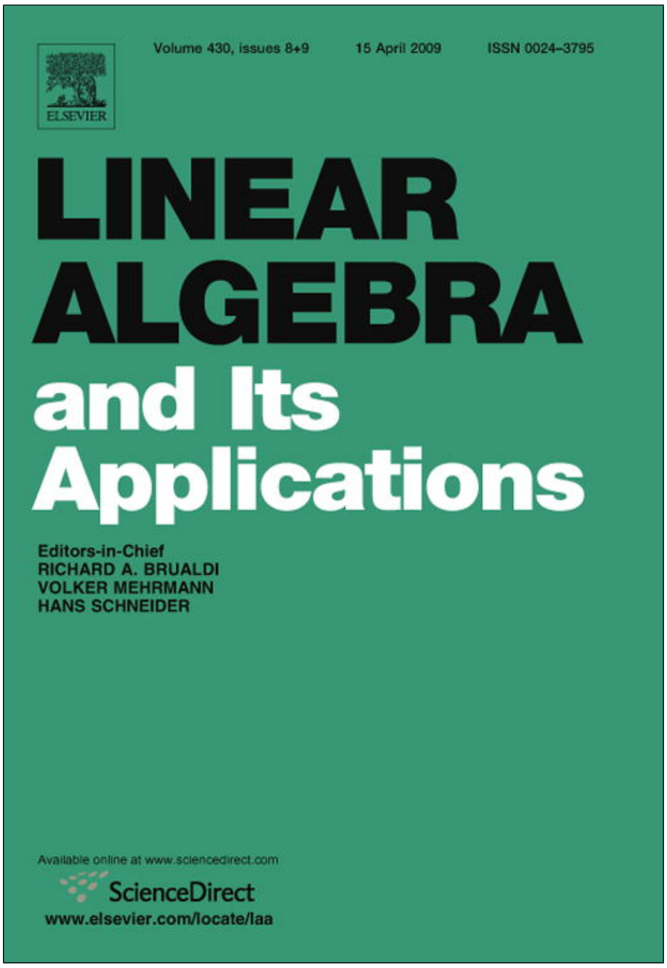
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Some inertia theorems in Euclidean Jordan algebras

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A B S T R A C T

This paper deals with some inertia theorems in Euclidean Jordan algebras. First, based on the continuity of eigenvalues, we give an alternate proof of Kaneyuki's generalization of Sylvester's law of inertia in simple Euclidean Jordan algebras. As a consequence, we show that the cone spectrum of any quadratic representation with respect to a symmetric cone is finite. Second, we present Ostrowski–Schneider type inertia results in Euclidean Jordan algebras. In particular, we relate the inertias of objects a and x in a Euclidean Jordan algebra when $L_a(x) > 0$ or $S_a(x) > 0$, where L_a and S_a denote Lyapunov and Stein transformations, respectively.

1. Introduction

For a square matrix A with complex entries, the inertia is defined by

$$\mathrm{In}(A) := (\pi(A), \nu(A), \delta(A)),$$

where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of eigenvalues of A with positive, negative, and zero real parts, counting multiplicities.

In this paper, we consider some well known inertia theorems from matrix analysis and study their analogs in Euclidean Jordan algebras.

First, we consider Sylvester's law of inertia [12, Theorem 4.5.8] which states that for any Hermitian matrix X and any invertible (complex) matrix C,

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$$\mathrm{In}(CXC^*) = \mathrm{In}(X),$$

where C^* denotes the conjugate of C. In [18], Ostrowski extends this result by showing that for any complex matrix C,

$$\pi(CXC^*) \leqslant \pi(X) \quad \text{and} \quad \nu(CXC^*) \leqslant \nu(X).$$

Sylvester's law of inertia has been extended to simple Euclidean Jordan algebras by Kaneyuki [14]. To describe this extension, let $\mathscr{H}^n$ denote the set of all $n \times n$ (complex) Hermitian matrices and $\mathscr{H}^n_+$ denote the closed convex cone of positive semidefinite matrices in $\mathscr{H}^n$. Then $\mathscr{H}^n$ is a Euclidean Jordan algebra (see Section 2 for the definition) with the inner product and the Jordan product given by

$$\langle X, Y\rangle := \mathrm{trace}(XY) \quad \text{and} \quad X \circ Y := \frac{XY + YX}{2}.$$

In this setting, consider the linear transformation $\Gamma : \mathscr{H}^n \to \mathscr{H}^n$ defined by $\Gamma(X) = CXC^*$ where C is a (fixed) invertible complex matrix. By writing C in the polar form as $C = UA$ where $A \in \mathscr{H}^n$ is invertible (actually, positive definite) and U is unitary, we can write $\Gamma = \Lambda P_A$, where P_A (called the quadratic representation of A) and Λ (called an algebra automorphism) are invertible linear transformations on $\mathscr{H}^n$ with the defining properties

$$P_A(X) := 2A \circ (A \circ X) - A^2 \circ X \quad \text{and} \quad \Lambda(X \circ Y) = \Lambda(X) \circ \Lambda(Y)$$

for all $X, Y \in \mathscr{H}^n$. (We note that $P_A(X) = AXA$ for all $X \in \mathscr{H}^n$.)

Given this, Sylvester's law of inertia can be stated in the following way: For any invertible A in $\mathscr{H}^n$ and an algebra automorphism Λ on $\mathscr{H}^n$,

$$\mathrm{In}(\Lambda P_A(X)) = \mathrm{In}(X)$$

for all $X \in \mathscr{H}^n$.

A Euclidean Jordan algebra is a finite dimensional real Hilbert space with a compatible Jordan product. A Euclidean Jordan algebra is said to be simple if it cannot be decomposed as the product of two (non-trivial) Euclidean Jordan algebras. In addition to $\mathscr{S}^n$ and $\mathscr{H}^n$, examples of simple algebras include the algebra of all $n \times n$ Hermitian matrices with quaternion entries, and the algebra of all 3×3 Hermitian matrices with octonion entries. The space $R^n (n > 1)$ with the usual inner product and componentwise (Jordan) product is an example of a non-simple Euclidean Jordan algebra. In a Euclidean Jordan algebra, every element has a spectral decomposition which defines the (real) eigenvalues of that element. As in the case of a Hermitian matrix, we can then define the inertia of an element x by

$$\mathrm{In}(x) := (\pi(x), \nu(x), \delta(x)),$$

where $\pi(x), \nu(x)$, and $\delta(x)$ are, respectively, the number of positive, negative, and zero eigenvalues, counting multiplicities. Similar to $\mathscr{H}^n_+$, each Euclidean Jordan algebra has a corresponding cone of squares called the symmetric cone. Algebra automorphisms are those invertible linear transformations on the algebra that preserve the Jordan product. Corresponding to an element a in the algebra, we define the quadratic representation

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

Given the above, Kaneyuki's extension [14] of Sylvester's law of inertia can now be stated: In a simple Euclidean Jordan algebra V, for any invertible element $a \in V$ and any algebra automorphism Λ on V, we have

$$\mathrm{In}(\Lambda P_a(x)) = \mathrm{In}(x)$$

for all $x \in V$. Kaneyuki, in [14], proves this result by using Lie algebraic ideas and results, and in [15] further extends it to graded Lie algebras. In this paper, we give an alternate proof of Kaneyuki's result by relying on the continuity of eigenvalues (which itself comes from the min–max theorem of Hirzebruch on simple Euclidean Jordan algebras). Then using the continuity arguments again, we show that in any Euclidean Jordan algebra V, for any element a and any algebra automorphism Λ on V,

$$\pi(\Lambda P_a(x)) \leqslant \pi(x) \quad \text{and} \quad \nu(\Lambda P_a(x)) \leqslant \nu(x). \tag{1}$$

As an application of the above results, we study the finiteness of cone spectrum of algebra automorphisms and quadratic representations with respect to the symmetric cone. Given a linear transformation L on a finite-dimensional real Hilbert space H and a closed convex cone C (with dual C^*) in H, the *cone spectrum* [23] of L with respect to C, denoted by $\sigma(L, C)$, is the set of all real λ for which there exists $x \in H$ such that

$$0 \neq x \in C, L(x) - \lambda x \in C^*, \quad \text{and} \quad \langle x, L(x) - \lambda x \rangle = 0.$$

In [29], Zhou and Gowda proved the finiteness of cone spectrum of quadratic representations by analyzing the cone spectrum on each of the simple algebras and then using the structure theorem (see Section 2). In the present paper, we provide a short proof based on the inequalities in (1) and at the same time extend the Zhou–Gowda result to products of algebra automorphisms and quadratic representations.

Our next set of results deals with Ostrowski–Schneider type inertia results of matrix theory. The famous result of Lyapunov [17] states that a real (complex) $n \times n$ matrix A is positive stable – which means that $\mathrm{In}(A) = (n, 0, 0)$ – if and only if for some (equivalently, every) symmetric (Hermitian) positive definite matrix W, the system

$$L_A(X) := \frac{1}{2}(AX + XA^*) = W$$

has a symmetric (Hermitian) positive definite solution X. Ostrowski and Schneider [19], see also Taussky [25], extended this result by showing the following: Given $A \in R^{n\times n}(C^{n\times n})$, there exists an $X \in \mathcal{S}^n$ (respectively, $\mathcal{H}^n$) such that $AX + XA^T \succ 0$ if and only if $\delta(A) = 0$, in which case

$$\mathrm{In}(A) = \mathrm{In}(X).$$

Carlson and Schneider [3] further extended this result by showing that if $\delta(A) = 0$ and $AX + XA^T \succeq 0$ for some $X \in \mathcal{S}^n(\mathcal{H}^n)$, then

$$\pi(X) \leqslant \pi(A) \quad \text{and} \quad \nu(X) \leqslant \nu(A).$$

Analogous results hold for Stein transformations. It is known, see [19,28], that for any real (complex) $n \times n$ matrix A, the system

$$S_A(X) := X - AXA^T \succ 0$$

has a solution $X \in \mathcal{S}^n(\mathcal{H}^n)$ if and only if A has no eigenvalues on the unit circle in the complex plane, in which case

$$\mathrm{In}(X) = \mathrm{In}_0(A),$$

where $\mathrm{In}_0(A)$ is the circle inertia of A defined as the triple consisting of eigenvalues of A that lie within the unit circle, outside the unit circle, and on the unit circle. Analogous to the Carlson–Schneider result above, Datta [5] extended this to the semidefinite case.

In this paper, we extend the above inertia results to certain linear transformations defined on Euclidean Jordan algebras, see Theorems 18 and 19. Besides deducing the above results from our inertia results, we show, in particular, the following:

- Given an element a in a Euclidean Jordan algebra, the system

  $$L_a(x) := a \circ x > 0$$

  has a solution x if and only if $\delta(x) = 0$, in which case

  $$\mathrm{In}(a) = \mathrm{In}(x).$$

- Given an element a in a Euclidean Jordan algebra, the system

  $$S_a(x) := x - P_a(x) > 0$$

  has a solution x if and only if a has no eigenvalues on the unit circle, in which case,

$$\text{In}(x) = \text{In}_0(a)$$

where $\text{In}_0(a)$ denotes the circle inertia of a.

The paper is organized as follows. In Section 2, we describe some concepts from the theory of Euclidean Jordan algebras. Section 3 deals with the min–max theorem of Hirzebruch and its consequences. In Section 4, we prove our main inequality (1) for products of quadratic representations and algebra automorphisms. In Section 5, we present our cone spectrum result. Finally, in Section 6 we describe Ostrowski–Schneider type inertia theorems.

2. Euclidean Jordan algebras

In this section, we describe some concepts, properties, and results from the Euclidean Jordan algebra theory that are relevant to our study. Most of these can be found in Faraut and Korányi [6], Schmieta and Alizadeh [21], and Gowda et al. [9].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$ where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions for all x and y : $x \circ y = y \circ x, x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, and $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$. It is known that in such an algebra, there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

Henceforth, V denotes a Euclidean Jordan algebra. In V, the set of squares

$$K := \{x \circ x : x \in V\}$$

is a *symmetric cone*. This means that K is a self-dual closed convex cone and for any two elements $x, y \in \text{interior}(K)$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$, see Faraut and Korányi [6, p. 46].

For an element $z \in V$, we write

$$z \geqslant 0 (z > 0) \quad \text{if and only if } z \in K (z \in K^o)$$

where K^o denotes the interior of K.

For $x \in V$, we define

$$m(x) := \min\{k > 0 : \{e, x, \ldots, x^k\} \text{ is linearly dependent}\}$$

and *rank* of V by $r = \max\{m(x) : x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{if } i \neq j \quad \text{and} \quad \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j\rangle = \langle e_i \circ e_j, e\rangle = 0$ whenever $i \neq j$.

Theorem 1 (The spectral decomposition theorem [6]). *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2}$$

In (2), the expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the *spectral decomposition (or the spectral expansion)* of x. The real numbers λ_i (also written as $\lambda_i(x)$) are called the *eigenvalues* of x; these are uniquely defined, even though the Jordan frame corresponding to x need not be unique. For any $x \in V$ given by (2), we define the *trace* and *spectrum* of x by

$$\text{trace}(x) := \lambda_1 + \lambda_2 + \cdots + \lambda_r$$

and

$$\sigma(x) := \{\lambda_1, \lambda_2, \ldots, \lambda_r\}.$$

We also define the *inertia* of x by

$$\text{In}(x) = (\pi(x), \nu(x), \delta(x))$$

where $\pi(x)$, $\nu(x)$, and $\delta(x)$ are, respectively, the number of eigenvalues of x which are positive, negative, and zero, counting multiplicities. We note that

$$\pi(x) + \nu(x) + \delta(x) = r$$

for all x. If x has no zero eigenvalues, then we say that x is invertible; in this case, following (2), we define the inverse of x by

$$x^{-1} = \frac{1}{\lambda_1} e_1 + \cdots + \frac{1}{\lambda_r} e_r.$$

We note that $\langle u, v\rangle_t := \text{trace}(u \circ v)$ defines another inner product on V so that $(V, \circ, \langle\cdot,\cdot\rangle_t)$ is also a Euclidean Jordan algebra.

The Peirce decomposition

Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = Re_i$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\right\}.$$

Then we have the following

Theorem 2 ([6, Theorem IV.2.1]). *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leqslant j)$. Furthermore,*

$V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj}$,
$V_{ij} \circ V_{jk} \subset V_{ik}$ *if* $i \neq k$, *and*
$V_{ij} \circ V_{kl} = \{0\}$ *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

Thus, given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j\leqslant r} x_{ij},$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$. This expression is the *Peirce decomposition* of x with respect to $\{e_1, e_2, \ldots, e_r\}$.

A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two (non-trivial) Euclidean Jordan algebras. The classification theorem (Chapter V, [6]) says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathcal{L}^n = (R^n, \circ, \langle\cdot,\cdot\rangle)$, where $R^n = R \times R^{n-1}$ $(n > 1)$, $\langle\cdot,\cdot\rangle$ is the usual inner product, and
$$(x_0, \bar{x}) \circ (y_0, \bar{y}) = (x_0 y_0 + \langle \bar{x}, \bar{y}\rangle, x_0 \bar{y} + y_0 \bar{x}).$$
(2) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(3) The algebra $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(4) The algebra $\mathcal{Q}^n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(5) The algebra $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 3 ([6, Propositions III.4.4 and III.4.5, Theorem V.3.7]). *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

Suppose V is a direct sum of simple algebras V_i: $V = V_1 \oplus V_2 \oplus \cdots \oplus V_N$. Then any primitive idempotent in V is of the form $(0, 0, \ldots, e_i, 0, \ldots, 0)$ for some primitive idempotent e_i in V_i. Consequently, for any $x = (x_1, x_2, \ldots, x_N)$ in the direct sum,

(i) The spectrum of x is the union of the spectra of x_i,
(ii) x is invertible in V if and only if x_i is invertible in V_i, and
(iii) $\mathrm{In}(x) = \left(\sum_i^N \pi(x_i), \sum_i^N \nu(x_i), \sum_i^N \delta(x_i)\right)$.

Index of a simple algebra

For any primitive idempotent c, let

$$V\left(c, \frac{1}{2}\right) := \left\{x : x \circ c = \frac{1}{2}x\right\}.$$

Now let V be simple. It is known (see Corollary IV.2.6 in [6]) that for any two orthogonal primitive idempotents c_1 and c_2, the dimension of the (nonzero) space

$$V\left(c_1, \frac{1}{2}\right) \cap V\left(c_2, \frac{1}{2}\right)$$

is independent of $\{c_1, c_2\}$. This common dimension will be called the *index* of V and is denoted by d. Thus

$$d := \dim V\left(c_1, \frac{1}{2}\right) \cap V\left(c_2, \frac{1}{2}\right).$$

We remark that the index of $\mathscr{S}^n$ is one and index of $\mathscr{L}^n$ $(n > 2)$ is $n - 2$.

Lyapunov transformation and quadratic representations

For a given $a \in V$, we define the corresponding *Lyapunov transformation* $L_a : V \to V$ by

$$L_a(x) = a \circ x$$

and the *quadratic representation* P_a by

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

We say that elements a and b *operator commute* if L_a and L_b commute, i.e.,

$$L_a L_b = L_b L_a.$$

It is known that a and b operator commute if and only if a and b have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2, [6] or Theorem 27, [21]).

We recall the following from Gowda, Sznajder and Tao [9]:

Proposition 4. *For $x, y \in V$, the following conditions are equivalent:*

(i) $x \geqslant 0, y \geqslant 0$, *and* $\langle x, y \rangle = 0$.

(ii) $x \geqslant 0, y \geqslant 0$, *and* $x \circ y = 0$.

In each case, elements x and y operator commute.

An easy consequence is the following.

Proposition 5. *Let the Peirce decomposition of an element x with respect to a Jordan frame* $\{e_1, e_2, \ldots, e_r\}$ *be given by*

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}.$$

If $x \geqslant 0$ *and* $x_i = 0$ *for some i, then* $x_{li} = x_{ij} = 0$ *for all l, j with* $l < i$ *and* $i < j$.

This can be easily seen by noting that when $x \geqslant 0, 0 = \langle x, e_i \rangle \Rightarrow x \circ e_i = 0$ and $0 = x \circ e_i = x_i e_i + \frac{1}{2}\left(\sum_{l<i} x_{li} + \sum_{i<j} x_{ij}\right)$ implies $x_{ij} = 0$ for all j due to the orthogonality of the spaces V_{ij}.

The following result describes the effect of L_a and P_a on any element x.

Proposition 6. *Suppose that* $\{e_1, e_2, \ldots, e_r\}$ *is a Jordan frame,*

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r$$

and

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

(*with* $x_i \in R$ *and* $x_{ij} \in V_{ij}$) *be the Peirce decomposition of x with respect to this Jordan frame. Then*

$$L_a(x) = \sum_{i=1}^{r} a_i x_i e_i + \sum_{i<j} \frac{a_i + a_j}{2} x_{ij}$$

and

$$P_a(x) = \sum_{i=1}^{r} {a_i}^2 x_i e_i + \sum_{i<j} a_i a_j x_{ij}.$$

Thus, when $\sigma(a) = \{a_1, a_2, \ldots, a_r\}$, *we have*

$$\sigma(L_a) \subseteq \left\{ \frac{a_i + a_j}{2} : i, j = 1, 2, \ldots, r \text{ and } i \leqslant j \right\}$$

and

$$\sigma(P_a) \subseteq \{a_i a_j : i, j = 1, 2, \ldots, r \text{ and } i \leqslant j\}$$

with equality when V is simple.

Proof. From Theorem 2, we have $a \circ e_i = a_i e_i$ and $a \circ x_{ij} = \frac{a_i + a_j}{2} x_{ij}$. From these, the stated expressions for L_a and P_a follow. The spectrum containments also follow. Finally, when V is simple, V_{ij} (in Theorem 2) is nonzero for each $i \leqslant j$ (see Corollary IV.2.4 in [6]) and hence equality holds in the spectrum inclusions. □

We list below some properties of P_a. These can be found in [6].

Proposition 7. *The following statements hold*:

(1) P_a *is a self-adjoint linear transformation*.
(2) $P_a(K) \subseteq K$ *for all* $a \in V$, *with equality if a is invertible*.
(3) P_a *is invertible if and only if a is invertible. In this case*, $(P_a)^{-1} = P_{a^{-1}}$.
(4) $P_{P_a(x)} = P_a P_x P_a$. *In particular, if a and x are invertible, then so is* $P_a(x)$.

Algebra and cone automorphisms

An invertible linear transformation Λ on V is an algebra automorphism if

$$\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y) \quad \forall x, y \in V.$$

Since such an automorphism takes a Jordan frame to a Jordan frame, we see that inertia remains invariant under an algebra automorphism. It is known [6, Theorem IV.2.5] that in a simple algebra, any two Jordan frames can be mapped onto each other by an algebra automorphism.

A linear transformation $\Gamma : V \to V$ is called a cone automorphism if $\Gamma(K) = K$. We denote by $\mathrm{Aut}(K)$ the set of all cone automorphisms of V. It is known that in a simple algebra, any cone automorphism Γ can be written as

$$\Gamma = \Lambda P_a$$

(or equivalently, as $\Gamma = P_a \Lambda$) where $a > 0$ and Λ is an algebra automorphism, see page 56 in [6].

If Γ is a cone automorphism on $\mathscr{H}^n$, then there exists an invertible matrix C with $\Gamma(X) = CXC^*$ for all $X \in \mathscr{H}^n$; conversely, each invertible matrix C induces a cone automorphism of $\mathscr{H}^n$ that takes any X to CXC^* (see [22]).

3. Min–max theorem of Hirzebruch and its consequences

In this section, we state the min–max theorem of Hirzebruch and derive some consequences. These results are needed for our subsequent analysis and are also of independent interest.

In a simple Euclidean Jordan algebra V, let

$$\mathscr{J}(V) := \{c : c \text{ is a primitive idempotent in V}\}.$$

Then $\mathscr{J}(V)$ is a compact set in V (see Exercise 5, p. 78 in [6]). If $\langle\cdot,\cdot\rangle$ is the inner product in V, then there exists a positive number α such that $\langle x, y\rangle = \alpha \,\mathrm{trace}(x \circ y)$ (see [6, Prop. III.4.1]. In particular, we have

$$\alpha = \langle c, e\rangle = \|c\|^2 \quad \forall c \in \mathscr{J}(V).$$

In what follows, we use the following notation: Given any set of numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, we rearrange the objects and write the set as

$$\{\lambda_1^{\downarrow}, \lambda_2^{\downarrow}, \ldots, \lambda_k^{\downarrow}\}$$

where $\lambda_1^{\downarrow} \geqslant \lambda_2^{\downarrow} \geqslant \cdots \geqslant \lambda_k^{\downarrow}$. Thus for any $x \in V$, $\lambda_i^{\downarrow}(x) (i = 1, 2, \ldots, r)$ denote the eigenvalues of x written in the decreasing order.

We also write

$$\lambda^{\downarrow}(x) := (\lambda_1^{\downarrow}(x), \lambda_2^{\downarrow}(x), \ldots, \lambda_r^{\downarrow}(x)).$$

Theorem 8 (Min–max theorem of Hirzebruch [11]).
Let V be simple. Then for any $x \in V$ *we have*

$$\lambda_1^{\downarrow}(x) = \max_{c \in \mathscr{J}(V)} \frac{\langle x, c\rangle}{\langle e, c\rangle},$$

$$\lambda_r^{\downarrow}(x) = \min_{c \in \mathscr{J}(V)} \frac{\langle x, c\rangle}{\langle e, c\rangle},$$

and

$$\lambda^{\downarrow}_{k+1}(x) = \min_{\{f_1,f_2,\ldots,f_k\}\subset \mathscr{J}(V)} \max_{c\in\mathscr{J}(V), c\perp\{f_1,f_2,\ldots,f_k\}} \frac{\langle x,c\rangle}{\langle e,c\rangle}$$

for $k = 1, \ldots, r-2$.

This min–max theorem immediately yields the following result on the continuity of eigenvalues.

Theorem 9. *Let V be a Euclidean Jordan algebra. Then the following statements hold*:

(1) *When V is simple, for any two elements $a, b \in V$ we have*

$$\|\lambda^{\downarrow}(a) - \lambda^{\downarrow}(b)\|_\infty := \max_{1\leqslant i\leqslant r} |\lambda^{\downarrow}_i(a) - \lambda^{\downarrow}_i(b)| \leqslant \frac{1}{\|c\|}\|a-b\|,$$

where c is any primitive idempotent in V.

(2) *When V is not simple, V can be written as a product of simple algebras*: $V = V_1 \times \cdots \times V_N$. *In this case, there is a positive number Δ such that for all $a = (a_1, a_2, \ldots, a_N)$ and $b = (b_1, b_2, \ldots, b_N)$ in $V = V_1 \times V_2 \times \cdots \times V_N$,*

$$\|\lambda^{\downarrow}(a_l) - \lambda^{\downarrow}(b_l)\|_\infty \leqslant \Delta\|a-b\| \quad (l = 1, 2, \ldots, N).$$

Proof. Item (1) follows from the Hirzebruch theorem and the inequality

$$\frac{\langle a,c\rangle}{\langle e,c\rangle} \leqslant \frac{\langle b,c\rangle}{\langle e,c\rangle} + \frac{\|a-b\|}{\|c\|}$$

with the observation that $\|c\|$ is a constant on $\mathscr{J}(V)$. For Item (2), we decompose (the general) V into simple algebras (see Theorem 3) as $V = V_1 \times V_2 \times \cdots \times V_N$ and apply Item (1). □

Remarks. We note that the inequality in item (1) above is analogous to the Weyl's perturbation inequality in matrix theory [2]. Baes [1, Corollary 24] based on a generalization of von Neumann's trace inequality, proves a stronger Hoffman–Wielandt type inequality:

$$\|\lambda^{\downarrow}(a) - \lambda^{\downarrow}(b)\|_2 \leqslant \frac{1}{\|c\|}\|a-b\|.$$

As a consequence of the above theorem, we prove the continuity and invariance of Inertia.

Theorem 10. *Let V be a Euclidean Jordan algebra with* $\mathrm{In}(x)$ *denoting the inertia of an element x in V. Let* $\mathrm{Inv}(V)$ *denote the set of all invertible elements in V. Then the following hold*:

(a) *On* $\mathrm{Inv}(V)$, *inertia (as a function from* $\mathrm{Inv}(V) \to \{0, 1, 2, \ldots\}^3$) *is continuous. In fact, if* $a \in \mathrm{Inv}(V)$, *then for any x in some neighborhood of a, we have* $\mathrm{In}(x) = \mathrm{In}(a)$.

(b) *On each (connected) component of* $\mathrm{Inv}(V)$, *inertia is a constant.*

(c) *If* $H(t) : [0, 1] \to V$ *is continuous with $H(t)$ invertible for all t, then* $\mathrm{In}(H(0)) = \mathrm{In}(H(1)) = \mathrm{In}(H(t))$.

(d) *π and ν are lower semicontinuous on V; In fact, if $a \in V$, then for any x in some neighborhood of a, we have*

$$\pi(a) \leqslant \pi(x) \quad \text{and} \quad \nu(a) \leqslant \nu(x).$$

Proof. (a) Fix $a \in \mathrm{Inv}(V)$. Then $\mathrm{In}(a) = (\pi(a), \nu(a), 0)$. By Item (2) in Theorem 9, for all x near a, $\pi(x) = \pi(a)$ and $\nu(x) = \nu(a)$; thus, $\mathrm{In}(x) = \mathrm{In}(a)$ for all x near a. This gives the continuity of inertia at a.

(b) Let Ω be a connected component of $\mathrm{Inv}(V)$. (Note that $\mathrm{Inv}(V)$ is an open set in V.) As $\pi : \mathrm{Inv}(V) \to \{0, 1, 2, \ldots\}$ is continuous, $\pi(\Omega)$ is connected and hence is a singleton set. Similarly, $\nu(\Omega)$ is a singleton set. Item (b) follows.

(c) As the image of H is contained in a connected component of $\mathrm{Inv}(V)$, the result follows from Item (b).

(d) The inequalities in (d) follow immediately from Item (2) in the previous theorem. The lower semicontinuity properties follow since the sets $\{x : \pi(x) > \alpha\}$ and $\{x : \nu(x) > \alpha\}$ are open for any real number α. □

Remarks. Although we have stated the above theorem in the setting of Euclidean Jordan algebras, the results are valid (with minor modifications) in some other settings also. For example, we could replace V in the above theorem by $C^{n\times n}$ (the space of all $n \times n$ complex matrices) with $\mathrm{In}(A)$ denoting the inertia of A. In this setting, the continuity of inertia on invertible elements of $C^{n\times n}$ follows from the continuity of eigenvalues (see [2, Theorem VI.1.2]) or continuity of roots of a complex polynomial [27].

4. Inertia results for quadratic representations

We come to the main inertia result which generalizes the theorems of Sylvester [12] and Ostrowski [18]. As noted previously, the invariance of inertia was first established by Kaneyuki [14] by Lie algebraic means.

Theorem 11. *Let V be any Euclidean Jordan algebra and Λ be an algebra automorphism of V. Then for any invertible $a \in V$ we have,*

$$\mathrm{In}(\Lambda P_a(x)) = \mathrm{In}(x) \quad \text{for all } x \in V.$$

More generally, for all $a, x \in V$,

$$\pi(\Lambda P_a(x)) \leqslant \pi(x) \quad \text{and} \quad \nu(\Lambda P_a(x)) \leqslant \nu(x).$$

Proof. As Λ is an algebra automorphism of V, it maps a Jordan frame onto a Jordan frame; thus, inertia, π, and ν are invariant under Λ. Hence it is enough to prove the above statements with $\Lambda = I$ (identity). We first assume that *V is simple and a is invertible*. We show that $\mathrm{In}(\Lambda P_a(x)) = \mathrm{In}(x)$. Now, $P_a(K) = K$ and hence we can write P_a as

$$P_a = \Lambda_1 P_d,$$

where Λ_1 is an algebra automorphism of V and $d > 0$. Once again, as inertia is invariant under Λ_1, it is enough to show that

$$\mathrm{In}(P_d(x)) = \mathrm{In}(x)$$

for all $x \in V$.

Case 1: Let x be invertible. Then (by Prop. 7) for all $t \in [0, 1]$,

$$H(t) := P_{(1-t)d+te}(x)$$

is invertible. Hence by Theorem 10(c), $\mathrm{In}(P_d(x)) = \mathrm{In}(x)$.

Case 2: Let x be non-invertible. In this case, x will have zero eigenvalues. Then, by the spectral decomposition theorem, for all small positive $\varepsilon, \pi(x - \varepsilon e) = \pi(x)$ and $\nu(x + \varepsilon e) = \nu(x)$. As $x + \varepsilon e$ and $x - \varepsilon e$ are invertible for all small ε, by Theorem 10(d) and Case 1, we have

$$\pi(P_d(x)) \leqslant \pi(P_d(x - \varepsilon e)) = \pi(x - \varepsilon e) = \pi(x)$$

and

$$\nu(P_d(x)) \leqslant \nu(P_d(x + \varepsilon e)) = \nu(x + \varepsilon e) = \nu(x).$$

Now applying these inequalities to d^{-1}, we get

$$\pi(x) = \pi(P_{d^{-1}}(P_d(x))) \leqslant \pi(P_d(x))$$

and

$$\nu(x) = \nu(P_{d^{-1}}(P_d(x))) \leqslant \nu(P_d(x)).$$

Thus we have $\pi(P_d(x)) = \pi(x)$ and $\nu(P_d(x)) = \nu(x)$. This proves that $\text{In}(P_a(x)) = \text{In}(P_d(x)) = \text{In}(x)$.

Now let V be *any Euclidean Jordan algebra and a be invertible in V*. Then we can decompose V into a direct sum of simple algebras $V_i : V = V_1 \oplus V_2 \oplus \cdots \oplus V_N$. Writing $a = (a_1, a_2, \ldots, a_N), P_a = (P_{a_1}, \ldots, P_{a_N})$ and $x = (x_1, x_2, \ldots, x_N)$ we see that each a_i is invertible in V_i. From the previous case, $\text{In}(P_{a_i}(x_i)) = \text{In}(x_i)$ for all i.

Since $\text{In}(x) = \left(\sum_i^N \pi(x_i), \sum_i^N \nu(x_i), \sum_i^N \delta(x_i)\right)$, we get $\text{In}(P_a(x)) = \text{In}(x)$. This concludes the proof when a is invertible.

Now suppose a is not invertible and let

$$a = a_1e_1 + a_2e_2 + \cdots + a_ke_k + 0e_{k+1} + \cdots + 0e_r$$

be the spectral decomposition of a where $a_i \neq 0$ for $1 \leqslant i \leqslant k < r$. For $t > 0$, define

$$b := a_1e_1 + a_2e_2 + \cdots + a_ke_k + t(e_{k+1} + \cdots + e_r).$$

Then, b is invertible and

$$\text{In}(P_b(x)) = \text{In}(x)$$

for all x. For any given x, pick a small t so that $P_b(x)$ is close to $P_a(x)$ and (hence) by Theorem 10(d),

$$\pi(P_a(x)) \leqslant \pi(P_b(x)).$$

Thus we have $\pi(P_a(x)) \leqslant \pi(x)$. Now by working with $-x$, we get $\nu(P_a(x)) \leqslant \nu(x)$.

At this stage, we have proved that for any general V and $a \in V$, $\pi(P_a(x)) \leqslant \pi(x)$ and $\nu(P_a(x)) \leqslant \nu(x)$ for all x with $\text{In}(P_a(x)) = \text{In}(x)$ when a is invertible. This completes the proof. $\square$

Remarks. For an algebra automorphism Λ and $a \in V$, we note that

$$P_a(\Lambda(x)) = \Lambda P_{\Lambda^{-1}(a)}(x)$$

and so $\pi(P_a(\Lambda(x))) = \pi(P_{\Lambda^{-1}(a)}(x)) \leqslant \pi(x)$. Similarly, $\nu(P_a(\Lambda(x))) = \nu(P_{\Lambda^{-1}(a)}(x)) \leqslant \nu(x)$. In addition, $\text{In}(P_a(\Lambda(x))) = \text{In}(x)$ when a is invertible. It should be remarked that when a is not invertible, $\pi(\Lambda P_a(x))$ may be different from $\pi(P_a\Lambda(x))$. This can be seen as follows.

Let $V = \mathscr{S}^2, A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $P_A(X) = AXA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus $\pi(P_A(X)) = 1$. Take $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $B = \Lambda^{-1}(A) = UAU^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Hence $P_B(X) = BXB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\pi(P_B(X)) = 0$. Therefore $\pi(P_A(X)) \neq \pi(P_B(X))$.

Corollary 12. *Suppose V is simple and Γ is a cone automorphism of V. Then for any $x \in V$,*

$$\text{In}(\Gamma(x)) = \text{In}(x).$$

Proof. For any cone automorphism Γ, we can write $\Gamma = P_a\Lambda$ for some $a > 0$ and an algebra automorphism Λ. In this case, the result follows from the above theorem. $\square$

We now describe a converse of Theorem 11 which can also be found in [14].

Theorem 13. *Suppose that V is simple and for two elements x and y, $\text{In}(x) = \text{In}(y)$. Then there is an invertible a in V and an algebra automorphism Λ of V such that $P_a\Lambda(x) = y$.*

Proof. We write the spectral decompositions

$$x = (\lambda_1e_1 + \cdots + \lambda_ke_k) + (0e_{k+1} + \cdots + 0e_l) + (\lambda_{l+1}e_{l+1} + \cdots + \lambda_re_r)$$

and

$$y = (\mu_1f_1 + \cdots + \mu_kf_k) + (0f_{k+1} + \cdots + 0f_l) + (\mu_{l+1}f_{l+1} + \cdots + \mu_rf_r),$$

where λ_i and μ_i are positive for $i = 1, 2, \ldots, k$ and negative for $i = l + 1, \ldots, r$. Since V is simple, there is an algebra automorphism Λ of V such that $\Lambda(e_i) = f_i$ for all i. Then

$$z := \Lambda(x) = (\lambda_1 f_1 + \cdots + \lambda_k f_k) + (0f_{k+1} + \cdots + 0f_l) + (\lambda_{l+1} f_{l+1} + \cdots + \lambda_r f_r).$$

Let $a = a_1 f_1 + a_2 f_2 + \cdots + a_r f_r$ with $a_i = \sqrt{\frac{\mu_i}{\lambda_i}}$ for $i \in \{1, 2, \ldots, k\} \cup \{l + 1, \ldots, r\}$ and $a_i = 1$ for $i \in \{k + 1, k + 2, \ldots, l\}$. Then a is invertible, and it is easily seen that

$$P_a(z) = y.$$

Thus $P_a \Lambda(x) = y$, proving the result. □

Suppose that x and y are elements in a Euclidean Jordan algebra such that $P_a \Lambda(x) = y$ for some algebra automorphism Λ of V and invertible $a \in V$. Then

$$\mathrm{In}(P_x) = \mathrm{In}(P_y).$$

This can be seen as follows: Let $z := \Lambda(x)$ so that $P_a(z) = y$. Then by Proposition 7, $P_a P_z P_a = P_y$. As P_a is invertible, by Sylvester's law of inertia (applicable to self-adjoint transformation P_z and invertible transformation P_a), $\mathrm{In}(P_z) = \mathrm{In}(P_y)$. Now $P_z(u) = 2z \circ (z \circ u) - z^2 \circ u = \Lambda P_x \Lambda^{-1}(u)$ for all $u \in V$ implies that $\mathrm{In}(P_z) = \mathrm{In}(\Lambda P_x \Lambda^{-1}) = \mathrm{In}(P_x)$. Thus, $\mathrm{In}(P_x) = \mathrm{In}(P_y)$.

The following result establishes a relation between the inertias of an object and its corresponding quadratic representation. Recall that d denotes the index of V.

Proposition 14. *Let V be simple. Then the following statements hold:*

(i) *If* $\mathrm{In}(a) = (k, l, r - k - l)$, *then*

$$\mathrm{In}(P_a) = (\bar{k} + \bar{l}, kld, \bar{r} - \bar{k} - \bar{l} - kld)$$

where $\bar{k} = k + \frac{dk(k-1)}{2}$, $\bar{l} = l + \frac{dl(l-1)}{2}$, *and* $\bar{r} = r + \frac{dr(r-1)}{2}$.

(ii) $\mathrm{In}(x) = \mathrm{In}(\pm y)$ *if and only if* $\mathrm{In}(P_x) = \mathrm{In}(P_y)$.

Proof. Item (i) follows from the expression for $P_a(x)$ in Proposition 6 and the fact that on V_{ij}, P_a has the eigenvalue $a_i a_j$ that is repeated $d (= \dim V_{ij})$ times.

(ii) Suppose $\mathrm{In}(x) = \mathrm{In}(y) = (k, l, r - k - l)$. Then Item (i) gives the equality of inertias of P_x and P_y. When $\mathrm{In}(x) = \mathrm{In}(-y)$, we have $\mathrm{In}(P_x) = \mathrm{In}(P_{-y}) = \mathrm{In}(P_y)$ as $P_y = P_{-y}$.

Now suppose $\mathrm{In}(P_x) = \mathrm{In}(P_y)$. Let $\mathrm{In}(x) = (k, l, r - k - l)$ and $\mathrm{In}(y) = (k', l', r - k' - l')$. Then, using Item (i) for x and y,

$$k + \frac{dk(k-1)}{2} + l + \frac{dl(l-1)}{2} = k' + \frac{dk'(k'-1)}{2} + l' + \frac{dl'(l'-1)}{2}$$

and

$$kld = k'l'd.$$

This leads to

$$\begin{aligned} &k(k+1) + (d-1)k(k-1) + l(l+1) + (d-1)l(l-1) \\ &\quad = k'(k'+1) + (d-1)k'(k'-1) + l'(l'+1) + (d-1)l'(l'-1) \end{aligned}$$

and $kl = k'l'$. From these we get

$$\begin{aligned} &(k+l)^2 + (k+l) + (d-1)[(k+l)^2 - (k+l)] \\ &\quad = (k'+l')^2 + (k'+l') + (d-1)[(k'+l')^2 - (k'+l')]. \end{aligned}$$

Further simplification leads to

$$[k+l-(k'+l')][d(k+l+k'+l'-1)+2]=0.$$

If both x and y are zero, then $\mathrm{In}(x)=\mathrm{In}(\pm y)$. When one of them is nonzero, $(k+l+k'+l'-1)\geqslant 0$ and hence $k+l=k'+l'$. Now, $k+l=k'+l'$ and $kl=k'l'$ imply $(k-l)^2=(k'-l')^2$ and $k-l=\pm(k'-l')$. It follows that either $k=k'$ and $l=l'$ or $k=l'$ and $l=k'$. Hence $\mathrm{In}(x)=\mathrm{In}(\pm y)$. $\square$

The following example shows that Theorem 13 need not hold for a non-simple algebra.

Example 1. In $\mathscr{L}^n$ $(n>2)$, take objects x_1, x_2, y_1, and y_2 with

$$\mathrm{In}(x_1)=(1,1,0),\quad \mathrm{In}(x_2)=(1,0,1),\quad \mathrm{In}(y_1)=(2,0,0)\quad\text{and}\quad \mathrm{In}(y_2)=(0,1,1).$$

Define $V:=\mathscr{L}^n\oplus\mathscr{L}^n, x=(x_1,x_2), y=(y_1,y_2)$. Then $P_x=(P_{x_1},P_{x_2})$ and $P_y=(P_{y_1},P_{y_2})$. From Section 2,

$$\mathrm{In}(x)=(\pi(x_1)+\pi(x_2),\nu(x_1)+\nu(x_2),\delta(x_1)+\delta(x_2))$$

and

$$\mathrm{In}(P_x)=(\pi(P_{x_1})+\pi(P_{x_2}),\nu(P_{x_1})+\nu(P_{x_2}),\delta(P_{x_1})+\delta(P_{x_2})).$$

As the index of $\mathscr{L}^n$ is $n-2$, we have $\mathrm{In}(x)=(2,1,1)=\mathrm{In}(y), \mathrm{In}(P_{x_1})=(2,n-2,*), \mathrm{In}(P_{x_2})=(1,0,*)$, $\mathrm{In}(P_{y_1})=(n,0,*)$, and $\mathrm{In}(P_{y_2})=(1,0,*)$. Thus we have $\mathrm{In}(P_x)=(3,n-2,*)$ and $\mathrm{In}(P_y)=(n+1,0,*)$, hence $\mathrm{In}(P_x)\neq\mathrm{In}(P_y)$.

5. Cone spectrum results

As an application of the results of the previous section, we consider the finiteness of cone spectrum of quadratic representations.

In the definition below, H denotes a finite dimensional real Hilbert space and C denotes a closed convex cone in H with dual C^*.

Definition 15 [23]. Given a linear transformation $L: H\to H$, the *cone spectrum* of L with respect to C is the set of all $\lambda\in R$ for which there exists $x\in H$ such that

$$0\neq x\in C,\quad L(x)-\lambda x\in C^*,\quad\text{and}\quad \langle x, L(x)-\lambda x\rangle=0.$$

We denote the cone spectrum of L *with respect to* C by $\sigma(L,C)$ and its cardinality by $|\sigma(L,C)|$.

In [24], Seeger and Torki discuss the nonemptyness, continuity, and finiteness of the cone spectrum. In particular, they show that the cone spectrum is nonempty for any proper cone C (that is, when $C\cap(-C)=\{0\}$) and that it is finite for any polyhedral cone. They also show that every symmetric linear transformation on the Lorentz cone (this is the symmetric cone of $\mathscr{L}^n$) has finite cone spectrum. Motivated by this, they raise the problem of describing cones for which every symmetric transformation has finite cone spectrum. In a recent paper, Iusem and Seeger [13] construct an example of a symmetric transformation on a cone which has infinite cone spectrum. Influenced by these, Zhou and Gowda [29] studied this finite cone spectrum problem for some special linear transformations on symmetric cones. They showed that for **Z**-transformations (see Section 6 for the definition) the cone spectrum with respect to the symmetric cone is always finite. In addition, by studying quadratic representations on each of the simple algebras and then by using the structure theorem they showed that on any Euclidean Jordan algebra, quadratic representations have finite cone spectrum. In what follows, we present a proof of this result that avoids case by case analysis and that allows a generalization to transformations which are products of algebra automorphisms and quadratic representations.

In what follows, for any linear transformation $L: V\to V, \sigma(L)$ denotes the spectrum of L.

Theorem 16. *Let V be any Euclidean Jordan algebra. Then for any algebra automorphism Λ of V and any $a\in V$, we have*

$$\sigma(\Lambda P_a, K) \subseteq \sigma(\Lambda P_a) \cup \{0\}.$$

Hence $|\sigma(\Lambda P_a, K)| < \infty$. *In particular, on any simple algebra, every cone automorphism has finite cone spectrum.*

Note: A similar result holds for $P_a \Lambda$.

Proof. Let Λ be an algebra automorphism of V and $a \in V$. Without loss of generality, let $a \neq 0$ and $0 \neq \lambda \in \sigma(\Lambda P_a, K)$. Then there exists a nonzero x in K, y in $K^* = K$ with $\Lambda P_a(x) = \lambda x + y$ and $\langle x, y \rangle = 0$. Note that $\Lambda P_a(x) \geqslant 0$ and so λ is positive. In view of Prop. 4, there exists Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_k e_k \quad \text{and} \quad y = y_{k+1} e_{k+1} + \cdots + y_r e_r,$$

where each x_i is positive and each y_i is nonnegative. Therefore,

$$\pi(\lambda x + y) = \pi(x) + \pi(y) = k + \pi(y).$$

From Theorem 11,

$$\pi(\Lambda P_a(x)) \leqslant \pi(x) = k.$$

As $\Lambda P_a(x) = \lambda x + y$, we see that $k \geqslant k + \pi(y)$. This yields $\pi(y) = 0$. Since $y \geqslant 0$, we must have $y = 0$ and $\Lambda P_a(x) = \lambda x$. The inclusion relation between the cone spectrum and the spectrum follows. The finiteness of the cone spectrum is obvious. Finally, when V is simple, every cone automorphism can be written as a product of a quadratic representation and an algebra automorphism. The finiteness of cone spectrum for such an automorphism follows. □

6. Ostrowski–Schneider type inertia theorems

In this section, we extend Ostrowski–Schneider type inertia results mentioned in Section 1 to certain linear transformations on Euclidean Jordan algebras. Throughout this section, we assume that V denotes a general Euclidean Jordan algebra.

First, we provide an alternate description of the existence of element $\bar{x}$ in V such that $L(\bar{x}) > 0$.

Proposition 17. *Let* $L : V \to V$ *be linear. Then the following statements are equivalent*:

(i) *There exists an* $\bar{x}$ *in* V *such that* $L(\bar{x}) > 0$.
(ii) *The implication* $[u \geqslant 0, L^T(u) = 0] \Rightarrow u = 0$ *holds*.

Proof. Suppose there exist $\bar{x}$ in V such that $L(\bar{x}) > 0$ and a nonzero $u \geqslant 0$ with $L^T(u) = 0$. Then

$$0 < \langle L(\bar{x}), u \rangle = \langle \bar{x}, L^T(u) \rangle = 0$$

gives a contradiction. Hence (i) $\Rightarrow$ (ii). Now suppose that (i) does not hold. Then the convex sets $L(V)$ and K^o are disjoint. By a separation theorem (see e.g., Theorem 3.4 in [20]), there exists a nonzero u and a number α such that

$$\langle L(v), u \rangle \leqslant \alpha \leqslant \langle z, u \rangle$$

for all $v \in V$ and $z \in K^o$. This leads to $u \geqslant 0$ and $L^T(u) = 0$. Thus negation of (i) implies the negation of (ii). This proves (ii) $\Rightarrow$ (i). □

We now state our inertia theorem.

Theorem 18. *Suppose* $L : V \to V$ *is linear with the following properties*:

(1) *There exists an* $\bar{x}$ *in* V *such that* $L(\bar{x}) > 0$.

(2) *Every x with* $L(x) > 0$ *is invertible*.

Then for any z with $L(z) \geqslant 0$, *we have*

$$\pi(z) \leqslant \pi(\bar{x}) \quad \text{and} \quad \nu(z) \leqslant \nu(\bar{x}).$$

In particular, for any invertible z with $L(z) \geqslant 0$, *we have* $\text{In}(z) = \text{In}(\bar{x})$; *Also, for any y with* $L(y) > 0$, *we have* $\text{In}(y) = \text{In}(\bar{x})$.

Proof. Let $L(z) \geqslant 0$. As $L(\bar{x}) > 0$, we see that $L((1-t)z + t\bar{x}) > 0$ for all $0 < t \leqslant 1$. By Property (2), $(1-t)z + t\bar{x}$ is invertible for all $0 < t \leqslant 1$. For all such t, by Theorem 10(c), $\text{In}(\bar{x}) = \text{In}((1-t)z + t\bar{x})$. For t close to zero, $(1-t)z + t\bar{x}$ is close to z, and by Theorem 10(d), $\pi(z) \leqslant \pi((1-t)z + t\bar{x}) = \pi(\bar{x})$. A similar inequality holds for $\nu(z)$. Now when z is invertible,

$$r = \pi(z) + \nu(z) \leqslant \pi(\bar{x}) + \nu(\bar{x}) \leqslant r$$

implies that $\pi(z) = \pi(\bar{x})$ and $\nu(z) = \nu(\bar{x})$. This proves that z and $\bar{x}$ have the same inertia. Finally, when $L(y) > 0, y$ will be invertible (by our assumption) and so $\text{In}(y) = \text{In}(\bar{x})$. □

In what follows, we say that a linear transformation $Q : V \to V$ is *Lyapunov-like* [8] if

$$x, y \in K, \langle x, y\rangle = 0 \Rightarrow \langle Q(x), y\rangle = 0.$$

In view of the spectral decomposition theorem, the above condition can be stated as: for any Jordan frame $\{e_1, e_2 \ldots, e_r\}$, the equality $\langle Q(e_i), e_j\rangle = 0$ holds for all $i \neq j$. Examples of such transformations include

(a) L_A on $\mathcal{S}^n$ for any $A \in R^{n\times n}$ (see Section 1),
(b) L_A on $\mathcal{H}^n$ for any $A \in C^{n\times n}$,
(c) L_a on any Euclidean Jordan algebra V with $a \in V$, and
(d) any matrix of the form $Q = \begin{bmatrix} a & b^T \\ b & D \end{bmatrix}$ with $D + D^T = 2aI$ on $\mathcal{L}^n$.

Remarks. Recently, Damm [4] has proved that every Lyapunov-like transformation on $\mathcal{S}^n$ or $\mathcal{H}^n$ is of the form L_A for some square matrix A. Tao [26] has shown that on $\mathcal{L}^n$, matrices given in (d) are the only Lyapunov-like transformations. However, the form of a Lyapunov-like transformation on a general Euclidean Jordan algebra is not known.

Theorem 19. *Suppose* $L : V \to V$ *is linear with the following properties*:

(a) *There exists* $\bar{x}$ *in V such that* $L(\bar{x}) > 0$;
(b) $L = PQ$ *where* $P : V \to V$ *is an invertible linear transformation with* $P^{-1}(K^o) \subseteq K^o$ *and* $Q : V \to V$ *is Lyapunov-like*.

Then the conclusions of the previous theorem hold.

Proof. In order to apply the previous theorem, we need only to show that Item (b) implies condition (2) of the theorem. To this end, let $L(x) > 0$. Then $PQ(x) > 0$. By the imposed condition on P, we have $Q(x) > 0$. Assume, if possible, x is not invertible so that the spectral decomposition of x is given by

$$x = 0e_1 + x_2e_2 + \cdots + x_re_r.$$

Then $Q(x) = 0Q(e_1) + x_2Q(e_2) + \cdots + x_rQ(e_r) > 0$. We have

$$0 < \langle Q(x), e_1\rangle = 0\langle Q(e_1), e_1\rangle + x_2\langle Q(e_2), e_1\rangle + \cdots + x_r\langle Q(e_r), e_1\rangle = 0$$

where the last equality comes from the defining property of the Lyapunov-like transformation Q. This contradiction proves that x is invertible. The conclusion follows. □

Remarks. Lyapunov-like transformations are special cases of **Z**-transformations defined by

$$x, y \in K, \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leqslant 0.$$

The negative of a **Z**-transformation is known in the literature as a cross-positive transformation. Examples of **Z**-transformations include **Z**-matrices, Lyapunov and Stein transformations L_A and S_A on $\mathscr{S}^n$ (or $\mathscr{H}^n$), the transformations L_a and S_a on any Euclidean Jordan algebra, see [10] for further details. The result below generalizes the theorems of Lyapunov and Stein to **Z**-transformations:

Proposition 20 [10]. *The following are equivalent for a* **Z**-*transformation on any Euclidean Jordan algebra*:

(i) *There exists an* $\bar{x} > 0$ *with* $L(\bar{x}) > 0$.
(ii) L^{-1} *exists and* $L^{-1}(K) \subseteq K$.
(iii) L *is positive stable*.

Motivated by this result, we may ask if a Ostrowski–Schneider type inertia result holds for **Z**-transformations. The following example shows that this is not the case.

Example 2. On the Euclidean Jordan algebra R^2, let

$$L = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

be an invertible Z-matrix.

For this matrix, $L(\bar{x}) > 0$ and $L(z) > 0$ where

$$\bar{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

yet $\text{In}(z) \neq \text{In}(\bar{x})$.

We now show that the Ostrowski–Schneider type results mentioned earlier are special cases of Theorem 19. In addition, we state two new results in general Euclidean Jordan algebras.

Corollary 21 [3]. *Let* $A \in C^{n\times n}$ *and* $L_A(X) = \frac{1}{2}(AX + XA^*)$ *on* $\mathscr{H}^n$. *Then there exists* $\overline{X} \in \mathscr{H}^n$ *such that* $L_A(\overline{X}) \succ 0$ *if and only if* $\delta(A) = 0$. *If* $\delta(A) = 0$, *then for any* Z *with* $L_A(Z) \succeq 0$, *we have* $\pi(Z) \leqslant \pi(\overline{X}) = \pi(A)$ *and* $\nu(Z) \leqslant \nu(\overline{X}) = \nu(A)$.

Proof. The first part of the corollary is well-known. For completeness, we provide a proof in Appendix I.

Now assume that $\delta(A) = 0$ so that there exists an $\overline{X}$ with $L_A(\overline{X}) \succ 0$. Let $L_A(Z) \succeq 0$. We can apply Theorem 19 to the Lypaunov-like transformation L_A in the Euclidean Jordan algebra $\mathscr{H}^n$ and get $\pi(Z) \leqslant \pi(\overline{X})$ etc. To complete the proof, we show that $\text{In}(\overline{X}) = \text{In}(A)$. Firstly, we extend $L_A : C^{n\times n} \to C^{n\times n}$ by

$$L_A(B) = \frac{1}{2}(AB + B^*A^*).$$

Secondly, if $L_A(B) \succ 0$, then AB is positive definite on C^n, and hence B is invertible. Finally, putting $H(t) := t\overline{X} + (1-t)A^{-1} : C^{n\times n} \to C^{n\times n}$ and applying the remark following Theorem 10, we get $\text{In}(\overline{X}) = \text{In}(A^{-1})$. Since $\text{In}(A^{-1}) = \text{In}(A)$, the result follows. □

Corollary 22. *Consider a Euclidean Jordan algebra* V *and let* $a \in V$. *Then we have the following:*

(1) *There exists* $\bar{x} \in V$ *such that* $a \circ \bar{x} > 0$ *if and only if* a (*also* $\bar{x}$) *is invertible* (*that is*, $\delta(a) = 0$).
(2) *Let* a *be invertible. Then for any* z *with* $a \circ z \geqslant 0$, *we have* $\pi(z) \leqslant \pi(a)$ *and* $\nu(z) \leqslant \nu(a)$. *In particular, if* z *is invertible, then* $\text{In}(z) = \text{In}(a)$.

Proof. The Lyapunov transformation L_a, defined by $L_a(x) = a \circ x$ on V, is Lyapunov-like: For any two elements $u, v \geqslant 0$ with $\langle u, v\rangle = 0$, we have $u \circ v = 0$ and so $\langle a \circ u, v\rangle = \langle a, u \circ v\rangle = 0$.

(1) If a is invertible, then $a \circ a^{-1} = e > 0$. Conversely, suppose $a \circ \bar{x} = w > 0$. If a not invertible, then there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that

$$a = 0e_1 + a_2e_2 + \cdots + a_re_r.$$

Then

$$0 < \langle w, e_1\rangle = \langle a \circ \bar{x}, e_1\rangle = \langle \bar{x}, a \circ e_1\rangle = 0$$

is a contradiction. Hence a is invertible. Similarly, $\bar{x}$ is also invertible. Thus we have item (1). Now Item (2) follows from Theorem 19 by taking $P = I$ (identity), $Q = L_a, \bar{x} = a^{-1}$ and noting $\mathrm{In}(a^{-1}) = \mathrm{In}(a)$. □

As an application of the above corollary, we describe the inertia of an element when its Peirce decomposition has a particular form. For the matrix case (in $\mathscr{S}^n$ and $\mathscr{H}^n$), the result was proved by Wimmer [28]. In what follows, for any Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and $1 \leqslant k \leqslant r$, we let

$$V_{e_1+e_2+\cdots+e_k} := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_k) = x\}.$$

It is known [7] that this is a subalgebra of V. Also, for any $y \in V$ with Peirce decomposition $y = \sum_1^r y_ie_i + \sum_{i<j\leqslant r} y_{ij}$, the object $y' := \sum_1^k y_ie_i + \sum_{i<j\leqslant k} y_{ij}$ belongs to $V_{e_1+e_2+\cdots+e_k}$. Furthermore, if $y \geqslant 0$ in V, then $y' \geqslant 0$ in $V_{e_1+e_2+\cdots+e_k}$. Similarly, one can define $V_{e_{k+1}+\cdots+e_r}$ and the corresponding element y''.

Corollary 23. *Corresponding to a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, let the Peirce decomposition of x be given by*

$$x = \sum_1^r x_ie_i + \sum_{i<j\leqslant r} x_{ij}.$$

If $x' := \sum_1^k x_ie_i + \sum_{i<j\leqslant k} x_{ij} > 0$ in $V_{e_1+e_2+\cdots+e_k}$ and $x'' := \sum_{k+1}^r x_ie_i + \sum_{k+1\leqslant i<j} x_{ij} < 0$ in $V_{e_{k+1}+\cdots+e_r}$, then $In(x) = (k, r-k, 0)$.

Proof. Let $a = \sum_1^k e_i - \sum_{k+1}^r e_i$. Then

$$\begin{aligned} a \circ x &= \sum_1^k x_ie_i - \sum_{k+1}^r x_ie_i + \frac{1}{2}\sum_{i<j}(a_i + a_j)x_{ij} \\ &= \left(\sum_1^k x_ie_i + \sum_{i<j\leqslant k} x_{ij}\right) - \left(\sum_{k+1}^r x_ie_i + \sum_{k+1\leqslant i<j} x_{ij}\right) \\ &= x' - x''. \end{aligned}$$

For any nonzero $y \geqslant 0$ consider y' and y'' which are nonnegative in their respective (sub) algebras. Because of Proposition 5, we note that at least one of them is nonzero. Because of the orthogonality of the Peirce spaces, it is easily verified that

$$\langle a \circ x, y\rangle = \langle x' - x'', y' + y''\rangle = \langle x', y'\rangle + \langle -x'', y''\rangle > 0.$$

Since the symmetric cone K in V is self-dual, we see that $a \circ x > 0$. By the previous inertia theorem, $\mathrm{In}(x) = \mathrm{In}(a) = (k, r-k, 0)$. □

Corollary 24. *Let $A \in C^{n\times n}$ and $S_A(X) = X - AXA^*$ on $\mathscr{H}^n$. Then there exists $\overline{X} \in \mathscr{H}^n$ such that $S_A(\overline{X}) \succ 0$ if and only if $\delta_0(A) = 0$. If $\delta_0(A) = 0$, then for any Z with $S_A(Z) \succeq 0$, we have $\pi(Z) \leqslant \pi_0(A)$ and $\nu(Z) \leqslant \nu_0(A)$; In particular, when Z is invertible $\mathrm{In}(Z) = \mathrm{In}_0(A)$.*

Proof. Suppose that $S_A(\overline{X}) \succ 0$ for some $\overline{X} \in \mathscr{H}^n$. If A^* has an eigenvalue λ with $|\lambda| = 1$ and corresponding eigenvector u, then $\langle S_A(\overline{X})u, u\rangle = 0$ gives a contradiction. Hence $\delta_0(A) = \delta_0(A^*) = 0$.

Now suppose that $\delta_0(A) = 0$. Then $I + A$ is invertible. Let

$$B := (I + A)^{-1}(I - A).$$

Then $\text{In}(B) = \text{In}_0(A)$ and

$$\frac{1}{2}(I + A)(BX + XB^*)(I + A^*) = X - AXA^*$$

holds for all $X \in \mathscr{H}^n$. This shows that S_A is of the form PQ where P, defined by $P(Y) := (I + A)Y(I + A^*)$, is invertible with $P^{-1}(\mathscr{H}^n_+) \subseteq \mathscr{H}^n_+$ and $Q(X) := L_B(X)$ is Lyapunov-like. As $\delta(B) = 0$, there is an $\overline{X}$ with $L_B(\overline{X}) \succ 0$. This implies that $S_A(\overline{X}) \succ 0$. Now, if $S_A(Z) \succeq 0$, then $L_B(Z) \geqslant 0$. The result follows by applying Corollary 21 to B. (Note that we can also apply Theorem 19.) $\square$

Now we state an analog of this result in Euclidean Jordan algebras. First we have a definition.

Definition 25. Let $a \in V$. The unit circle inertia $\text{In}_0(a)$ is defined by

$$\text{In}_0(a) = (\pi_0(a), \nu_0(a), \delta_0(a)),$$

where $\pi_0(a)$, $\nu_0(a)$, and $\delta_0(a)$ are, respectively, the number of eigenvalues of a inside, outside, and on the unit circle, counting multiplicities.

Corollary 26. *For an element $a \in V$, we define the Stein transformation S_a by*

$$S_a = I - P_a.$$

Then we have the following statements:

(i) *There exists an $\bar{x}$ such that $S_a(\bar{x}) > 0$ if and only if $\pm 1 \notin \sigma(a)$ (that is, $\delta_0(a) = 0$).*
(ii) *Let $\delta_0(a) = 0$. Then for any z with $S_a(z) \geqslant 0$ we have*

$$\pi(z) \leqslant \pi_0(a) \quad \text{and} \quad \nu(z) \leqslant \nu_0(a).$$

In particular, when z is invertible, we have $\text{In}(z) = \text{In}_0(a)$.

Proof. (i) Suppose that $S_a(\bar{x}) = w > 0$ and $\varepsilon = \pm 1 \in \sigma(a)$. Without loss of generality, let $a = \varepsilon e_1 + \sum_2^r a_i e_i$. Then

$$\begin{aligned} 0 < \langle w, e_1 \rangle &= \langle \bar{x}, e_1 \rangle - \langle P_a(\bar{x}), e_1 \rangle \\ &= \langle \bar{x}, e_1 \rangle - \langle \bar{x}, P_a^T(e_1) \rangle \\ &= \langle \bar{x}, e_1 \rangle - \langle \bar{x}, P_a(e_1) \rangle \\ &= \langle \bar{x}, e_1 \rangle - \langle \bar{x}, e_1 \rangle = 0. \end{aligned}$$

This is a contradiction. Thus $\varepsilon \notin \sigma(a)$.

Conversely, suppose that $\sigma(a)$ does not contain $+1$ and -1. Given $a = \sum_1^r a_i e_i$ (with $a_i \neq 1$ for all i), we choose real numbers $\bar{x}_i$ so that $\bar{x}_i$ and $1 - a_i{}^2$ keep the same sign and define $\bar{x} := \sum_1^r \bar{x}_i e_i$. Then $P_a(\bar{x}) = \sum_1^r a_i{}^2 \bar{x}_i e_i$ and $\bar{x} - P_a(\bar{x}) = \sum_1^r (1 - a_i{}^2)\bar{x}_i e_i > 0$.

(ii) Now let $\pm 1 \notin \sigma(a)$ so that $S_a(\bar{x}) > 0$ for some $\bar{x}$. Then from the Lemma in Appendix II, we have $P_{e+a}(b \circ x) = x - P_a(x) = S_a(x)$ where $b := (e + a)^{-1} \circ (e - a)$ and $\text{In}(b) = \text{In}_0(a)$. As P_{e+a} invertible with $P_{e+a}^{-1}(K^o) \subseteq K^o$, we see that $S_a(\bar{x}) > 0$ implies $b \circ \bar{x} > 0$ and $S_a(z) \geqslant 0$ implies $b \circ z \geqslant 0$. Applying Corollary 22 to b and noting $\text{In}(b) = \text{In}_0(a)$, we get the required conclusions. $\square$

Appendix I

Here we prove the first part of Corollary 21.

Suppose $\delta(A) \neq 0$. Then there exists λ, an eigenvalue of A, such that $\lambda + \bar{\lambda} = 0$. Let $y \neq 0$ in C^n be an eigenvector of A^* corresponding to $\bar{\lambda}$ and let $U := yy^*$. Then

$$L_A^T(U) = L_{A^*}(U) = A^*U + UA = \lambda yy^* + \bar{\lambda}yy^* = 0.$$

As $0 \neq U \succeq 0$, Proposition 17 shows that there cannot be any $\overline{X} \in H^n$ such that $L_A(\overline{X}) \succ 0$.

For the converse, suppose that $\delta(A) = 0$. To prove the existence of $\overline{X} \in H^n$ such that $L_A(\overline{X}) \succ 0$, it is enough to verify condition (2) in Proposition 17. Suppose there exists $U \neq 0$ such that $U \succeq 0$ and $L_A^T(U) = L_{A^*}(U) = 0$. Then without loss of generality

$$A^*D + DA = 0, \quad \text{where } D \text{ is diagonal}.$$

Let

$$A^* = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $D_1 0$ is positive definite and diagonal. Then

$$A_1D_1 + (A_1D_1)^* = 0 \quad \text{and} \quad A_3D_1 = 0,$$

hence $A_3 = 0$.

Next, multiplying both sides of $A_1D_1 + (A_1D_1)^* = 0$ by $\left(\sqrt{D_1}\right)^{-1}$ and letting

$$E_1 = \left(\sqrt{D_1}\right)^{-1} A_1\sqrt{D_1},$$

we get $E_1 + E_1^* = 0$. This implies $\delta(E_1) \neq 0$.

Now, $\sigma(E_1) = \sigma(A_1) \subseteq \sigma(A^*)$ implies that $\delta(A) = \delta(A^*) \neq 0$ leading to a contradiction.

Appendix II

Lemma 27. *For $a \in V$, with $\pm 1 \notin \sigma(a)$, define $b := (e + a)^{-1} \circ (e - a)$. Then $P_{e+a}(b \circ x) = x - P_a(x)$. Moreover*, $\text{In}(b) = \text{In}_0(a)$.

Proof. Let the spectral decomposition of a be given by $a = \sum_1^r a_ie_i$ and let $x = \sum_1^r x_ie_i + \sum_{i<j} x_{ij}$ denote the corresponding Peirce decomposition of x. Then $e + a = \sum_1^r (1 + a_i)e_i$, $(e + a)^2 = \sum_1^r (1 + a_i)^2 e_i$, $(e + a)^{-1} = \sum_1^r (1 + a_i)^{-1}e_i$, and $e - a = \sum_1^r (1 - a_i)e_i$.

Since

$$e + a = \sum_1^r (1 + a_i)e_i,$$

$$b = \sum_1^r \frac{1 - a_i}{1 + a_i} e_i, \text{ and}$$

$$y = b \circ x = \sum_1^r \frac{1 - a_i}{1 + a_i} x_ie_i + \frac{1}{2}\sum_{i<j}\left(\frac{1 - a_i}{1 + a_i} + \frac{1 - a_j}{1 + a_j}\right)x_{ij},$$

we have (via Proposition 6),

$$P_{e+a}(y) = \sum_1^r (1 - a_i{}^2)x_ie_i + \sum_{ij}(1 - a_ia_j)x_{ij}$$
$$= x - P_a(x) = S_a(x).$$

The equality $\text{In}(b) = \text{In}_0(a)$ follows from the observations that if λ is an eigenvalue of a, then $\frac{1-\lambda}{1+\lambda}$ is an eigenvalue of b and that $-1 < \lambda < 1$ if and only if $\frac{1-\lambda}{1+\lambda} > 0$. $\square$

Concluding remarks. In this paper, we gave an alternate proof of Kaneyuki's generalization of Sylvester's law of inertia in simple Euclidean Jordan algebras. We considered the finiteness of cone spectrum of quadratic representations and presented Ostroski–Schneider type inertia results in Euclidean Jordan algebras. We conclude this paper by mentioning a result of Loewy [16]: For $A \in C^{n\times n}$, let X be a Hermitian matrix such that $AX + XA^* = M \succeq 0$. Then

$$|\pi(A) - \pi(X)| \leqslant n - l \quad \text{and} \quad |\nu(A) - \nu(X)| \leqslant n - l,$$

where l is the dimension of the column space of the matrix

$$[M\ AM\ A^2M \cdots A^{n-1}M].$$

It will certainly be interesting to see if this result can be extended to the Euclidean Jordan algebra setting.

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