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On the **P**-property of **Z** and Lyapunov-like transformations on Euclidean Jordan algebras

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ABSTRACT

The **P**, **Z**, and **S** properties of a linear transformation on a Euclidean Jordan algebra are generalizations of the corresponding properties of a square matrix on R^n. Motivated by the equivalence of **P** and **S** properties for a **Z**-matrix [2] and a similar result for Lyapunov and Stein transformations on the space of real symmetric matrices [6,5], in this paper, we present two results supporting the conjecture that **P** and **S** properties are equivalent for a **Z**-transformation on a Euclidean Jordan algebra. We show that the conjecture holds for Lyapunov-like transformations and **Z**-transformations satisfying an additional condition.

1. Introduction

Consider a Euclidean Jordan algebra $(V, \circ, \langle \cdot, \cdot \rangle)$ with the corresponding symmetric cone K [4]. In the setting of linear complementarity problems over symmetric cones, the following properties of a linear transformation are of fundamental importance and have been well studied. They are generalizations of various matrix properties studied in the setting of standard linear complementarity problems [3].

A linear transformation L on V is said to have the

- **GUS**-*property if for every* $q \in V$, *the symmetric cone linear complementarity problem* LCP(L, K, q) *has a unique solution, that is, there is a unique* $x \in V$ *such that*

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$$x \geqslant 0, \quad L(x) + q \geqslant 0, \ \ and \ \ \langle L(x) + q, x \rangle = 0,$$

where $x \geqslant 0$ *means that* $x \in K$*;*
- **P**-*property if* $[x$ *and* $L(x)$ *operator commute,* $x \circ L(x) \leqslant 0] \Rightarrow x = 0$*;*
- **Q**-*property if for every* $q \in V$*, LCP*(L, K, q) *has a solution;*
- **S**-*property if there exists* $d \in V$ *such that* $d > 0$ *and* $L(d) > 0$*, where* $d > 0$ *means that* $d \in interior(K)$*;*
- **Z**-*property if* $[x \geqslant 0, \ y \geqslant 0, \ and \ \langle x, y \rangle = 0] \Rightarrow \langle L(x), y \rangle \leqslant 0$*;*
- *Lyapunov-like property if both L and* $-L$ *have the* **Z**-*property;*
- *Positive stable property if all the eigenvalues of L have positive real parts.*

It is known that in any Euclidean Jordan algebra, the following implications hold [8,9]:

GUS-property $\Rightarrow$ **P**-property $\Rightarrow$ **Q**-property $\Rightarrow$ **S**-property

and when L has the **Z**-property,

Q-property $\Leftrightarrow$ **S**-property $\Leftrightarrow$ Positive stable property.

Since **GUS** , **P** and **Q** properties are difficult to characterize for a general linear transformation, as in the current state of affairs, we limit ourselves to certain special classes of transformations. In this paper, we consider transformations with the **Z**-property (in particular, those with the Lyapunov-like property) and investigate the validity of the following conjecture [9]:

Conjecture. *For any linear transformation with the* **Z**-*property,* **P** *and* **S** *properties are equivalent.*

The above conjecture is known to hold in the following special cases/settings:

(1) $V = R^n$ and $K = R^n_+$; see [2, Chapter 6, Theorem 2.3, 3, Theorems 3.3.4, 3.3.7, and 3.11.10].
(2) $V = \mathcal{S}^n$ (the space of all real $n \times n$ symmetric matrices), $K = \mathcal{S}^n_+$, and $L = L_A$ (Lyapunov transformation), where for any matrix $A \in R^{n\times n}$, $L_A(X) := AX + XA^T$; see [6]. We note that L_A is Lyapunov-like.
(3) $V = \mathcal{S}^n$, $K = \mathcal{S}^n_+$, and $L = S_A$ (Stein transformation), where for any matrix $A \in R^{n\times n}$, $S_A(X) := X - AXA^T$; see [5]. We note that S_A has the **Z**-property.
(4) Rank of V is 2; see [9].

In this paper, going beyond the settings of Items (1) and (2) above, we prove the following results on a *general* Euclidean Jordan algebra.

(a) *If L has the* **Z** *and* **S** *properties, then for some* $c > 0$*,* $P_c^{-1} L P_c$ *has the* **P**-*property, where* P_c *denotes the quadratic representation of c.*
(b) *If L is Lyapunov-like, then* **P** *and* **S** *properties are equivalent.*

These results are novel and significant in the sense that they are valid on any Euclidean Jordan algebra and for transformations more general than matrices over R^n (of Item (1)) and Lyapunov transformations on $\mathcal{S}^n$ (of Item (2)). In addition, our proofs are coordinate and classification free, and the proof of Item (b) uses the new result (which we shall prove using Lie-algebraic ideas) that a transformation L is Lyapunov-like on V if and only if L is of the form $L_a + D$ where $a \in V$, $L_a(x) := a \circ x$ and D is a (inner) derivation on V.

2. Preliminaries

Throughout this paper, we let V denote a Euclidean Jordan algebra of rank r and follow the notation and basic results from [4] or [8]. The symmetric cone of V is denoted by K; the inner product and Jordan product of two elements x and y in V are denoted, respectively, by $\langle x, y \rangle$ and $x \circ y$. In V, e

denotes the unit element. We use the notation $x \geqslant 0 \quad (x > 0)$ to indicate that $x \in K$ (respectively, $x \in interior(K)$). In V we write $x \perp y$ if $\langle x, y \rangle = 0$. The *canonical or trace inner product* on V is given by

$$\langle x, y \rangle_{tr} := tr(x \circ y).$$

When V is simple, there exists a positive number α such that, $\langle x, y \rangle = \alpha\, tr(x \circ y)$ for all x and y (see [4, Proposition III.4.1]). More generally, if V is a product of simple algebras V_i $(i = 1, 2, \ldots, N)$ (where the Jordan product is computed componentwise and the inner product is the sum of the inner products in each V_i), then for two objects $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$ in V,

$$\langle x, y \rangle = \sum_1^N \langle x_i, y_i \rangle = \sum_1^N \alpha_i\, tr(x_i \circ y_i) = \sum_1^N \alpha_i\, \langle x_i, y_i \rangle_{tr}. \tag{1}$$

We also note that

$$\langle x, y \rangle_{tr} = \sum_1^N \langle x_i, y_i \rangle_{tr}. \tag{2}$$

It follows from these that $x \geqslant 0,\ y \geqslant 0,\ \langle x, y \rangle = 0 \Leftrightarrow x_i \geqslant 0,\ y_i \geqslant 0$, and $\langle x_i, y_i \rangle = 0\ \forall\, i$. Given any $a \in V$, we let L_a and P_a denote the corresponding Lyapunov transformation and quadratic representation of a on V:

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

We say that elements a and b in V *operator commute* if $L_a L_b = L_b L_a$ (equivalently, a and b have their spectral decompositions with respect to a common Jordan frame).

From now on, we use a boldface letter to denote either a class of linear transformations or a property satisfied by a linear transformation; for example, $L \in \mathbf{P}$ if and only if L has the **P**-property.

3. Z-transformations and the P-property

We recall the following from [11,9].

Theorem 1. *The following are equivalent:*

(*i*) *L has the* **Z**-*property.*
(*ii*) *For every Jordan frame* $\{e_1, e_2, \ldots, e_r\}$ *in V, it holds that*

$$\langle L(e_i), e_j \rangle \leqslant 0 \quad \text{for all } i \neq j.$$

(*iii*) $e^{-tL}(K) \subseteq K$ *for all* $t \geqslant 0$ *in R.*

The result below shows that the **Z**-property is “independent" of the inner product.

Proposition 2. *Suppose L has the* **Z**-*property on the Euclidean Jordan algebra V which carries the inner product* $\langle \cdot, \cdot \rangle$. *Then L has the* **Z**-*property with respect to the canonical inner product* $\langle \cdot, \cdot \rangle_{tr}$ *and conversely.*

Proof. The result is obvious when V is simple, as any inner product on a simple algebra is a positive multiple of the trace inner product, see [4, Proposition III.4.1]. Now assume that V is a product of simple algebras; for simplicity, let $V = V_1 \times V_2$, where V_1 and V_2 are simple. Assume that L has the **Z**-property with respect to $\langle \cdot, \cdot \rangle$ and consider $x = (x_1, x_2) \geqslant 0$, $y = (y_1, y_2) \geqslant 0$ with $0 = \langle x, y \rangle_{tr} = \langle x_1, y_1 \rangle_{tr} + \langle x_2, y_2 \rangle_{tr}$. This implies that $\langle x_i, y_i \rangle_{tr} = 0$ for $i = 1, 2$ and hence (via (1)), $0 \leqslant x \perp (y_1, 0) \geqslant 0$ with respect to $\langle \cdot, \cdot \rangle$. Hence $\langle L(x), (y_1, 0) \rangle \leqslant 0$, which yields $\langle L(x)_1, y_1 \rangle_{tr} \leqslant 0$. A similar inequality ensues when we replace the subscript 1 by 2. By adding these inequalities and using (2), we get $\langle L(x), y \rangle_{tr} \leqslant 0$. This proves that L has the **Z**-property with respect to the trace inner product. The converse statement is proved in a similar way. $\square$

In the result below, without loss of generality, *V carries the trace inner product, in which case, the norm of any primitive idempotent is one. Note that changing the inner product has no effect on the* **P**-*property.*

Theorem 3. *If $L \in \mathbf{Z}$ and $L(e) > 0$, then $L \in \mathbf{P}$. More generally, if $L \in \mathbf{Z} \cap \mathbf{S}$, then there is a $c > 0$ such that $P_c^{-1} L P_c \in \mathbf{P}$.*

Proof. First suppose that $L \in \mathbf{Z}$ and $L(e) > 0$. Let $x \in V$ be such that

$$x \text{ and } L(x) \text{ operator commute with } x \circ L(x) \leqslant 0.$$

Then for some Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V we have the spectral decompositions,

$$x = \sum_1^r x_i e_i \quad \text{and} \quad L(x) = \sum_1^r y_i e_i,$$

where $x_i y_i \leqslant 0$ for all $i = 1, 2, \ldots, r$. Hence,

$$\sum_1^r x_i L(e_i) = L(x) = \sum_1^r y_i e_i,$$

which leads to

$$\sum_1^r x_i \langle L(e_i), e_j \rangle = y_j \quad \forall j = 1, 2, \ldots, r \tag{3}$$

as e_is are orthogonal and $||e_j|| = 1$ for all j. Now, in view of Theorem 1(ii), the matrix $A := [a_{ij}]$ with $a_{ij} = \langle L(e_i), e_j \rangle$ is a **Z**-matrix and $A^T p = q$, where p (q) is the column vector in R^r whose components are x_i (respectively, y_i). As $x_i y_i \leqslant 0$ for all $i = 1, 2, \ldots, r$, we see that $p * A^T p \leqslant 0$, where '$*$' denotes the componentwise product. Now the assumption $L(e) > 0$ implies that

$$\sum_{i=1}^r \langle L(e_i), e_j \rangle = \langle L(e), e_j \rangle > 0 \quad \forall j = 1, 2, \ldots, r.$$

This leads to

$$A^T \mathbf{1} > 0 \ (\text{in } R^r),$$

where $\mathbf{1}$ is the vector in R^r with all components equal to one. This means that the **Z**-matrix A^T is also an **S**-matrix. It follows that, see [3, Theorem 3.11.10], A^T is a **P**-matrix and hence $p * A^T p \leqslant 0 \Rightarrow p = 0$. This proves that $x_i = 0$ for all i. Hence L has the **P**-property.

Now for the general case, assume that $L \in \mathbf{Z}$ and that there is a $d > 0$ such that $L(d) > 0$. Put $c := \sqrt{d} > 0$ and consider the quadratic representation P_c. Then P_c is self-adjoint, invertible, $(P_c)^{-1} = P_{c^{-1}}$, $P_c(K) = K$, and $P_c(e) = c^2 = d$. It is easily seen that

$$\widetilde{L} := P_c^{-1} L P_c$$

is a **Z**-transformation with $\widetilde{L}(e) = (P_c)^{-1}(L(d)) > 0$. By the first part of the proof, $\widetilde{L}$ has the **P**-property. This completes the proof. $\square$

4. Lyapunov-like transformations

In this section, we give a characterization of Lyapunov-like transformations on Euclidean Jordan algebras by using Lie-algebraic results. Let $Aut(K)$ denote the set of all (invertible) linear transformations which map K onto K. We recall that the corresponding Lie algebra is given by [1]

$$Lie(Aut(K)) = \{L \in \mathcal{B}(V) : e^{tL} \in Aut(K), \ \forall t \in R\},$$

where $\mathcal{B}(V)$ is the set of all linear transformations on V. A linear transformation D on V is said to be a **derivation** if for all $x, y \in V$,

$$D(x \circ y) = D(x) \circ y + x \circ D(y).$$

It is said to be an *inner derivation* if it is a linear combination of commutators of the form

$$[L_a, L_b] := L_aL_b - L_bL_a$$

for some a, b in V.

Theorem 4. *The following are equivalent:*

(i) *L is Lyapunov-like on V.*
(ii) *For any Jordan frame $\{e_1, e_2, \dots, e_r\}$, $\langle L(e_i), e_j\rangle = 0\ \forall\, i \neq j$.*
(iii) *$e^{tL} \in Aut(K)$ for all $t \in R$.*
(iv) *$L \in Lie(Aut(K))$.*
(v) *$L = L_a + D$, where $a \in V$ and D is an inner derivation.*

Proof. The equivalence of (i) and (ii) comes from Theorem 1 applied to L and $-L$. Since the condition $e^{tL}(K) \subseteq K\ \ \forall\, t \in R$ can be written as $e^{tL}(K) = K$ for all $t \in R$, it follows that (i) and (iii) are equivalent, once again, by Theorem 1. The equivalence of (iii) and (iv) is just the definition. Finally, the equivalence of (iv) and (v) follows from [4, Proposition VIII.2.6] and the fact that on a Euclidean Jordan algebra, every derivation is inner, see [4, Proposition VI.1.2] or [10, Theorem 8, p. 87]. $\square$

It follows from the above theorem that every derivation is Lyapunov-like and (since derivations are skew-symmetric, see [4, Proposition VIII.2.6]) every symmetric Lyapunov-like transformation on V is of the form L_a for some $a \in V$. In addition, every Lyapunov-like transformation on $\mathcal{S}^n$ is of the form L_A for some $A \in R^{n\times n}$ (with similar statements in the matrix algebras over complex numbers and quaternions).

5. The P-property of Lyapunov-like transformations

In this section, we show that **P** and **S** properties are equivalent for Lyapunov-like transformations. *Henceforth, we assume that the inner product is given by the trace inner product.* Given any idempotent $c \in V$, we let

$$V(c, 1) := \{x \in V : x \circ c = x\}.$$

Theorem 5. *For a Lyapunov-like transformation, the* **P** *and* **S** *properties are equivalent.*

Since the proof is somewhat involved, we first give a sketch of the proof. Assume that L is Lyapunov-like with the **S**-property, but not the **P**-property. Then there is a nonzero element $\overline{x}$ and a Jordan frame $\{e_1, e_2, \dots, e_r\}$ such that $\overline{x} = \sum_1^k \overline{x}_i e_i$ with $\overline{x}_i \neq 0$ for all $i = 1, 2, \dots, k$, $L(\overline{x}) = \sum_1^k \overline{y}_i e_i$ with $\overline{x}_i\overline{y}_i \leqslant 0$ for all i. With $W := V(e_1 + e_2 + \cdots + e_k, 1)$, we show that $L(W) \subseteq W$. As L is positive stable, it turns out that the restriction L' of L to W is also positive stable, hence has positive trace. From the inequalities $\overline{x}_i\overline{y}_i \leqslant 0$ we show that the trace of L' is non-positive, leading to a contradiction. Thus the **S**-property implies the **P**-property when L is Lyapunov-like. The other implication is always true.

Before giving a detailed proof of this result, we prove several lemmas leading up to Proposition 12 which deals with the inclusion $L(W) \subseteq W$ (mentioned above). *In these lemmas, we fix a Jordan frame $\{e_1, e_2, \dots, e_r\}$ in V and consider the corresponding Peirce decomposition $V = \sum_{i \leqslant j} V_{ij}$ and properties of V_{ij} given in [4, Theorem IV.2.1]. Furthermore, in each of these lemmas, we assume that V has appropriate rank to make the lemma non-vacuous. Throughout, we assume that L is Lyapunov-like.*

Lemma 6. *For $i \neq j$, let $x_{ij} \in V_{ij}$ with $||x_{ij}|| = 1$. Then*

(a) $x := e_i + e_j + \sqrt{2}x_{ij} \geqslant 0$,
(b) $y := e_i + e_j - \sqrt{2}x_{ij} \geqslant 0$, *and*
(c) $\langle x, y\rangle = 0$.

Proof. (a) In V, consider any $d = \sum d_i e_i + \sum d_{ij} \geqslant 0$. We note that $2d_i d_j \geqslant ||d_{ij}||^2$ for all $i \neq j$, see [4, Exercise 7, p. 80]. Then,

$$\begin{aligned}\langle x, d\rangle &= d_i + d_j + \sqrt{2}\langle x_{ij}, d_{ij}\rangle \geqslant d_i + d_j - \sqrt{2}||x_{ij}||||d_{ij}|| \\ &= d_i + d_j - \sqrt{2}||d_{ij}|| \geqslant d_i + d_j - 2\sqrt{d_i}\sqrt{d_j} = (\sqrt{d_i} - \sqrt{d_j})^2 \geqslant 0.\end{aligned}$$

As K is self-dual, $x \geqslant 0$. The proof of (b) is similar, and (c) is obvious. □

Lemma 7. *The following hold:*

(a) $i \neq j, x_{ij} \in V_{ij},\ ||x_{ij}|| = 1 \Rightarrow \langle L(x_{ij}), x_{ij}\rangle = \frac{1}{2}(\langle L(e_i), e_i\rangle + \langle L(e_j), e_j\rangle)$.
(b) $\langle L(e_i), e_i\rangle \leqslant 0\ \forall i = 1, 2, \ldots, r \Rightarrow Tr(L) \leqslant 0$, *where $Tr(L)$ denotes the trace of the linear transformation L on V.*

Proof. (a) For a given x_{ij}, let x and y be as in the previous lemma. Since L is Lyapunov-like, we have $\langle L(x), y\rangle = 0 = \langle L(y), x\rangle$. Now,

$$\begin{aligned}\langle L(x), y\rangle = 0 \Rightarrow &\langle L(e_i), e_i\rangle + \langle L(e_j), e_j\rangle - 2\langle L(x_{ij}), x_{ij}\rangle \\ &= -\sqrt{2}\langle L(x_{ij}), e_i + e_j\rangle + \sqrt{2}\langle L(e_i) + L(e_j), x_{ij}\rangle. \end{aligned} \quad (4)$$

$$\begin{aligned}\langle L(y), x\rangle = 0 \Rightarrow &\langle L(e_i), e_i\rangle + \langle L(e_j), e_j\rangle - 2\langle L(x_{ij}), x_{ij}\rangle \\ &= \sqrt{2}\langle L(x_{ij}), e_i + e_j\rangle - \sqrt{2}\langle L(e_i) + L(e_j), x_{ij}\rangle. \end{aligned} \quad (5)$$

Adding these equations, we get the desired result.

(b) Now suppose $\langle L(e_i), e_i\rangle \leqslant 0$ for all $i = 1, 2, \ldots, r$. Recall that the trace of the linear transformation L on the (real) Hilbert space V is the sum of numbers of the form $\langle L(u), u\rangle$, as u varies over any orthonormal basis in V. (This can be seen by writing the matrix representation of L with respect to an orthonormal basis and adding the diagonal elements of that matrix.) As V is an orthogonal direct sum of spaces V_{ij} for $i \leqslant j$, the union of orthonormal bases in various V_{ij}s is an orthonormal basis in V. By our assumption and Item (a), the sum of numbers of the form $\langle L(u), u\rangle$ as u varies over an orthonormal basis in any V_{ij} is non-positive. Thus, the sum of these, namely, $Tr(L)$, is also non-positive. This completes the proof. □

Lemma 8. *Suppose i, k, and l are distinct and $x_{kl} \in V_{kl}$. Then $\langle L(e_i), x_{kl}\rangle = 0$ and $\langle L(x_{kl}), e_i\rangle = 0$.*

Proof. Without loss of generality, we assume that $||x_{kl}|| = 1$. Then $e_k + e_l - \sqrt{2}x_{kl} \geqslant 0$ by Lemma 6, and $\langle e_i, e_k + e_l - \sqrt{2}x_{kl}\rangle = 0$ for $i \notin \{k, l\}$. Since L is a Lyapunov-like transformation, we have $\langle L(e_i), e_k + e_l - \sqrt{2}x_{kl}\rangle = 0 \Rightarrow \langle L(e_i), x_{kl}\rangle = 0$. Similarly, $\langle L(e_k + e_l - \sqrt{2}x_{kl}), e_i\rangle = 0 \Rightarrow \langle L(x_{kl}), e_i\rangle = 0$. □

Lemma 9. *Suppose i, j, k, and l are all distinct, $x_{ij} \in V_{ij}$, and $x_{kl} \in V_{kl}$. Then $\langle L(x_{ij}), x_{kl}\rangle = 0$.*

Proof. Following Theorem 4, we let $L = L_a + D$, where $a \in V$ and D is a derivation on V. Then $\langle L_a(x_{ij}), x_{kl}\rangle = \langle a, x_{ij} \circ x_{kl}\rangle = 0$. Further, $\langle D(x_{ij}), x_{kl}\rangle = 2\langle D(e_i \circ x_{ij}), x_{kl}\rangle = 2\,[e_i \circ D(x_{ij}) + D(e_i) \circ x_{ij}, x_{kl}\rangle = 2\,[\langle D(x_{ij}), e_i \circ x_{kl}\rangle + \langle D(e_i), x_{ij} \circ x_{kl}\rangle] = 0$. Thus, $\langle L(x_{ij}), x_{kl}\rangle = \langle L_a(x_{ij}), x_{kl}\rangle + \langle D(x_{ij}), x_{kl}\rangle = 0$. □

Lemma 10. *Suppose there exists* $\bar{x} = \sum_1^k \bar{x}_i e_i$ *such that* $\bar{x}_i \neq 0$ *for all* $i = 1, 2, \ldots, k$ *and* $\bar{y} = L(\bar{x}) = \sum_1^k \bar{y}_i e_i$. *Then*

$$\langle L(e_i), z_{ij}\rangle = 0, \quad \forall\, i \in \{1, 2, \ldots, k\},\ j \in \{k+1, k+2, \ldots, r\},\ \ z_{ij} \in V_{ij}.$$

Proof. Fix $i \in \{1, 2, \ldots, k\}$, $j \in \{k+1, k+2, \ldots, r\}$ and $z_{ij} \in V_{ij}$. Since the spaces V_{ij} are mutually orthogonal and $z_{ij} \in V_{ij}$, we have $\langle L(\bar{x}), z_{ij}\rangle = \langle \bar{y}, z_{ij}\rangle = 0$. On expanding, we get

$$\bar{x}_1\langle L(e_1), z_{ij}\rangle + \bar{x}_2\langle L(e_2), z_{ij}\rangle + \cdots + \bar{x}_i\langle L(e_i), z_{ij}\rangle + \bar{x}_{i+1}\langle L(e_{i+1}), z_{ij}\rangle + \cdots + \bar{x}_k\langle L(e_k), z_{ij}\rangle = 0.$$

By Lemma 8, $\langle L(e_l), z_{ij}\rangle = 0$ for all $l \neq i$. For $l = i$, $\bar{x}_i \neq 0$ and so $\langle L(e_i), z_{ij}\rangle = 0$. □

By writing $L = L_a + D$ and using $e_i \circ z_{ij} = \frac{1}{2} z_{ij}$, we may write the conclusion in the above lemma as:

$$\frac{1}{2}\langle a, z_{ij}\rangle + \langle D(e_i), z_{ij}\rangle = 0, \quad \forall\, i \in \{1, 2, \ldots, k\},\ j \in \{k+1, k+2, \ldots, r\},\ \ z_{ij} \in V_{ij}. \tag{6}$$

Lemma 11. *Suppose that the conditions of the previous lemma are in place. Then* $\langle L(x_{ij}), y_{il}\rangle = 0$, *for all* $i \neq j \in \{1, 2, \ldots, k\}$, $l \in \{k+1, k+2, \ldots, r\}$, $x_{ij} \in V_{ij}$, *and* $y_{il} \in V_{il}$.

Proof. We write $L = L_a + D$ and let $z_{jl} := x_{ij} \circ y_{il} \in V_{jl}$. Then

$$\langle L_a(x_{ij}), y_{jl}\rangle = \langle a, x_{ij} \circ y_{jl}\rangle = \langle a, z_{jl}\rangle.$$

Also, using $D(x_{ij}) = 2D(e_j \circ x_{ij}) = 2\,[e_j \circ D(x_{ij}) + D(e_j) \circ x_{ij}]$, we have

$$\langle D(x_{ij}), y_{il}\rangle = 2\,[\langle e_j \circ D(x_{ij}), y_{il}\rangle + \langle D(e_j) \circ x_{ij}, y_{il}\rangle = 2\langle D(e_j) \circ x_{ij}, y_{il}\rangle$$

since $\langle e_j \circ D(x_{ij}), y_{il}\rangle = \langle D(x_{ij}), e_j \circ y_{il}\rangle = 0$. Adding these two expressions, we get

$$\langle L(x_{ij}), y_{il}\rangle = 2\,[\,\frac{1}{2}\langle a, z_{jl}\rangle + \langle D(e_j), z_{jl}\rangle] = 0$$

by (6). This completes the proof. □

The above lemmas lead to the following result which may be of independent interest.

Proposition 12. *Let L be Lyapunov-like on V and suppose that corresponding to a Jordan frame* $\{e_1, e_2, \ldots, e_r\}$, *there exists* $\bar{x} = \sum_1^k \bar{x}_i e_i$ *such that* $\bar{x}_i \neq 0$ *for all* $i = 1, 2, \ldots, k$, *and* $\bar{y} = L(\bar{x}) = \sum_1^k \bar{y}_i e_i$. *Let* $W := V(e_1 + e_2 + \cdots + e_k, 1)$. *Then* $L(W) \subseteq W$.

Proof. The result is obvious if $k = 1$ (as $V(e_1, 1) = \mathcal{R}e_1$) or $k = r$. We assume that $1 < k < r$. This implies that $r \geqslant 3$ and Lemma 11 is applicable. Consider the Peirce decomposition of any $x \in W$ with respect to the Jordan frame $\{e_1, e_2, \ldots, e_k\}$: $x = \sum_1^k x_i e_i + \sum_{1\leqslant i<j\leqslant k} x_{ij}$. We write the Peirce decomposition of $z := L(x)$ with respect to the Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V:

$$\sum_1^k x_i L(e_i) + \sum_{1\leqslant i<j\leqslant k} L(x_{ij}) = \left(\sum_1^k z_i e_i + \sum_{1\leqslant i<j\leqslant k} z_{ij}\right) + \left(\sum_{1\leqslant i\leqslant k<k+1\leqslant l\leqslant r} z_{il}\right) + \left(\sum_{k+1}^r z_i e_i + \sum_{k+1\leqslant i<l\leqslant r} z_{il}\right).$$

(Note that the last summation is vacuous when $r = 3$.) Now, taking the inner product of any term on the left-hand side of the above expression with z_{il} for $1 \leqslant i \leqslant k < k+1 \leqslant l \leqslant r$ or with $z_i e_i$ for $i \geqslant k+1$, or with z_{il} for $k+1 \leqslant i < l \leqslant r$, and using the previous lemmas, we deduce that the last two (grouped) terms on the right-hand side of the above expression are both zero. Thus,

$$L(x) = \sum_1^k z_i e_i + \sum_{1\leqslant i<j\leqslant k} z_{ij} \in W.$$

This completes the proof. □

Proof of Theorem 5. Since the **P**-property always implies the **S**-property, we (only) prove the reverse implication by assuming that L is Lyapunov-like on V and $L \in \mathbf{S}$. Assume, if possible, there is a nonzero $\bar{x}$ such that

$$\bar{x} \text{ and } \bar{y} := L(\bar{x}) \text{ operator commute, and } \bar{x} \circ \bar{y} \leqslant 0.$$

Then for some Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V,

$$\bar{x} = \sum_1^r \bar{x}_i e_i \quad \text{and} \quad L(\bar{x}) = \sum_1^r \bar{y}_i e_i,$$

where $\bar{x}_i \bar{y}_i \leqslant 0$ for all $i = 1, 2, \ldots, r$. As $\bar{x} \neq 0$, we may assume that $\bar{x}_i \neq 0$ for $1 \leqslant i \leqslant k$ and $\bar{x}_i = 0$ for $i > k$, where $k \leqslant r$. Then $\bar{x} = \sum_1^k \bar{x}_i e_i$ and

$$\sum_1^k \bar{x}_i L(e_i) = L(\bar{x}) = \sum_1^r \bar{y}_i e_i,$$

which leads to

$$\sum_1^k \bar{x}_i \langle L(e_i), e_j \rangle = \bar{y}_j \quad \forall j = 1, 2, \ldots, r, \tag{7}$$

as $||e_j|| = 1$ for all j. Since L is Lyapunov-like, these yield $\bar{y}_j = 0$ for $j \geqslant k + 1$ so that

$$L(\bar{x}) = L\left(\sum_1^k \bar{x}_i e_i\right) = \sum_1^k \bar{y}_i e_i.$$

Moreover,

$$\langle L(e_i), e_i \rangle = \frac{\bar{y}_i}{\bar{x}_i} \leqslant 0 \quad \forall\, i = 1, 2, \ldots, k. \tag{8}$$

Let $W := V(e_1 + e_2 + \cdots + e_k, 1)$. Then by Proposition 12, $L(W) \subseteq W$. Denote the restriction of L to W by L'. Then L' is Lyapunov-like on the (subalgebra) W. Also, the matrix representation of L on $V = W \bigoplus W^{\perp}$ is of the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where A is the matrix representation of L' on W. Since L is Lyapunov-like and hence a **Z**-transformation, the **S**-property is equivalent to the positive stable property, see [9, Theorems 6 and 7]. This implies, from the above matrix representation, that L' also has the positive stable property and so its trace must be positive. However, (8) implies that $\langle L'(e_i), e_i \rangle \leqslant 0$ for all $i = 1, 2, \ldots, k$ and by Lemma 7, applied to L' and W, $Tr(L') \leqslant 0$. This is a contradiction. Hence no $\bar{x}$ exists, proving the **P**-property of L. This completes the proof of the theorem. □

6. Concluding remarks

In this paper, we have presented two results in support of the conjecture that **P** and **S** properties are equivalent for a **Z**-transformation. Regarding the **GUS**-property, it is known that a Lyapunov-like transformation has the **GUS**-property if and only if it is positive stable and monotone [7]. However, a characterization of the **GUS**-property for a general **Z**-transformation is still open.

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