_i, B_i]}.$$

It follows that $D = L_B$, where $B := \sum_1^m [A_i, B_i] \in \mathcal{F}^{n\times n}$. Hence, $L = L_A + L_B = L_{A+B} = L_C$, where $C := A + B \in \mathcal{F}^{n\times n}$. This completes the proof. □

We next show that a result of the previous type is not valid for matrices over octonions.

Theorem 3. *There exists* $A \in \mathcal{O}^{3\times 3}$ *such that* L_A *is not Lyapunov-like on* $\mathrm{Herm}(\mathcal{O}^{3\times 3})$.

Proof. By Remark 3 in Moldovan and Gowda [2009], there exists a Jordan frame $\{E_1, E_2, E_3\}$ in $\mathrm{Herm}(\mathcal{O}^{3\times 3})$ such that $E_1E_2 \neq 0$. (Such a Jordan frame comes, for example, from the spectral decomposition of the matrix given in Remark 2 in Moldovan and Gowda [2009].)

Now, $E_1 \circ E_2 = 0 \Rightarrow E_1E_2 + E_2E_1 = 0$. Let

$$E_1E_2 := \begin{bmatrix} p & a & b \\ \alpha & q & c \\ \beta & \gamma & r \end{bmatrix}.$$

Then

$$E_2E_1 = (E_1E_2)^* = \begin{bmatrix} \bar{p} & \bar{\alpha} & \bar{\beta} \\ \bar{a} & \bar{q} & \bar{\gamma} \\ \bar{b} & \bar{c} & \bar{r} \end{bmatrix}.$$

From $E_1E_2 + E_2E_1 = 0$, we get $\mathrm{Re}(p) = \mathrm{Re}(q) = \mathrm{Re}(r) = 0$, $a + \bar{\alpha} = 0$, $b + \bar{\beta} = 0$, and $c + \bar{\gamma} = 0$. We will construct an octonion matrix

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

such that $\langle L_A(E_1), E_2\rangle \neq 0$.

As $E_1E_2 \neq 0$, some row of E_1E_2 is nonzero. Without loss of generality, assume that the first row $[p \;\; a \;\; b]$ is nonzero. In this case, we take $a_{ij} = 0$ for $i \in \{2, 3\}$ and $j \in \{1, 2, 3\}$. Then, using (4),

$$\mathrm{Re}\,\mathrm{tr}((AE_1)E_2) = \mathrm{Re}\,\mathrm{tr}((A(E_1E_2)) = \mathrm{Re}\,\mathrm{tr}((AE_1)^*E_2)$$

and so

$$\begin{aligned} \langle L_A(E_1), E_2\rangle = \mathrm{Re}\,\mathrm{tr}(L_A(E_1)E_2) &= \mathrm{Re}\,\mathrm{tr}((AE_1)E_2 + (AE_1)^*E_2) \\ &= 2\mathrm{Re}(a_{11}p + a_{12}\alpha + a_{13}\beta). \end{aligned}$$

As $[p \quad a \quad b]$ is nonzero, the vector $[p \;\; \alpha \;\; \beta]$ is also nonzero. Now, if $p \neq 0$, we can take $a_{11} = \frac{1}{p}$, $a_{12} = a_{13} = 0$. Then $\langle L_A(E_1), E_2\rangle = 2$. A similar construction can be made if α or β is nonzero. Thus, A can be constructed so that $\langle L_A(E_1), E_2\rangle \neq 0$. This means that L_A is not Lyapunov-like. □

3.2. *Lyapunov-like transformations on $\mathcal{L}^n$*

Consider the Jordan spin algebra $\mathcal{L}^n$ whose underlying space is R^n, $n > 1$. We write any element x in the form

$$x = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \tag{5}$$

with $x_0 \in R$ and $\overline{x} \in R^{n-1}$. The inner product in $\mathcal{L}^n$ is the usual inner product on R^n. The Jordan product $x \circ y$ in $\mathcal{L}^n$ is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \overline{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0\overline{y} + y_0\overline{x} \end{bmatrix}.$$

Then $\mathcal{L}^n$ is a Euclidean Jordan algebra of rank 2. In this section, we give two characterizations of Lyapunov-like transformations on $\mathcal{L}^n$. Our second result, Theorem 5, can also be deduced from Theorem 1.

Recalling that the underlying space of $\mathcal{L}^n$ is R^n $(n > 1)$, we fix a matrix $A \in \mathcal{R}^{n\times n}$ and let $J \in \mathcal{R}^{n\times n}$ be defined by

$$J := \operatorname{diag}(1, -1, \ldots, -1).$$

We recall the following from Gowda and Tao [2009]:

Lemma 1. *The matrix A has the Z-property on $\mathcal{L}^n$ if and only if there exists $\gamma \in \mathcal{R}$ such that $\gamma J - (JA + A^{\mathrm{T}}J)$ is positive semidefinite on R^n.*

As a consequence, we prove

Theorem 4. *The matrix $A \in R^{n\times n}$ is Lyapunov-like on $\mathcal{L}^n$ if and only if there exists $\beta \in \mathcal{R}$ such that $\beta J + (JA + A^{\mathrm{T}}J) = 0$.*

Proof. Suppose A is Lyapunov-like, in which case, A and $-A$ have the Z-property. Then by Lemma 1, there exist α and β such that $\alpha J - (JA + A^{\mathrm{T}}J)$ and $\beta J + (JA + A^{\mathrm{T}}J)$ are positive semidefinite. Therefore, $\alpha J - (JA + A^{\mathrm{T}}J) + \beta J + (JA + A^{\mathrm{T}}J) = (\alpha + \beta)J$ is positive semidefinite. Thus, we have $\alpha = -\beta$. Now, $-\beta J - (JA + A^{\mathrm{T}}J) = -(\beta J + (JA + A^{\mathrm{T}}J))$ is positive semidefinite and symmetric, hence $\beta J + (JA + A^{\mathrm{T}}J) = 0$. The 'if' part is obvious because of the previous lemma. □

Theorem 5. *A matrix $A \in R^{n\times n}$ is Lyapunov-like on $\mathcal{L}^n$ if and only if it is of the form*

$$A = \begin{bmatrix} a & b^{\mathrm{T}} \\ b & D \end{bmatrix}, \tag{6}$$

where $a \in R$, $D \in R^{(n-1)\times(n-1)}$, with $D + D^{\mathrm{T}} = 2aI$.

Proof. To see the 'if' part, take x and y in $\mathcal{L}^n$ such that $0 \leq x \perp y \geq 0$. Assuming that x and y are nonzero, we may write $x = [1 \quad u]^{\mathrm{T}}$, $y = [1 \quad -u]^{\mathrm{T}}$, where $\|u\| = 1$

(see e.g., Tao [2004]). Then $\langle Ax, y\rangle = a - u^{\mathrm{T}}Du = 0$, where the last equality comes from $D + D^{\mathrm{T}} = 2aI$. This proves the Lyapunov-like property of A. Now for the "only if" part. Suppose the matrix A is Lyapunov-like on $\mathcal{L}^n$ and is given by

$$A = \begin{bmatrix} a & b^{\mathrm{T}} \\ c & D \end{bmatrix}.$$

Putting $x = [1 \quad u]^{\mathrm{T}}$, $y = [1 \quad -u]^{\mathrm{T}}$ with $\|u\| = 1$, we see that $0 \leq x \perp y \geq 0$. Since A is Lyapunov-like, we have $\langle Ax, y\rangle = 0$ and so

$$a + (b - c)^{\mathrm{T}}u - u^{\mathrm{T}}Du = 0. \tag{7}$$

Replacing u by $-u$ in (7), we have

$$a - (b - c)^{\mathrm{T}}u - u^{\mathrm{T}}Du = 0. \tag{8}$$

The above two equations lead to $(b - c)^{\mathrm{T}}u = 0$ for all u with $\|u\| = 1$. Thus, $b = c$. Now from the previous result, we have, $\beta J + (JA + A^{\mathrm{T}}J) = 0$ (for some β). This leads to $\beta = -2a$ and $D + D^{\mathrm{T}} = -\beta I$. Therefore, $D + D^{\mathrm{T}} = 2aI$. This completes the proof. □

References

Damm, T. [2004] Positive groups on H^n are completely positive, *Linear Algebra Appl.* **393**, 127–137.

Dray, T. and Manogue, C. A. [1998] The octonionic eigenvalue problem, *Adv. Appl. Clifford Algebras* **8**, 341–364.

Faraut, J. and Korányi, A. [1994] *Analysis on Symmetric Cones* (Oxford University Press, Oxford).

Gowda, M. S. and Sznajder, R. [2007] Some global uniqueness and solvability results for linear complementarity problems over symmetric cones, *SIAM J. Optim.* **18**, 461–481.

Gowda, M. S., Sznajder, R. and Tao, J. [2004] Some **P**-properties for linear transformations on Euclidean Jordan algebras, *Linear Algebra Appl.* **393**, 203–232.

Gowda, M. S., Sznajder, R. and Tao, J. [2013] The automorphism group of a completely positive cone and its Lie algebra, *Linear Algebra Appl.* **438**, 3862–3871.

Gowda, M. S. and Tao, J. [2009] *Z*-transformations on proper and symmetric cones, *Math. Program., Ser. B* **117**, 195–222.

Gowda, M. S. and Tao, J. [2011] On the bilinearity rank of a proper cone and Lyapunov-like transformations, Research Report, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250.

Gowda, M. S. Tao, J. and Ravindran, G. [2012] On the **P**-property of **Z** and Lyapunov-like transformations on Euclidean Jordan algebras, *Linear Algebra Appl.* **436**, 2201–2209.

Koecher, M. [1999] *The Minnesota Notes on Jordan Algebras and Their Applications*, Lecture Notes in Mathematics (Springer, Berlin).

Moldovan, M. M. and Gowda, M. S. [2009] Strict diagonal dominance and a Geršgorin type theorem in Euclidean Jordan algebras, *Linear Algebra Appl.* **431**, 148–161.

Moldovan, M. M. and Gowda, M. S. [2010] On common linear/quadratic Lyapunov functions for switched linear systems, in *Nonlinear Analysis and Variational*

Problems, Springer Optimization and its Applications, Vol. 35 (Springer, New York), pp. 415–429.

Tao, J. [2004] Some P-properties for linear transformations on the Lorentz cone, Ph.D. Thesis, University of Maryland Baltimore County, Baltimore, Maryland.

Schneider, H. and Vidyasagar, M. [1970] Cross-positive matrices, *SIAM J. Numer. Anal.* **7**, 508–519.