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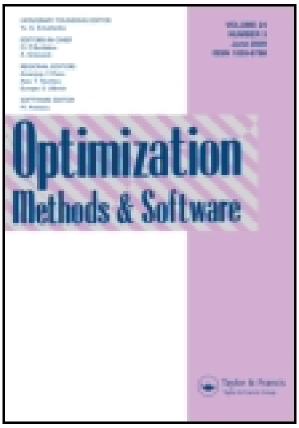

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Complementarity properties of Peirce-diagonalizable linear transformations on Euclidean Jordan algebras†

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Complementarity properties of Peirce-diagonalizable linear transformations on Euclidean Jordan algebras†

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Peirce-diagonalizable linear transformations on a Euclidean Jordan algebra are of the form $L(x) = A \cdot x := \sum a_{ij}x_{ij}$, where $A = [a_{ij}]$ is a real symmetric matrix and $\sum x_{ij}$ is the Peirce decomposition of an element x in the algebra with respect to a Jordan frame. Examples of such transformations include Lyapunov transformations and quadratic representations on Euclidean Jordan algebras. Schur (or Hadamard) product of symmetric matrices provides another example. Motivated by a recent generalization of the Schur product theorem, we study general and complementarity properties of such transformations.

Keywords: Peirce-diagonalizable transformation; linear complementarity problem; Euclidean Jordan algebra; symmetric cone; Schur/Hadamard product; Lypaunov transformation; quadratic representation

2010 AMS Subject Classifications: 15A33; 17C20; 17C65; 90C33

1. Introduction

Let L be a linear transformation on a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$ of rank r and let $\mathcal{S}^r$ denote the set of all real $r \times r$ symmetric matrices. We say that L is *Peirce-diagonalizable* if there exist a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ (with a specified ordering of its elements) in V and a matrix $A = [a_{ij}] \in \mathcal{S}^r$ such that for any $x \in V$ with its Peirce decomposition $x = \sum_{i\leq j} x_{ij}$ with respect to $\{e_1, e_2, \ldots, e_r\}$, we have

$$L(x) = A \cdot x := \sum_{1\leq i\leq j\leq r} a_{ij}x_{ij}. \tag{1}$$

The above expression defines a *Peirce-diagonal transformation* and a *Peirce-diagonal representation* of L.

Our first example is obtained by taking $V = \mathcal{S}^r$ with the canonical Jordan frame (see Section 2 for various notations and definitions). In this case, $A \cdot x$ reduces to the well-known Schur (also

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†We dedicate this paper to our friend, colleague, and teacher Florian Potra on the occasion of his 60th birthday.

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known as Hadamard) product of two symmetric matrices and L is the corresponding induced transformation.

On a general Euclidean Jordan algebra, we have two basic examples. The Lyapunov transformation L_a and the quadratic representation P_a corresponding to any element $a \in V$ are defined, respectively, by

$$L_a(x) = a \circ x \quad \text{and} \quad P_a(x) = 2a \circ (a \circ x) - a^2 \circ x.$$

If the spectral decomposition of a is given by

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r,$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame and $a_1, a_2, \ldots, a_r$ are the eigenvalues of a, then with respect to this Jordan frame and the induced Peirce decomposition $x = \sum_{i \le j} x_{ij}$ of any element, we have [19]

$$L_a(x) = \sum_{i \le j} \left(\frac{a_i + a_j}{2} \right) x_{ij} \quad \text{and} \quad P_a(x) = \sum_{i \le j} (a_i a_j) x_{ij}. \tag{2}$$

Thus, both L_a and P_a have the form (1) and the corresponding matrices are given, respectively, by $[(a_i + a_j)/2]$ and $[a_i a_j]$. We shall see (in Section 3) that every transformation L given by (1) is a linear combination of quadratic representations.

Our third example deals with Löwner functions. Given a differentiable function $\phi : \mathcal{R} \to \mathcal{R}$, consider the corresponding Löwner function $\Phi : V \to V$ defined by (the spectral decompositions)

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r \quad \text{and} \quad \Phi(a) = \phi(\lambda_1) e_1 + \phi(\lambda_2) e_2 + \cdots + \phi(\lambda_r) e_r.$$

Then, the directional derivative of Φ at $a = \sum \lambda_i e_i$ in the direction of $x = \sum x_{ij}$ (Peirce decomposition written with respect to $\{e_1, e_2, \ldots, e_r\}$) is given by

$$\Phi'(a; x) := \sum_{i \le j} a_{ij} x_{ij},$$

where $a_{ij} := (\phi(\lambda_i) - \phi(\lambda_j))/(\lambda_i - \lambda_j)$ (which, by convention, is the derivative of ϕ when $\lambda_i = \lambda_j$). We now note that (for a fixed a), $\Phi'(a; x)$ is of the form (1). Korányi [15] studies the operator monotonicity of Φ based on such expressions.

Expressions like (1) also appear in connection with the so-called *uniform non-singularity property*; see the recent paper [2] for further details.

Our objective in this paper is to study Peirce-diagonalizable transformations and, in particular, to describe their complementarity properties. Consider a Euclidean Jordan algebra V with the corresponding symmetric cone K. Given a linear transformation L on V and an element $q \in V$, the (*symmetric cone*) *linear complementarity problem*, denoted by LCP(L, K, q), is to find an $x \in V$ such that

$$x \in K, \quad L(x) + q \in K, \quad \text{and} \quad \langle L(x) + q, x \rangle = 0. \tag{3}$$

This problem is a generalization of the *standard linear complementarity problem* [3] which corresponds to a square matrix $M \in \mathcal{R}^{n \times n}$, the cone $\mathcal{R}^n_+$, and a vector $q \in \mathcal{R}^n$; it is a special case of a variational inequality problem [4].

In the theory of complementarity problems, global uniqueness and solvability issues are of fundamental importance. A linear transformation L is said to have the **Q**-property (respectively, **GUS**-property) if for every $q \in V$, LCP(L, K, q) has a solution (respectively, unique solution); it is said to have the $\mathbf{R}_0$-property if LCP$(L, K, 0)$ has a unique solution (namely zero) and the **S**-property if there is a $d > 0$ in V with $L(d) > 0$, where $d > 0$ means that d belongs to the interior

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of symmetric cone of V. In the context of Lyapunov and quadratic representations, we have the following known results.

(i) For any transformation of the form L_a,

$$\text{strict monotonicity} \Longleftrightarrow \mathbf{GUS} \Longleftrightarrow \mathbf{P} \Longleftrightarrow \mathbf{Q} \Longleftrightarrow \mathbf{S} \Longleftrightarrow a > 0.$$

(ii) For any transformation of the form P_a,

$$\text{strict monotonicity} \Longleftrightarrow \mathbf{GUS} \Longleftrightarrow \mathbf{P} \Longleftrightarrow \mathbf{R}_0 \Longleftrightarrow \mathbf{Q}.$$

In this paper, we will extend item (i) to Peirce-diagonalizable transformations satisfying the so-called **Z**-property (see Corollary 6.2 and Remark 11). We will also show how to deduce item (ii) from general complementarity results on Peirce-diagonalizable transformations (see Theorem 6.4 and Remark 14).

Our motivation for this study also comes from a recent result which generalizes the well-known Schur product theorem to Euclidean Jordan algebras [18]: If A in (1) is positive-semidefinite and $x \in K$, then $A \cdot x \in K$.

The organization of the paper is as follows. Section 2 covers some basic material. In Section 3, we cover general properties of Peirce-diagonalizable transformations. Section 4 deals with the copositivity property. In this section, we introduce certain cones between the cones of completely positive matrices and doubly non-negative matrices. Section 5 deals with the **Z**-property, and finally in Section 6, we cover the complementarity properties.

2. Some preliminaries

Throughout this paper, we fix a Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ of rank r with the corresponding symmetric cone (of squares) K [5,9]. It is well known that V is a product of simple algebras, each isomorphic to one of the matrix algebras $\mathcal{S}^n$, $\mathcal{H}^n$, $\mathcal{Q}^n$ (which are, respectively, the spaces of $n \times n$ Hermitian matrices over real numbers/complex numbers/quaternions), $\mathcal{O}^3$ (the space of 3×3 Hermitian matrices over octonions), or the Jordan spin algebra $\mathcal{L}^n$ ($n \geq 3$). In the matrix algebras, the (canonical) inner product is given by

$$\langle X, Y \rangle = \operatorname{Re}\operatorname{trace}(XY),$$

and in $\mathcal{L}^n$, it is the usual inner product on $\mathcal{R}^n$. The symmetric cone of $\mathcal{S}^n$ will be denoted by $\mathcal{S}^n_+$ (with a similar notation in other spaces). In any matrix algebra, let E_{ij} be the matrix with ones in the (i,j) and (j,i) slots and zeros elsewhere; we write, $E_i := E_{ii}$. In a matrix algebra of rank r, $\{E_1, E_2, \ldots, E_r\}$ is the *canonical* Jordan frame. We use the notation $A = [a_{ij}]$ to say that A is a matrix with components a_{ij}; we also write

$$A \succeq 0 \Longleftrightarrow A \in \mathcal{S}^r_+ \quad \text{and} \quad x \geq 0\ (> 0) \Longleftrightarrow x \in K\ \ (\operatorname{int}(K)).$$

For a given Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V, we consider the corresponding *Peirce decomposition* [5, Theorem IV.2.1]

$$V = \sum_{1 \leq i \leq j \leq r} V_{ij},$$

where $V_{ii} = \mathcal{R}e_i$ and $V_{ij} = \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}$, when $i \neq j$. For any $x \in V$, we get the corresponding Peirce decomposition $x = \sum_{i \leq j} x_{ij}$. We note that this is an orthogonal direct sum and $x_{ii} = x_i e_i$ for all i, where $x_i \in \mathcal{R}$.

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We recall a few new concepts and notation. Given a matrix $A \in \mathcal{S}^r$, a Euclidean Jordan algebra V, and a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ with a specific ordering, we define the linear transformation $D_A : V \to V$ and Schur (or Hadamard) products in the following way: for any two elements $x, y \in V$ with Peirce decompositions $x = \sum_{i \le j} x_{ij}$ and $y = \sum_{i \le j} y_{ij}$,

$$D_A(x) = A \cdot x := \sum_{1 \le i \le j \le r} a_{ij} x_{ij} \in V \tag{4}$$

and

$$x \triangle y := \sum_{1}^{r} \langle x_{ii}, y_{ii} \rangle E_i + \frac{1}{2} \sum_{1 \le i < j \le r} \langle x_{ij}, y_{ij} \rangle E_{ij} \in \mathcal{S}^r. \tag{5}$$

We immediately note that

$$\langle D_A(x), x \rangle = \langle A \cdot x, x \rangle = \sum_{i \le j} a_{ij} \|x_{ij}\|^2 = \langle A, x \triangle x \rangle, \tag{6}$$

where the inner product on the far right is taken in $\mathcal{S}^r$.

When V is $\mathcal{S}^r$ with the canonical Jordan frame, the above products $A \cdot x$ and $x \triangle y$ coincide with the well-known Schur (or Hadamard) product of two symmetric matrices. This can be seen as follows. Consider matrices $x = [\alpha_{ij}]$ and $y = [\beta_{ij}]$ in $\mathcal{S}^r$ so that $x = \sum_{i \le j} \alpha_{ij} E_{ij}$ and $y = \sum_{i \le j} \beta_{ij} E_{ij}$ are their corresponding Peirce decompositions with respect to the canonical Jordan frame. Then simple calculations show that

$$A \cdot x = \sum_{i \le j} a_{ij} \alpha_{ij} E_{ij} = [a_{ij} \alpha_{ij}] \quad \text{and} \quad x \triangle y = [\alpha_{ij} \beta_{ij}],$$

where the latter expression is obtained by noting that in $\mathcal{S}^r$, $\langle E_{ij}, E_{ij} \rangle = \mathrm{trace}(E_{ij}^2) = 2$.

It is interesting to note that

$$(A \cdot x) \triangle x = A \cdot (x \triangle x), \tag{7}$$

where the 'mid-dot' product on the right is taken in $\mathcal{S}^r$ with respect to the canonical Jordan frame.

DEFINITION 2.1 *A linear transformation L on V is said to be Peirce-diagonalizable if there exist a Jordan frame and a symmetric matrix A such that $L = D_A$.*

Remark 1 Note that in Definition 2.1, both L and D_A use a specific ordering of the Jordan frame. Any permutation of the elements of the Jordan frame results in a row and column permutation of the matrix A.

Remark 2 In Definition 2.1 of D_A, we say that an entry a_{ij} in A is *relevant* if $V_{ij} \neq \{0\}$. (When a_{ij} is non-relevant, it plays no role in D_A.) By replacing all non-relevant entries of A by zero, we may rewrite $D_A = D_B$, where every non-relevant entry of B is zero. We also note that when V is simple, every V_{ij} is non-trivial [5, Corollary IV.2.4] and thus every entry in A is relevant. We use the notation $\mathrm{rel}(A) > 0$ to mean that all relevant entries of A are positive.

Remark 3 Suppose V is a product of simple algebras. Then with appropriate grouping of the given Jordan frame, any element x in V can be decomposed into components so that the Peirce decomposition of x splits into Peirce decompositions of components in each of the simple algebras. Because of the uniqueness of the matrix B (of the previous remark), the transformation D_B splits into component transformations of the same form, each of which is defined on a simple algebra.

Any such component transformation corresponds to a block submatrix of a matrix obtained by permuting certain rows and columns of A. This type of decomposition will allow us, in certain instances, to *restrict our analysis to transformations on simple algebras.*

Remark 4 Let V be a simple algebra and $\Gamma : V \to V$ be an algebra automorphism, that is, Γ is an invertible linear transformation on V with $\Gamma(x \circ y) = \Gamma(x) \circ \Gamma(y)$ for all $x, y \in V$. Since such a Γ preserves the inner product in V [5, p. 57], it is easily verified that

$$\Gamma(x)\triangle'\Gamma(y) = x\triangle y, \tag{8}$$

where $\Gamma(x)\triangle'\Gamma(y)$ is computed via the Peirce decompositions with respect to the Jordan frame $\{\Gamma(e_1), \Gamma(e_2), \ldots, \Gamma(e_r)\}$.

From now on, unless otherwise specified, Peirce decompositions and D_A are considered with respect to the (fixed, but arbitrary) Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V.

Properties of the products $A \cdot x$ and $x\triangle y$ are described in [18] and [11], respectively. In particular, we have the following generalization of the classical Schur's theorem [13, Theorem 5.2.1].

PROPOSITION 2.2 *Suppose $A \succeq 0$ and $x, y \geq 0$. Then,*

$$A \cdot x \geq 0 \quad and \quad x\triangle y \succeq 0.$$

The following proposition is a key to many of our results. In the earlier versions of the paper, the conclusion of this proposition was noted for each of the simple algebras and used in the proofs of several results. The following explicit formulation – suggested by a referee – streamlined and simplified our proofs. The classification-free proof of the lemma given below is due to this referee; a rank/classification-dependent proof is outlined in Remark 5.

PROPOSITION 2.3 *Suppose that V is simple. Then, for any vector $u \in \mathcal{R}^r_+$, there exists an $x \in K$ such that*

$$x\triangle x = uu^{\mathrm{T}}. \tag{9}$$

Proof Let $u \in \mathcal{R}^r_+$. We assume without loss of generality that $r \geq 2$ and that the inner product is given by $\langle x, y\rangle = \mathrm{tr}(x \circ y)$ so that the norm of any primitive element is one. (This is because, in a simple algebra, the given inner product is a multiple of the trace inner product [5, Proposition III.4.1.].) For $u \in \mathcal{R}^r_{++}$, the element x constructed in the lemma below will satisfy (9). The general case follows from a limiting argument. ■

LEMMA 2.4 (Due to a referee) *Let V be a simple Jordan algebra, let $\{e_1, \ldots, e_r\}$ be a Jordan frame, and let $u \in R^r_{++}$ be a vector and $v = \sqrt{u}$. Choose for $1 < i \leq r$ an element $x_{1i} \in V_{1i}$ of norm $\sqrt{2}v_1v_i$. Define $x_{ij} := 2x_{1i} \circ x_{1j}/v_1^2$ for $1 < i < j$, $x_{ii} := v_i^2 e_i$ for all i, and $x := \sum_{i \leq j} x_{ij}$. Then, $x \in K$.*

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Proof We show that $x = x^2/\|v\|^2$. In view of the standard properties of elements in Pierce subspaces V_{ij}, we have when $1 < i < j$ [5, Lemma IV.2.2]:

$$x_{ij} \in V_{ij},\ \|x_{ij}\| = \frac{2\|x_{1i} \circ x_{1j}\|}{v_1^2} = \frac{2\|x_{1i}\|\,\|x_{1j}\|}{\sqrt{8}v_1^2} = \sqrt{2}v_iv_j, \quad x_{ij}^2 = v_i^2v_j^2(e_i + e_j).$$

Also when $1 < i < j < k$,

$$x_{1j} \circ x_{jk} = x_{1j} \circ \left(\frac{2x_{1j} \circ x_{1k}}{v_1^2}\right) = \frac{\|x_{1j}\|^2}{4v_1^2}x_{1k} = \frac{v_j^2}{2}x_{1k},$$

$$x_{ik}\frac{v_1^4v_j^2}{8} = (x_{1i} \circ x_{1j}) \circ (x_{1j} \circ x_{1k}) = \frac{x_{ij}v_1^2}{2} \circ \frac{x_{jk}v_1^2}{2} = \frac{x_{ij} \circ x_{jk}v_1^4}{4}.$$

The first equality above comes directly from a polarization of Jordan's axiom:

$$\begin{aligned}&(a \circ b) \circ (c \circ d) + (a \circ c) \circ (b \circ d) + (a \circ d) \circ (b \circ c)\\ &\quad = (a \circ (b \circ c)) \circ d + b \circ (a \circ (c \circ d)) + c \circ (a \circ (b \circ d)),\end{aligned}$$

with $a = d = x_{1j}, b = x_{1k}$, and $c = x_{1i}$. Hence, $2x_{ij} \circ x_{jk} = v_j^2x_{ik}$; obviously, similar relations hold for other orderings of i, j, k. Now

$$\begin{aligned}x^2 &= \left(\sum_{i\le j} x_{ij}\right)^2 = \sum_1^r x_{ii}^2 + \sum_{i<k} x_{ik}^2 + 2\sum_{i<k} x_{ik} \circ x_{ik} + 2\sum_{k<i} x_{ii} \circ x_{ki}\\ &\quad + 2\sum_{i<j<k} x_{ij} \circ x_{jk} + 2\sum_{i<k<j} x_{ij} \circ x_{kj} + 2\sum_{k<i<j} x_{ij} \circ x_{kj}\\ &= \sum_1^r v_i^4e_i + \sum_{i<k} v_i^2v_k^2(e_i + e_k) + \sum_{i<k} v_i^2x_{ik} + \sum_{k<i} v_i^2x_{ki}\\ &\quad + \sum_{i<j<k} v_j^2x_{ik} + \sum_{i<k<j} v_j^2x_{ik} + \sum_{k<i<j} v_j^2x_{ki}\\ &= \sum_1^r \|v\|^2v_i^2e_i + \sum_{i<k} \|v\|^2x_{ik} = \|v\|^2x.\end{aligned}$$

■

Remark 5 We now outline a rank/classification-dependent proof of Proposition 2.3. We assume once again that V is simple and carries the trace inner product.

Case 1 Rank of V is 2.

Let $y_{12} \in V_{12}$ with norm $\sqrt{2}$. Define

$$y = e_1 + e_2 + y_{12} \quad \text{and} \quad x := P_a(y) = u_1e_1 + u_2e_2 + \sqrt{u_1u_2}y_{12},$$

where $a = \sqrt{u_1}e_1 + \sqrt{u_2}e_2$. It follows from Lemma 1 in [12] that $y \in K$ and hence $x \in K$. Now, a direct verification leads to (9).

We note that (as in the original version of the paper) the verification of the inclusion $x \in K$ can be done by explicit calculations in the Jordan spin algebra $\mathcal{L}^n$.

Case 2 Rank of V is more than 2.

In this case, by the classification theorem, V must be isomorphic to a matrix algebra. By Remark 4, we may take the canonical Jordan frame as the given frame. Let $v = \sqrt{u}$ and $x = vv^{\mathrm{T}}$. Then, $x \triangle x = uu^{\mathrm{T}}$.

By taking u to be the vector of ones in Proposition 2.3, and using (7), we get the following.

COROLLARY 2.5 *When V is simple, for any $A \in \mathcal{S}^r$, there exists an $x \in K$ such that*

$$A = (A \cdot x) \triangle x.$$

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3. Some general properties

Recall that Peirce decompositions and D_A are considered with respect to the given Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V. Our first result describes some general properties of D_A.

THEOREM 3.1 *For any $A \in \mathcal{S}^r$, the following statements hold*:

(1) *D_A is self-adjoint.*
(2) *The spectrum of D_A is $\sigma(D_A) = \{a_{ij} : 1 \le i \le j \le r,\ a_{ij}$ is relevant$\}$.*
(3) *D_A is diagonalizable.*
(4) *If Γ is an algebra automorphism of V, then $\Gamma^{-1} D_A \Gamma$ is Peirce-diagonalizable with respect to the Jordan frame $\{\Gamma^{-1}(e_1), \Gamma^{-1}(e_2), \ldots, \Gamma^{-1}(e_r)\}$ and with the same matrix A.*
(5) *D_A is a finite linear combination of quadratic representations corresponding to mutually orthogonal elements which have their spectral decompositions with respect to $\{e_1, e_2, \ldots, e_r\}$.*
(6) *If $A \succeq 0$, then $D_A = \sum_1^k P_{a_i}$ with $1 \le k \le r$, where a_is are mutually orthogonal and have their spectral decompositions with respect to $\{e_1, e_2, \ldots, e_r\}$. In particular, if A is completely positive* (*that is, A is a finite sum of matrices of the form uu^{T}, where u is a non-negative vector*), *we may assume that $a_i \in K$ for all i.*
(7) *If $A \succeq 0$, then $D_A(K) \subseteq K$; converse holds if V is simple.*

Proof (1) This follows from the equality $\langle D_A(x), y\rangle = \sum_{i \le j} a_{ij} \langle x_{ij}, y_{ij}\rangle = \langle x, D_A(y)\rangle$.

(2) When $V_{ij} \ne \{0\}$, any non-zero $x_{ij} \in V_{ij}$ acts as an eigenvector of D_A with the corresponding eigenvalue a_{ij}. Hence, all relevant a_{ij}s form a subset of $\sigma(D_A)$. Also, if x is an eigenvector of D_A corresponding to an eigenvalue λ, by writing the Peirce decomposition of x with respect to $\{e_1, e_2, \ldots, e_r\}$ and using the orthogonality of the Peirce spaces V_{ij}, we deduce that λ must be equal to some relevant a_{ij}. This proves that the spectrum of D_A consists precisely of all relevant a_{ij}s.
(3) D_A is a multiple of the identity on each V_{ij}. By choosing a basis in each V_{ij}, we get a basis of V consisting of eigenvectors of D_A. This proves that D_A is diagonalizable.
(4) Let Γ be an algebra automorphism of V; let $\Gamma^{-1}(e_i) = f_i$ for all i, so that $\{f_1, f_2, \ldots, f_r\}$ is a Jordan frame in V. Consider the Peirce decomposition $y = \sum_{i \le j} y_{ij}$ of any element y with respect to $\{f_1, f_2, \ldots, f_r\}$. Then, $\Gamma(y) = \sum_{i \le j} \Gamma(y_{ij})$ is the Peirce decomposition of $\Gamma(y)$ with respect to $\{e_1, e_2, \ldots, e_r\}$. Hence, $D_A(\Gamma(y)) = \sum_{i \le j} a_{ij} \Gamma(y_{ij})$. This implies that

$$\Gamma^{-1} D_A \Gamma(y) = \sum_{i \le j} a_{ij} y_{ij}.$$

Thus, $\Gamma^{-1} D_A \Gamma$ is Peirce-diagonalizable with respect to the Jordan frame $\{f_1, f_2, \ldots, f_r\}$ and with the matrix A.

(5) Since $A \in \mathcal{S}^r$, we may write A as a linear combination of matrices of the form uu^{T}, where $u \in \mathcal{R}^r$. From (2), $(uu^{\mathrm{T}}) \cdot x = \sum_{i \leq j} u_i u_j x_{ij} = P_a(x)$, where $a = \sum_1^r u_i e_i$. We see that D_A is a linear combination of quadratic representations of the form P_a, where a has its spectral decomposition with respect to $\{e_1, e_2, \ldots, e_r\}$.
(6) When $A \succeq 0$, we can write $D_A = \sum \gamma_k P_{a_k} = \sum P_{\sqrt{\gamma_k} a_k}$ whenever γ_k are non-negative. When A is completely positive, we may assume that u (which appears in the proof of Item (5)) is a non-negative vector. In this case, the corresponding a belongs to K.
(7) If $A \succeq 0$, then by Proposition 2.2, $D_A(K) \subseteq K$. For the converse, suppose V is simple. By Corollary 2.5, $A = (A \cdot x) \triangle x$ for some $x \in K$. When $D_A(K) \subseteq K$, by Proposition 2.2, $A \cdot x \geq 0$ and so $(A \cdot x) \triangle x \succeq 0$. Thus, $A \succeq 0$.

■

Corollary 3.2 *Suppose V is simple and for some $a \in V$, $L_a(K) \subseteq K$. Then, a is a non-negative multiple of the unit element.*

Proof Using the spectral decomposition of $a = \sum_1^r a_i e_i$, we may assume that $L_a = D_A$, where $A = [(a_i + a_j)/2]$. Since $L_a(K) \subseteq K$, by the previous theorem, A is positive-semidefinite. By considering the non-negativity of any 2×2 principal minor of A, we conclude that all diagonal elements of A are equal. This gives the stated result. ■

4. The copositivity property

In this section, we address the question of when D_A is copositive on K. Recall that an $n \times n$ real matrix A (symmetric or not) is a *copositive* (*strictly copositive*) *matrix* if $x^{\mathrm{T}} A x \geq 0$ (> 0) for all $0 \neq x \in \mathcal{R}^n_+$ and D_A is copositive (strictly copositive) on K if $\langle D_A(x), x \rangle \geq 0$ (> 0) for all $0 \neq x \in K$. We shall see that copositivity of D_A is closely related to the cone of completely positive matrices. We first recall some definitions and introduce some notation. For any set Ω, let cone(Ω) denote the convex cone generated by Ω (so that cone(Ω) is the set of all finite non-negative linear combinations of elements of Ω). Let

$$\begin{aligned} \mathcal{COP}_n &:= \text{The set of all copositive matrices in } \mathcal{S}^n, \\ \mathcal{CP}_n &:= \mathrm{cone}\{uu^{\mathrm{T}} : u \in \mathcal{R}^n_+\}, \\ \mathcal{NN}_n &:= \text{The set of all non-negative matrices in } \mathcal{S}^n, \\ \mathcal{DNN}_n &:= \mathcal{S}^n_+ \cap \mathcal{NN}_n, \text{ and} \\ \mathcal{D}(K) &:= \mathrm{cone}\{x \triangle x : x \in K\}. \end{aligned}$$

We note that $\mathcal{CP}_n$ is the cone of *completely positive* matrices and $\mathcal{DNN}_n$ is the cone of *doubly non-negative* matrices; for information on such matrices, see [1].

Remark 6 When V is simple, $\mathcal{D}(K)$ is independent of the Jordan frame used to define $x \triangle x$. This follows from (8). It is also independent of the underlying inner product in V. A similar statement can be made for general algebras (as they are products of simple ones).

It is well known [1, p. 71] that the cones $\mathcal{COP}_n$, $\mathcal{CP}_n$, and $\mathcal{DNN}_n$ are closed. We prove a similar result for $\mathcal{D}(K)$.

Proposition 4.1 *Let r denote the rank of V and $N = r(r+1)/2$. Then,*

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(i) $\mathcal{D}(K) = \{\sum_1^N x_i \triangle x_i : x_i \in K,\ 1 \le i \le N\}$,
(ii) *$\mathcal{D}(K)$ is closed in $\mathcal{S}^r$, and*
(iii) $\mathcal{D}(K) \subseteq \mathcal{DNN}_r$.
(iv) *When V is simple, $\mathcal{CP}_r \subseteq \mathcal{D}(K) \subseteq \mathcal{DNN}_r$ with equality for $r \le 4$.*

Proof (i) This follows from a standard application of Carathéodory's theorem for cones [1, p. 46].

(ii) To show that $\mathcal{D}(K)$ is closed, we take a sequence $\{z_k\} \in \mathcal{D}(K) \subseteq \mathcal{S}^r$ that converges to $z \in \mathcal{S}^n$. Then, with respect to the canonical Jordan frame in $\mathcal{S}^r$, we have $(z_k)_{ij} \to z_{ij}$ for all $i \le j$. Now, $z_k = \sum_{m=1}^N x_m^{(k)} \triangle x_m^{(k)}$ with $x_m^{(k)} \in K$ for all m and k. Then,

$$(z_k)_{ij} = \sum_{m=1}^{N} (x_m^{(k)} \triangle x_m^{(k)})_{ij} = \sum_{m=1}^{N} \varepsilon_{ij} \| (x_m^{(k)})_{ij} \|^2 E_{ij},$$

where $\sum_{i \le j} (x_m^{(k)})_{ij}$ is the Peirce decomposition of $x_m^{(k)}$ and ε_{ij} is a positive number that depends only on the pair (i,j). As $(z_k)_{ij}$ converges for each m and (i,j), $(x_m^{(k)})_{ij}$ is bounded as $k \to \infty$. We may assume, as V_{ij}s are closed, that $(x_m^{(k)})_{ij}$ converges to, say, $(x_m)_{ij}$ in V_{ij}. Defining $x_m := \sum_{i \le j} (x_m)_{ij}$, we see that $x_m^{(k)} \to x_m$. Then, $x_m \in K$ and $x_m^{(k)} \triangle x_m^{(k)} \to x_m \triangle x_m$ for each m. We conclude that $z = \sum_{m=1}^N x_m \triangle x_m \in \mathcal{D}(K)$. Thus, $\mathcal{D}(K)$ is closed.

(iii) Every matrix in $\mathcal{D}(K)$ is positive-semidefinite (from Proposition 2.2) and has non-negative entries (from the definition of $x \triangle x$). Thus, we have the stated inclusion.

(iv) When V is simple, the inclusion $\mathcal{CP}_r \subseteq \mathcal{D}(K)$ follows from Proposition 2.3. When $r \le 4$, it is known [1, p. 73] that $\mathcal{CP}_r = \mathcal{DNN}_r$.

■

Remark 7 For $r \ge 5$, we have the inclusions

$$\mathcal{CP}_r \subseteq \mathcal{D}(\mathcal{S}_+^r) \subseteq \mathcal{D}(\mathcal{H}_+^r) \subseteq \mathcal{D}(\mathcal{Q}_+^r) \subseteq \mathcal{DNN}_r.$$

While $\mathcal{CP}_r \neq \mathcal{DNN}_r$ for $r \ge 5$ (see the example given below), it is not clear if the other inclusions are indeed proper. In particular, it is not clear if $x \triangle x$ is completely positive for every $x \in \mathcal{S}_+^r$.

Remark 8 Consider the following matrices:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 6 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 1 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

It is known that M and N are doubly non-negative, while N is completely positive, M not completely positive [1, p. 63 and 79]. It can be easily shown that M and N are not of the form $x \triangle x$ for any $x \in \mathcal{S}_+^5$. (If $M = x \triangle x$ for some $x \in \mathcal{S}_+^5$, then the entries of x are the square roots of entries of M. The leading 3×3 principal minor of any such x is negative.) It may be interesting to see if M belongs to $\mathcal{D}(\mathcal{S}_+^5)$ or not.

THEOREM 4.2 *Suppose V is simple and D_A is copositive on K. Then, A is a copositive matrix.*

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Proof Since V is simple, we can apply Proposition 2.3: for any $u \in R^r_+$, there is an $x \in K$ such that $x \triangle x = uu^{\mathrm{T}}$. Then, by (6),

$$0 \leq \langle D_A(x), x\rangle = \langle A, x \triangle x\rangle = \langle A, uu^{\mathrm{T}}\rangle = u^{\mathrm{T}} Au.$$

Thus, A is a copositive matrix. ■

Remark 9 It is clear that the converse of the above result depends on whether the equality $\mathcal{CP}_r = \mathcal{D}(K)$ holds or not.

5. The Z-property

A linear transformation L on V is said to be a **Z**-transformation if

$$x \geq 0,\ y \geq 0, \quad \text{and} \quad \langle x, y\rangle = 0 \Longrightarrow \langle L(x), y\rangle \leq 0.$$

Being a generalization of a **Z**-matrix, such a transformation has numerous eigenvalue and complementarity properties [10,12]. In this section, we characterize the **Z**-property of a Peirce-diagonalizable transformation.

Corresponding to a Jordan frame $\{e_1, e_2, \dots, e_r\}$, we define two sets in $\mathcal{S}^r$:

$$\mathcal{C} := \{x \triangle y : x \geq 0,\ y \geq 0\}$$

and

$$\widehat{\mathcal{C}} := \{x \triangle y : 0 \leq x \perp y \geq 0\}.$$

Now, define the linear transformation $\Lambda : \mathcal{S}^r \to \mathcal{S}^r$ by $\Lambda(Z) := EZE = \delta(Z)E$, where $Z = [z_{ij}]$, $\delta(Z) = \sum_{i,j} z_{ij}$, and E denotes the matrix of ones. It is clear that Λ is self-adjoint on $\mathcal{S}^r$.

LEMMA 5.1 *When V is simple with rank r, we have $\mathcal{C} = \mathcal{S}^r_+$ and $\widehat{\mathcal{C}} = \mathcal{S}^r_+ \cap \operatorname{Ker}(\Lambda)$.*

Proof From Proposition 2.2, $\mathcal{C} \subseteq \mathcal{S}^r_+$. To see the reverse inclusion, let $A \in \mathcal{S}^r_+$. By Corollary 2.5, there is an $x \in K$ such that $A = (A \cdot x) \triangle x$. By Proposition 2.2, $y := A \cdot x \geq 0$ and so $A = y \triangle x$, where $x, y \in K$. Thus, $\mathcal{S}^r_+ \subseteq \mathcal{C}$ and the required equality holds.

Now, let $B = x \triangle y \in \widehat{\mathcal{C}}$. Then, $\delta(B) = \sum_{i,j} b_{ij} = \sum_{i \leq j} \langle x_{ij}, y_{ij}\rangle = \langle x, y\rangle = 0$ and so $B \in \operatorname{Ker}(\Lambda)$. Since $B \in \mathcal{C} = \mathcal{S}^r_+$, we have $B \in \mathcal{S}^r_+ \cap \operatorname{Ker}(\Lambda)$. For the reverse inclusion, let $B \in \mathcal{S}^r_+ \cap \operatorname{Ker}(\Lambda)$. As $\mathcal{C} = \mathcal{S}^r_+$, we can write $B = x \triangle y$, where $x, y \in K$. Then, $\langle x, y\rangle = \delta(B) = 0$ and so $B \in \widehat{\mathcal{C}}$. This completes the proof. ■

THEOREM 5.2 *Let V be simple and $A \in \mathcal{S}^r$. Then, D_A has the **Z**-property if and only if there exist $B_k \in \mathcal{S}^r_+$ and a sequence $\alpha_k \in \mathcal{R}$ for which $A = \lim_{k\to\infty}(\alpha_k E - B_k)$. In this case,*

$$D_A = \lim_{k\to\infty} (\alpha_k I - D_{B_k}).$$

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Proof From the previous lemma, $\widehat{\mathcal{C}} = \mathcal{S}^r_+ \cap \mathrm{Ker}(\Lambda)$. Then, the dual of $\widehat{\mathcal{C}}$ is given by [1, p. 48]

$$(\widehat{\mathcal{C}})^* = \overline{\mathcal{S}^r_+ + \mathrm{Ran}(\Lambda)},$$

where the overline denotes the closure. Now, D_A has the **Z**-property if and only if

$$0 \le x \perp y \ge 0 \Longrightarrow \langle D_A(x), y\rangle \le 0.$$

Writing $\langle D_A(x), y\rangle = \sum_{i\le j} a_{ij}\langle x_{ij}, y_{ij}\rangle = \mathrm{trace}(AZ)$, where $Z = x\triangle y \in \widehat{\mathcal{C}}$, we see that D_A has the **Z**-property if and only if

$$Z \in \widehat{\mathcal{C}} \Longrightarrow \langle A, Z\rangle \le 0.$$

This means that D_A has the **Z**-property if and only if $-A \in (\widehat{\mathcal{C}})^* = \overline{\mathcal{S}^r_+ + \mathrm{Ran}(\Lambda)}$. The stated conclusions follows. ■

Remark 10 It is known [17] that any **Z**-transformation is of the form $\lim_{k\to\infty}(\alpha_k I - S_k)$, where the linear transformation S_k keeps the cone K invariant. Our theorem above reinforces this statement with an additional information that Peirce-diagonalizability is preserved.

6. Complementarity properties

In this section, we present some complementarity properties of D_A.

THEOREM 6.1 *For any $A \in \mathcal{S}^r$, the following statements are equivalent*:

(i) D_A *is strictly (=strongly) monotone on* V : $\langle D_A(x), x\rangle > 0$ *for all* $x \ne 0$.
(ii) D_A *has the* **GUS**-*property: for all* $q \in V$, $\mathrm{LCP}(D_A, K, q)$ *has a unique solution.*
(iii) D_A *has the* **P**-*property*: *for any* x, *if* x *and* $D_A(x)$ *operator commute with* $x \circ D_A(x) \le 0$, *then* $x = 0$.
(iv) $\mathrm{rel}(A) > 0$, *that is*, *all relevant entries of* A *are positive*.

Proof The implications (i) ⇒ (ii) ⇒ (iii) are well known and true for any linear transformation [9, Theorem 11].

Now suppose (iii) holds. Since D_A is self-adjoint and satisfies the **P**-property, all its eigenvalues are positive in view of Theorem 11 in [9]. Item (iv) now follows from Theorem 3.1, item (2). Now suppose that $\mathrm{rel}(A) > 0$. Then, we have

$$\langle D_A(x), x\rangle = \sum_{i\le j} a_{ij}\|x_{ij}\|^2 > 0 \quad \forall x \ne 0.$$

(Note that when some a_{ij} is not relevant, the corresponding x_{ij} is zero.) This proves that (iv) ⇒ (i). ■

COROLLARY 6.2 *Suppose that D_A has the* **Z**-*property. Then*,

$$\textit{strict monotonicity} \Longleftrightarrow \mathbf{GUS} \Longleftrightarrow \mathbf{P} \Longleftrightarrow \mathbf{Q} \Longleftrightarrow \mathbf{S} \Longleftrightarrow \mathrm{diag}(A) > 0.$$

Proof The implications

$$\text{strict monotonicity} \Longrightarrow \mathbf{GUS} \Longrightarrow \mathbf{P} \Longrightarrow \mathbf{Q} \Longrightarrow \mathbf{S}$$

always hold for any linear transformation [9]. Suppose that D_A has the **S**-property. Then for some $d > 0$ in V, $A\cdot d = D_A(d) > 0$. By writing the Peirce decompositions of d and $A\cdot d$, we see that diag$(A) > 0$. Now suppose that diag$(A) > 0$. Then, $D_A(e) > 0$, where e is the unit element in V. This, together with the **Z**-property of D_A, implies the **P**-property of D_A (see Theorem 4 in [12]). We can now apply the previous theorem to get the strict monotonicity of D_A. ■

Remark 11 Consider an element a in V whose spectral decomposition is given by $a = a_1e_2 + a_2e_2 + \cdots + a_re_r$. Since $L_a = D_A$, where $A = [(a_i + a_j)/2]$, and L_a has the **Z**-property, Corollary 6.2 gives the equivalence of various properties of L_a mentioned in Section 1. Simple examples can be constructed (via Theorem 5.2) to see that Corollary 6.2 goes beyond transformations of the form L_a.

With a given above, we can apply the previous theorem to $P_a = D_B$, where $B = [a_ia_j]$ to get the equivalence of strict monotonicity, **GUS**-, and **P**-properties [8].

In what follows, we use the notation $A \in \mathbf{R}_0$ to mean that A has the $\mathbf{R}_0$-property, that is, LCP$(A, \mathcal{R}^n_+, 0)$ has a unique solution (namely zero). A similar notation is used for D_A in relation to LCP$(D_A, K, 0)$.

THEOREM 6.3 *Suppose $A \succeq 0$. If $A \in \mathbf{R}_0$, then $D_A \in \mathbf{R}_0$. Converse holds when V is simple.*

Proof Assume that $A \in \mathbf{R}_0$, but $D_A \notin \mathbf{R}_0$. Then, there is a non-zero $x \in V$ such that $x \geq 0$, $D_A(x) \geq 0$, and $\langle D_A(x), x\rangle = 0$. We have from (6)

$$0 = \langle D_A(x), x\rangle = \langle A, x\triangle x\rangle.$$

As $x \geq 0$ in V, $X := x\triangle x \succeq 0$ (by Proposition 2.2). Now in $\mathcal{S}^r$

$$X \succeq 0,\ A \succeq 0, \quad \text{and} \quad \langle A, X\rangle = 0 \Longrightarrow AX = 0.$$

Since X, which is non-zero, consists of non-negative entries, for any non-zero column u in X, we have $Au = 0$. This u is a non-zero solution of the problem LCP$(A, \mathcal{R}^r_+, 0)$, yielding a contradiction. Hence, $D_A \in \mathbf{R}_0$.

For the converse, suppose that V is simple and $D_A \in \mathbf{R}_0$. Assume, if possible, that $A \notin \mathbf{R}_0$. Then, there exists a non-zero $u \in R^r$ such that

$$u \geq 0,\ Au \geq 0, \quad \text{and} \quad \langle Au, u\rangle = 0.$$

As V is simple, for this u, by Proposition 2.3, there is an $x \in K$ such that $x\triangle x = uu^{\mathrm{T}}$. Since u is non-zero, x is also non-zero. Now

$$\langle D_A(x), x\rangle = \langle A, x\triangle x\rangle = \langle A, uu^{\mathrm{T}}\rangle = u^{\mathrm{T}}Au = 0.$$

We also have $D_A(x) = A \cdot x \geq 0$. Thus, even in this case, $D_A \notin \mathbf{R}_0$. Hence, when V is simple, $D_A \in \mathbf{R}_0 \Rightarrow A \in \mathbf{R}_0$. ■

Remark 12 Theorem 6.3 may not hold if A is not positive-semidefinite. For example, consider the matrices

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

with the corresponding transformations $D_A(x) = A \cdot x$ and $D_C(x) = C \cdot x$ on $\mathcal{S}^2$. It is easily seen that A is in $\mathbf{R}_0$ while $D_A \notin \mathbf{R}_0$ and D_C is in $\mathbf{R}_0$ while $C \notin \mathbf{R}_0$.

Before formulating our next result, we recall some definitions. Given a linear transformation L on V, its principal subtransformations are obtained in the following way: take any non-zero

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idempotent c in V. Then, the transformation $T_c := P_c L : V(c,1) \to V(c,1)$ is called a principal subtransformation of L, where $V(c,1) = \{x \in V : x \circ c = x\}$. (We note here that $V(c,1)$ is a subalgebra of V.) We say that L has the completely **Q**-property if for every non-zero idempotent c, the subtransformation T_c has the **Q**-property on $V(c,1)$. The transformation L is said to have the **S**-property if there exists a $d > 0$ in V such that $L(d) > 0$. The completely **S**-property is defined by requiring that all principal subtransformations have the **S**-property.

It is well known [6] that for a symmetric positive-semidefinite matrix A, the $\mathbf{R}_0$- and **Q**-properties are equivalent to the strict copositivity property on $\mathcal{R}^n$. A related result, given by Malik [16] for linear transformations on Euclidean Jordan algebras, says that for a self-adjoint, cone-invariant transformation, $\mathbf{R}_0$-, completely **Q**-, and completely **S**-properties are equivalent to the strict copositivity property. These two results yield the following.

THEOREM 6.4 *With* $A \succeq 0$, *consider the following statements*:

(a) *A has the* **Q**-*property*.
(b) *A is strictly copositive*.
(c) *A has the* $\mathbf{R}_0$-*property*.
(d) D_A *has the* $\mathbf{R}_0$-*property*.
(e) D_A *has the completely* **S**-*property*.
(f) D_A *has the completely* **Q**-*property*.
(g) D_A *is strictly copositive*.
(h) D_A *has the* **Q**-*property*.

Then,

$$\text{(a)} \Longleftrightarrow \text{(b)} \Longleftrightarrow \text{(c)} \Longrightarrow \text{(d)} \Longleftrightarrow \text{(e)} \Longleftrightarrow \text{(f)} \Longleftrightarrow \text{(g)} \Longrightarrow \text{(h)}.$$

Proof As A is symmetric and positive-semidefinite, the equivalence of (a)–(c) is given in [6]; the implication (c) $\Rightarrow$ (d) is given in the previous result. As A is positive-semidefinite, D_A (which is self-adjoint) keeps the cone-invariant. The equivalence of (d)–(g) now follows from Malik [16, Theorem 3.3.2 and Corollary 3.3.2]. Finally, the implication (g) $\Rightarrow$ (h) follows from the well-known theorem of Karamardian [14]. ■

Remark 13 One may ask if the reverse implications hold in Theorem 6.4. Since D_A involves only the relevant entries of A, easy examples (e.g. $V = \mathcal{R}^2 = \mathcal{S}^1 \times \mathcal{S}^1$) can be constructed in the non-simple case to show that (h) need not imply (a). While we do not have an answer to this question in the simple algebra case, in the result below we state some necessary conditions for D_A to have the **Q**-property.

PROPOSITION 6.5 *Suppose* D_A *has the* **Q**-*property. Then*, *the following statements hold*:

(i) *The diagonal elements of A are positive*.
(ii) *When V is simple*, *no positive linear combination of two rows of A can be zero*.
(iii) *If V has rank* 2 *and* $A \succeq 0$, *then* D_A *has the* $\mathbf{R}_0$-*property*.

Proof (i) Since D_A has the **Q**-property, it has the **S**-property, that is, there is a $u > 0$ in V such that $D_A(u) > 0$. Writing the Peirce decompositions $u = \sum_{i \le j} u_{ij}$ and $D_A(u) = \sum_{i \le j} a_{ij} u_{ij}$, we see that $u_i > 0$ and $a_{ii} u_i > 0$ for all i, where $u_{ii} = u_i e_i$. Thus, the diagonal of A is positive. Now, let $b := \sum_1^r (a_{ii}^{-1})^{1/4} e_i$ and consider $\widehat{D_A} = P_b D_A P_b$. It is easily seen that $\widehat{D_A}$ has the **Q**-property and Peirce-diagonalizable with the matrix whose entries are $a_{ij}/\sqrt{a_{ii}a_{jj}}$.

Note that *the diagonal elements of this latter matrix are one.*

(ii) Suppose V is simple and let, without loss of generality, $\lambda A_1 + \mu A_2 = 0$, where λ and μ are positive numbers, and A_1 and A_2, respectively, denote the first and second rows of A. Let $a = \sqrt{\lambda}e_1 + \sqrt{\mu}e_2 + e_3 + e_4 + \cdots + e_r$ and consider $\widehat{D_A} := P_a D_A P_a$. Then, $\widehat{D_A}$ has the **Q**-property and is Peirce-diagonalizable with a matrix in which the sum of the first two rows is zero. Thus, we may assume without loss of generality that in the given matrix A, the sum of first two rows is zero. As V is simple, the space V_{12} is non-zero. Let q_{12} be any non-zero element in V_{12}. We claim that $\mathrm{LCP}(D_A, K, q_{12})$ does not have a solution. Assuming the contrary, let x be a solution to this problem and let $y = D_A(x) + q_{12}$. Then, $x \geq 0$, $y \geq 0$, and $x \circ y = 0$. Writing the Peirce decompositions of x and y and using the properties of the spaces V_{ij} [5, Theorem IV.2.1], we see that

$$\begin{aligned} 0 = (x \circ y)_{12} &= \frac{1}{2}(a_{11} + a_{12})x_1 x_{12} + \frac{1}{2}(a_{12} + a_{22})x_2 x_{12} \\ &\quad + \sum_{2<i}(a_{1i} + a_{2i})x_{1i} \circ x_{2i} + \frac{1}{2}(x_1 + x_2)q_{12}, \end{aligned}$$

where $x_{11} = x_1 e_1$ and $x_{22} = x_2 e_2$. Since the sum of first two rows of A is zero, the above expression reduces to $0 = \frac{1}{2}(x_1 + x_2)q_{12}$. As q_{12} is non-zero, we get $x_1 + x_2 = 0$. Now, since $x \geq 0$, we must have $x_1 \geq 0$ and $x_2 \geq 0$. Thus, we have $x_1 = 0 = x_2$. This leads, from $x \geq 0$, to $x_{1j} = 0 = x_{2j}$ for all j [7, Proposition 3.2]. But then $y \geq 0$ together with $y_{11} = a_{11}x_{11} = 0$ leads to $y_{1j} = 0$ for all j, and, in particular, to $y_{12} = 0$. Now we see that $0 = y_{12} = a_{12}x_{12} + q_{12} = q_{12}$, which contradicts our assumption that q_{12} is non-zero. Hence, D_A does not have the **Q**-property, contrary to our assumption. This proves the stated assertion.

(iii) Now assume that V has rank 2. In this case, A is a 2×2 matrix. Without loss of generality (see the proof of Item (i)), assume that the diagonal of A consists of ones. Given that D_A has the **Q**-property, we claim that A has the $\mathbf{R}_0$-property and (via the previous theorem) D_A has the $\mathbf{R}_0$-property. Assume, if possible, that there is a non-zero vector $d \in \mathcal{R}^2$ such that $d \geq 0, Ad \geq 0$, and $\langle d, Ad \rangle = 0$. As A is symmetric and positive semidefinite, these conditions lead to $Ad = 0$. Since each diagonal element of A is one, d cannot have just one non-zero component. Thus, both components of d are non-zero. In this case, a positive linear combination of rows of A would be zero. This implies, see Item (ii), that V must be non-simple. In this case, V is isomorphic to $\mathcal{R}^2$ and we may write for any x, $x = x_1 e_1 + x_2 e_2$, $D_A(x) = a_{11}x_1 e_1 + a_{22}x_2 e_2$. Thus, we may regard D_A as a diagonal matrix acting on $\mathcal{R}^2$. Now, it is easy to show (from the **Q**-property) that D_A has the $\mathbf{R}_0$-property. This completes the proof. ■

Remark 14 From Remark 11, Theorem 6.4, and the well-known implication that $\mathbf{P} \Rightarrow \mathbf{R}_0$ always holds for any linear transformation, we see that for P_a

$$\text{strict monotonicity} \Longrightarrow \mathbf{GUS} \Longrightarrow \mathbf{P} \Longrightarrow \mathbf{R}_0 \Longrightarrow \mathbf{Q}.$$

We now prove the reverse implications. Suppose, for some $a \in V$, $D_A = P_a$ has the **Q**-property. We show that P_a is strictly monotone. Suppose first that V is simple. By writing the spectral decomposition $a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r$ we may let $A = [a_i a_j]$. Then, $a_i \neq 0$ for all i, by item (i) in Proposition 6.5. If some a_i and a_j have opposite signs, then a positive linear combination of the rows of A corresponding to i and j would be zero, contradicting item (ii). Thus, all a_i have the same sign. This means that $\pm a > 0$ and $A = [a_i a_j]$ is a matrix with positive entries; consequently, by Theorem 6.1, D_A is strictly monotone. Now suppose that V is a product of simple algebras V_i, $i = 1, 2, \ldots, m$. Then, we can write P_a as a product of transformations $P_{a^{(i)}}$, where $a^{(i)} \in V_i$ for all i. Since P_a has the **Q**-property, it follows that every $P_{a^{(i)}}$ has the **Q**-property on V_i. By our earlier argument, $P_{a^{(i)}}$ is strictly monotone on V_i and so P_a will also be strictly monotone on V.

We thus see the equivalence of various complementarity properties of P_a mentioned in Section 1.

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