SOME GLOBAL UNIQUENESS AND SOLVABILITY RESULTS FOR LINEAR COMPLEMENTARITY PROBLEMS OVER SYMMETRIC CONES

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Abstract. This article deals with linear complementarity problems over symmetric cones. Our objective here is to characterize global uniqueness and solvability properties for linear transformations that leave the symmetric cone invariant. Specifically, we show that for algebra automorphisms on the Lorentz space $\mathcal{L}^n$ and for quadratic representations on any Euclidean Jordan algebra, global uniqueness, global solability, and the $\mathbf{R}_0$-properties are equivalent. We also show that for Lyapunov-like transformations, global uniqueness property is equivalent to the transformation being positive stable and positive semidefinite.

Key Words. Euclidean Jordan algebra, symmetric cone, algebra/cone automorphism, $\mathbf{R}_0$-property, **Q**-property, **GUS**-property.

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1. Introduction. Given a finite dimensional real inner product space H, a closed convex set K in H, a continuous function $f : K \to H$, and a vector $q \in H$, the *variational inequality problem* $\mathrm{VI}(f, K, q)$ is to find an $x^* \in K$ such that

$$\langle f(x^*) + q, x - x^* \rangle \geq 0 \ \forall \ x \in K.$$

There is an extensive literature associated with this problem covering theory, applications, and computation of solutions, see e.g., [7]. When K is a closed convex cone, this problem reduces to the cone complementarity problem $\mathrm{CP}(f, K, q)$ which further reduces to linear complementarity problem $\mathrm{LCP}(f, K, q)$ when f is linear. In particular, when $H = R^n$ (with the usual inner product), $f (= M)$ is linear, and $K = R^n_+$, this reduces to the *standard linear complementarity problem* $\mathrm{LCP}(M, R^n_+, q)$ [3].

An unsolved problem in the variational inequality theory is the characterization of the global uniqueness property: given H and K, find a necessary and sufficient condition on f so that for all $q \in H$, $\mathrm{VI}(f, K, q)$ has a unique solution. This is related to the question of global invertibility of the normal map $F(x) := f(\Pi_K(x)) + x - \Pi_K(x)$ on H, see [7]. When K is polyhedral and f is linear, there is a well-known result of Robinson [21] that describes the invertibility of this map in terms of the determinants of a certain collection of matrices. This result, when specialized to the standard linear complementarity problem, says that for a square real matrix M, the standard linear complementarity problem $\mathrm{LCP}(M, R^n_+, q)$ has a unique solution for all q if and only if M is a **P**-matrix (which means that all principal minors of M are positive). The result of Robinson motivated researchers to consider (the more general) question of global invertibility of piecewise affine functions; see [25] and [8] for necessary and sufficient conditions.

Moving away from the polyhedral settings (where the underlying cone is the nonnegative orthant in R^n) and inspired by the recent interest in conic programming, various researchers have started looking at cone linear complementarity problems, particularly on semidefinite and second order cones, and more generally on symmetric cones. While symmetric cones are, in general, nonpolyhedral, they have a lot of structure. In spite of this, even in this special case (of a linear transformation on a symmetric cone) the global uniqueness problem remains unsolved. At present, various authors have studied this problem by restricting the symmetric cone and the linear transformation to specific

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classes. Here are some such results:

(1) Given a matrix $A \in R^{n \times n}$, consider the Lyapunov transformation L_A defined on the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices by,

$$L_A(X) := AX + XA^T.$$

Then it has been shown by Gowda and Song [10] that L_A has the global uniqueness property on the semidefinite cone $\mathcal{S}^n_+$ (i.e., for all $q \in \mathcal{S}^n$, LCP$(L_A, \mathcal{S}^n_+, q)$ has a unique solution) if and only if A is both positive stable (i.e., all its eigenvalues have positive real parts) and positive semidefinite.

(2) Given a matrix $A \in R^{n \times n}$, define the multiplication transformation M_A defined on $\mathcal{S}^n$ by

$$M_A(X) := AXA^T.$$

Then it has been shown by Bhimasankaram et al. [2] and Gowda, Song, and Ravindran [11] that M_A has the global uniqueness property on the semidefinite cone if and only if $\pm A$ is positive definite.

(3) On the Lorentz space $\mathcal{L}^n$ (see Section 2 for the definition), consider the quadratic representation P_a of an element $a \in \mathcal{L}^n$:

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

In this setting, Malik and Mohan [18] have shown that P_a has the global uniqueness property on the Lorentz cone if and only if $\pm a$ is in the interior of that cone.

Another issue in the variational inequality/complementarity theory is the global solvability: given H and K, find a necessary and sufficient condition on f so that VI(f, K, q) has a solution for all $q \in H$. We note that this remains unsolved even in the setting of standard linear complementarity problem. So, as in the uniqueness issue, one has to work within a class of cones/transformations to get meaningful results. Our motivation for this part comes from the following:

(a) For a nonnegative matrix M, Murty [19] has shown that M has the global solvability property with respect to the nonnegative orthant R^n_+ (i.e., LCP(M, R^n_+, q) has a solution for all $q \in R^n$) if and only if the diagonal of M is positive.

(b) The Lyapunov transformation L_A (defined earlier) has the global solvability property with respect to $\mathcal{S}^n_+$ if and only if A is positive stable [10].

(c) For matrix $A \in R^{n \times n}$, consider the Stein transformation S_A defined on the space $\mathcal{S}^n$ by,

$$S_A(X) := X - AXA^T.$$

Then, S_A has the global solvability property on $\mathcal{S}^n_+$ if and only if A is Schur stable (that is, all eigenvalues of A lie in the open unit disk of the complex plane) [9].

(d) Given a real matrix A, consider the multiplication transformation M_A (defined earlier) on $\mathcal{S}^n$. In [22], Sampangi Raman has shown that when A is symmetric, M_A has the global solvability property with respect to the semidefinite cone in $\mathcal{S}^n_+$ if and only if $\pm A$ is positive definite.

(e) In [18], Malik and Mohan have shown that P_a on $\mathcal{L}^n$ has the global solvability property with respect to the Lorentz cone if and only if $\pm a$ is in the interior of the Lorentz cone.

In keeping with the above global uniqueness/solvability issues, we consider linear complementarity problems over symmetric cones in Euclidean Jordan algebras. With the observation (elaborated in various sections of the paper) that all the transformations considered in items (1)-(3) and (a)-(e) either leave the symmetric cone invariant or related to one such, we characterize global uniqueness/solvability properties for algebra automorphisms, quadratic representations, and Lyapunov-like transformations.

Here is a brief description/outline of our paper. Let V be a Euclidean Jordan algebra V with the corresponding symmetric cone K. For a linear transformation $L : V \to V$ and a $q \in V$, we define the (cone) linear complementarity problem $\mathrm{LCP}(L, K, q)$ as the problem of finding $x \in V$ such that

$$x \in K, \ L(x) + q \in K, \quad \text{and} \quad \langle L(x) + q, x \rangle = 0.$$

Given L, we consider the following statements:
(α) For all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a unique solution.
(β) For all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a solution.
(γ) $\mathrm{LCP}(L, K, 0)$ has zero as the only solution.
• In Section 4, we show that for algebra automorphisms on $\mathcal{L}^n$ (these are invertible linear transformations satisfying $L(x \circ y) = L(x) \circ L(y) \ \ \forall \ x, y$) the above three properties are equivalent; this result may be regarded as an analog of item (2) above for algebra automorphisms on $\mathcal{L}^n$.
• In Section 5, we show that the above three properties are equivalent for any quadratic representation (given by $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$) on any Euclidean Jordan algebra, thereby extending Malik-Mohan's result (items (3) and (e) above) to arbitrary Euclidean Jordan algebras.
• In Section 6, we show that if L is Lyapunov-like, that is, if it satisfies the condition

$$x, y \in K, \ \text{and} \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0,$$

then the global uniqueness property (α) holds if and only if L is positive stable and positive semidefinite, thereby extending result of item (1) above to general Euclidean Jordan algebras.

2. Preliminaries.

2.1. Euclidean Jordan algebras. In this paper we deal with Euclidean Jordan algebras. For the sake of completeness, we provide a short introduction; for full details, see [6].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle \cdot, \cdot \rangle)$, where $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:
(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in V$.
We also assume that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

In a Euclidean Jordan algebra V, the set of squares

$$K := \{x \circ x : x \in V\}$$

is a *symmetric cone* (see Page 46, Faraut and Korányi [6]). This means that K is a self-dual closed convex cone (i.e., $K = K^* := \{x \in V : \langle x, y \rangle \geq 0 \ \forall y \in K\}$) and for any two elements

$x, y \in K^\circ (= \text{interior}(K))$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$. We use the notation

$$x \geq 0 \quad \text{and} \quad x > 0$$

when $x \in K$ and $x \in K^\circ$, respectively.

An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents.

We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of idempotents in V is a *a complete system of orthogonal idempotents* if

$$e_i \circ e_j = 0 \text{ for } i \neq j, \text{ and } \sum_1^m e_i = e.$$

(Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$.) Further, if each e_i is also primitive, we say that the system is a *Jordan frame*.

THEOREM 2.1. **(The Spectral decomposition theorem)** *(Faraut and Korányi [6], Thm. III.1.1 and Thm. III.1.2) Let V be a Euclidean Jordan algebra. Then there is a number r (called the rank of V) such that for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ with*

(2.1) $$x = \lambda_1 e_1 + \cdots + \lambda_r e_r.$$

Also, for each $x \in V$, there exists a unique set of distinct real numbers $\{\mu_1, \mu_2, \ldots, \mu_k\}$ and a unique complete system of orthogonal idempotents $\{f_1, f_2, \ldots, f_k\}$ such that

$$x = \mu_1 f_1 + \mu_2 f_2 + \cdots + \mu_k f_k.$$

For an x given by (2.1), the numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are called the eigenvalues of x. We say that x is *invertible* if every λ_i is nonzero. Corresponding to (2.1), we define

$$trace(x) := \lambda_1 + \lambda_2 + \cdots + \lambda_r.$$

In addition, when $x \geq 0$ (or equivalently, every λ_i is nonnegative), we define

$$\sqrt{x} := \sqrt{\lambda_1} e_1 + \cdots + \sqrt{\lambda_r} e_r.$$

In a Euclidean Jordan algebra V, for a given $x \in V$, we define the corresponding *Lyapunov transformation* $L_x : V \to V$ by

$$L_x(z) = x \circ z.$$

We say that elements x and y *operator commute* if $L_x L_y = L_y L_x$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2, Faraut and Korányi [6]).

Here are some standard examples.

Example 1. R^n is a Euclidean Jordan algebra with inner product and Jordan product defined respectively by

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i \quad \text{and} \quad x \circ y := (x_i y_i).$$

Here R_+^n is the corresponding symmetric cone.

Example 2. $\mathcal{S}^n$, the set of all $n \times n$ real symmetric matrices, is a Euclidean Jordan algebra with the inner and Jordan product given by

$$\langle X, Y \rangle := \text{trace}(XY) \quad \text{and} \quad X \circ Y := \frac{1}{2}(XY + YX).$$

In this setting, the symmetric cone $\mathcal{S}_+^n$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$. Also, X and Y operator commute if and only if $XY = YX$.

Example 3. Consider R^n ($n > 1$) where any element x is written as

$$x = \left[\begin{array}{c} x_0 \\ \overline{x} \end{array} \right]$$

with $x_0 \in R$ and $\overline{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \left[\begin{array}{c} x_0 \\ \overline{x} \end{array} \right] \circ \left[\begin{array}{c} y_0 \\ \overline{y} \end{array} \right] := \left[\begin{array}{c} \langle x, y \rangle \\ x_0 \overline{y} + y_0 \overline{x} \end{array} \right].$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}_+^n$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathcal{L}_+^n = \{x : x_0 \geq ||\overline{x}||\}$$

in which case, interior$(\mathcal{L}_+^n) = \{x : x_0 > ||\overline{x}||\}$.

We note the spectral decomposition of any x with $\overline{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2$$

where

$$\lambda_1 := x_0 + ||\overline{x}||, \ \lambda_2 := x_0 - ||\overline{x}||$$

and

$$e_1 := \frac{1}{2} \left[\begin{array}{c} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{array} \right], \text{ and } e_2 := \frac{1}{2} \left[\begin{array}{c} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{array} \right].$$

In $\mathcal{L}^n$, elements x and y operator commute if and only if either $\overline{x}$ is a multiple of $\overline{y}$ or $\overline{y}$ is a multiple of $\overline{x}$.

We recall the following from Gowda, Sznajder and Tao [12] (with the notation $x \geq 0$ when $x \in K$):

PROPOSITION 2.2. *For $x, y \in V$, the following conditions are equivalent:*
 1. $x \geq 0$, $y \geq 0$, and $\langle x, y \rangle = 0$.
 2. $x \geq 0$, $y \geq 0$, and $x \circ y = 0$.
In each case, elements x and y operator commute.

The Peirce decomposition. Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R \, e_i$$

and when $i \neq j$,

$$V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}.$$

Then we have the following result.

THEOREM 2.3. *(Theorem IV.2.1, Faraut and Korányi [6]) The space V is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

$$V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj}$$
$$V_{ij} \circ V_{jk} \subset V_{ik} \ \textit{if}\ i \neq k$$
$$V_{ij} \circ V_{kl} = \{0\} \ \textit{if}\ \{i, j\} \cap \{k, l\} = \emptyset.$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we have the *Peirce decomposition* of $x \in V$:

$$x = \sum_{i=1}^{r} x_{ii} + \sum_{i<j} x_{ij} = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two Euclidean Jordan algebras. The classification theorem (Chapter V, Faraut and Korányi [6]) says that every simple Euclidean Jordan algebra is isomorphic to one of the following:
   (1) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices.
   (2) The algebra $\mathcal{L}^n$.
   (3) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
   (4) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
   (5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
The following result characterizes all Euclidean Jordan algebras.

THEOREM 2.4. *(Prop. III.4.4, Prop. III.4.5, Thm. V.3.7, Faraut and Korányi [6]) Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

2.2. Complementarity concepts. Let V be a Euclidean Jordan algebra with the corresponding symmetric cone K. Recall that for a linear transformation $L : V \to V$ and $q \in V$, the *linear complementarity problem* $\mathrm{LCP}(L, K, q)$ is to find an $x \in V$ such that

$$x \in K, \ L(x) + q \in K, \quad \text{and} \quad \langle L(x) + q, x \rangle = 0.$$

As mentioned previously, this is a particular instance of a variational inequality problem. The standard linear complementarity problem (over the nonnegative orthant in R^n), the semidefinite linear complementarity problem, and the Lorentz cone (also known as the second order cone) linear complementarity problem are some of the special cases and have been well studied in the literature. Given L on V, we say that L has the
   (i) positive definite property if $\langle L(x), x \rangle > 0$ for all $x \neq 0$;
   (ii) **GUS** (globally uniquely solvable)-property if for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a unique solution;

(iii) **P**-property if
$$x \text{ and } L(x) \text{ operator commute and } x \circ L(x) \le 0 \Rightarrow x = 0;$$
(iv) $\mathbf{R}_0$-property if zero is the only solution of LCP$(L, K, 0)$;
(v) **Q**-property if for all $q \in V$, LCP(L, K, q) has a solution;
(vi) **S**-property if there exists a $d > 0$ such that $L(d) > 0$.

Henceforth, we will use $\mathbf{P}$, $\mathbf{R}_0$, $\mathbf{Q}$, $\mathbf{S}$, etc., to denote the set of maps L that satisfy the respective property.

The above properties have been well studied. In particular (see [12], Theorems 17, 14, and 12), we always have the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. That $(v) \Rightarrow (vi)$ follows from perturbing a solution of LCP$(L, K, -e)$ where e is the unit element of V.

The following well-known result shows that an under an additional assumption, $(iv) \Rightarrow (v)$.

THEOREM 2.5. *(Karamardian [16]) Suppose that $L : V \to V$ is a linear transformation such that for some $d > 0$, zero is the only solution of the problems $LCP(L, K, 0)$ and $LCP(L, K, d)$. Then L has the **Q**-property with respect to K.*

2.3. Automorphisms. Let V be a Euclidean Jordan algebra and K be the corresponding cone of squares. We consider the following sets of transformations:

- $Aut(V)$—set of all invertible linear transformations $L : V \to V$ such that

$$L(x \circ y) = L(x) \circ L(y) \; \forall x, y \in V,$$

- $Aut(K)$—set of all (invertible) linear transformations $L : V \to V$ such that $L(K) = K$,
- $\overline{Aut(K)}$—closure of $Aut(K)$ (with respect to the operator norm),
- $\Pi(K)$—set of all linear transformations $L : V \to V$ such that $L(K) \subseteq K$.

We note that

$$Aut(V) \subseteq Aut(K) \subseteq \overline{Aut(K)} \subseteq \Pi(K).$$

Also, if V is simple or if the inner product in V is given by $\langle x, y \rangle = c\, trace(x \circ y)$ (for some fixed c), then every L in $Aut(V)$ is orthogonal (that is, it preserves the inner product), see pp. 57, [6].

3. Cone invariant transformations. Recall that $\Pi(K)$ is the set of all linear transformations on V that leave K invariant. We begin by describing some complementarity properties of $\Pi(K)$ and $\overline{Aut(K)}$.

PROPOSITION 3.1. *For $L \in \Pi(K)$, the following are equivalent:*
(a) L has the $\mathbf{R}_0$-property.
(b) For any primitive idempotent $u \in V$, $\langle L(u), u \rangle > 0$.
(c) L is strictly copositive on K, i.e., $\langle L(x), x \rangle > 0$ for all $0 \ne x \ge 0$.
*In particular, if L has the $\mathbf{R}_0$-property, then it has the **Q**-property.*

Proof. From $L \in \Pi(K)$, we see that L is copositive on K, that is, $\langle L(x), x \rangle \ge 0$ for all $x \ge 0$. Now assume (a). For any primitive idempotent u, we have $L(u) \in K$ and so $\langle L(u), u \rangle \ge 0$. If $\langle L(u), u \rangle = 0$, then u will be a nonzero solution of LCP$(L, K, 0)$ contradicting condition (a). Hence (b) holds. Now suppose (b) holds. We know that L is copositive on K. Suppose if possible, $\langle L(x), x \rangle = 0$ for some nonzero $x \in K$. Let $x = \sum \lambda_i e_i$ be the spectral decomposition of x with some eigenvalue, say, $\lambda_k \ne 0$. As $\lambda_i \ge 0$ for all i, $\langle L(x), x \rangle = \sum_{i,j} \lambda_i \lambda_j \langle L(e_i), e_j \rangle \ge \lambda_k^2 \langle L(e_k), e_k \rangle > 0$ by condition (b). This is a contradiction. Hence (c) holds. Finally, the implication $(c) \Rightarrow (a)$ is obvious.

Now assume that (a) holds. Then L is strictly copositive on K and so the problems LCP$(L, K, 0)$ and LCP(L, K, e) have zero as the only solution. By Karamardian's theorem, LCP(L, K, q) has a solution for all $q \in V$. Thus L has the **Q**-property. $\quad\square$

PROPOSITION 3.2. *If $L \in \overline{Aut(K)}$ is invertible, then $L \in Aut(K)$.*

Proof. Let $L_k \in Aut(K)$ such that $L_k \to L$ (with respect to the operator norm) on V with L invertible. From $L_k(K) \subseteq K$, we get $L(K) \subseteq K$. Also, $L_k^{-1} \to L^{-1}$. From $(L_k)^{-1}(K) \subseteq K$, we get $L^{-1}(K) \subseteq K$. Thus we have $L(K) = K$.   ☐

PROPOSITION 3.3. $\overline{Aut(K)} \cap \mathbf{S} = Aut(K)$.

Proof. Recall that $L \in \mathbf{S}$ if there exists a $p > 0$ such that $L(p) > 0$. Then, clearly, $Aut(K)$ is contained in $\overline{Aut(K)} \cap \mathbf{S}$. Now suppose that $L \in \overline{Aut(K)} \cap \mathbf{S}$. Let $L = \lim L_k$ where $L_k \in Aut(K)$. As $L_k^T \in Aut(K)$ (See Proposition I.1.7, [6]) and $L^T = \lim L_k^T$, we have $L^T \in \overline{Aut(K)}$. We show that L^T is invertible (or equivalently, L is invertible) and then conclude (thanks to the previous proposition) that $L \in Aut(K)$. Now to show that L^T is invertible, we show that for each $d > 0$, there is an $x \in K$ such that $L^T(x) = d$. This then shows that the range of L^T contains the open set K° thus proving the invertibility of L^T. Now fix a $d > 0$. Since $L_k^T \in Aut(K)$ for each k, there exists a sequence $x_k \in K$ such that $L_k^T(x_k) = d$ for all k. Assume, if possible, that the sequence x_k is unbounded, say $\|x_k\| \to \infty$. As x_k is nonzero, we can let $\frac{x_k}{\|x_k\|} \to y \in K$ with $L^T(y) = 0$. Now because L has the $\mathbf{S}$-property, there exists a $p > 0$ such that $L(p) > 0$. If u is a suitable multiple of p, then $u \geq 0$ and $v = L(u) - d \geq 0$. Then,

$$0 \leq \langle y, v \rangle = \langle y, L(u) - d \rangle = \langle L^T(y), u \rangle - \langle y, d \rangle = -\langle y, d \rangle \leq 0.$$

Thus, $\langle y, d \rangle = 0$. Since $y \geq 0$ and $d > 0$, we must have $y = 0$ which is a contradiction. Hence the sequence x_k is bounded. Letting $x_k \to x \in K$, we have $L^T(x) = d$.   ☐

COROLLARY 3.4.

$$\overline{Aut(K)} \cap \mathbf{R}_0 \subseteq \overline{Aut(K)} \cap \mathbf{Q} \subseteq \overline{Aut(K)} \cap \mathbf{S} = Aut(K).$$

Proof. The first inclusion comes from Proposition 3.1. The second inclusion follows from the fact that the $\mathbf{Q}$-property implies the $\mathbf{S}$-property, see Section 2.2. The last equality comes from the previous proposition.   ☐

This corollary shows that

$$Aut(K) \cap \mathbf{Q} = Aut(K) \cap \mathbf{R}_0 \Rightarrow \overline{Aut(K)} \cap \mathbf{Q} = \overline{Aut(K)} \cap \mathbf{R}_0.$$

This means that to show the equivalence of $\mathbf{R}_0$ and $\mathbf{Q}$-properties in $\overline{Aut(K)}$, it is enough to prove such an equivalence in $Aut(K)$.

Motivated by Murty's result - item (a) in the Introduction - we may ask if $\mathbf{R}_0$ and $\mathbf{Q}$ properties are equivalent when $L \in \Pi(K)$. While the resolution of this question is our ultimate goal (or a road map), for lack of results describing objects of $\Pi(K)$, in the next two sections we deal with a subset of $\Pi(K)$, namely, $\overline{Aut(K)}$. In particular, we deal with algebra automorphisms and quadratic representations.

4. Algebra automorphisms. Recall that L is an algebra automorphism on V if L is invertible and

$$L(x \circ y) = L(x) \circ L(y)$$

for all x and y. In this section, we describe complementarity properties of such transformations.

To motivate our results, we first consider some examples.

Example 4. Consider $V = R^n$ with the usual inner product and Jordan product (= component-wise product). In this setting, every algebra automorphism of V is given by a permutation matrix. Then Murty's result (see item (a) of the Introduction), shows that such an automorphism has the $\mathbf{R}_0$-property if and only if the matrix is the identity matrix; thus **GUS**, **Q**, and $\mathbf{R}_0$ properties are equivalent for algebra automorphisms on R^n.

Example 5. Consider $V = \mathcal{S}^n$ with trace inner product and the usual Jordan product. Then it is known (as a consequence of Schneider's result in [24]) that every algebra automorphism on $\mathcal{S}^n$ is given by

$$L(X) = UXU^T$$

where U is a (real) orthogonal matrix. In this setting it is known ([2], [11]) that **GUS**, **Q**, and $\mathbf{R}_0$ properties are equivalent (to $\pm U$ being positive definite).

Example 6. Consider the Lorentz space $\mathcal{L}^n$. Since the underlying space is R^n, we think of a transformation L on $\mathcal{L}^n$ as given by a matrix

$$L = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix}$$

where $a \in R$, $b, c \in R^{n-1}$ and $D \in R^{(n-1) \times (n-1)}$.

Now suppose that $L \in Aut(\mathcal{L}^n)$. Since L preserves the unit element in $\mathcal{L}^n$, we must have

$$\begin{bmatrix} a & b^T \\ c & D \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

proving $a = 1$ and $c = 0$. As $L \in Aut(\mathcal{L}^n) \subset Aut(\mathcal{L}^n_+)$, by a result of Loewy and Schneider [17], there exists a $\mu > 0$ such that

$$L^T J_n L = \mu J_n$$

where $J_n = \text{diag}(1, -1, -1, \ldots, -1)$. A direct calculation shows that $b = 0$, $\mu = 1$ and $D^T D = I$ and so

(4.1) $$L = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$$

where D is an orthogonal matrix. (We note that $D = I$ and $D = -I$ are likely candidates.) In this section, we show that for such an automorphism, **GUS**, **Q**, and $\mathbf{R}_0$ properties are equivalent.

First we describe the real eigenvalues of an algebra automorphisms and a necessary condition for the $\mathbf{R}_0$-property. In what follows, $\sigma(L)$ denotes the spectrum of a linear transformation L.

PROPOSITION 4.1. *Let V be a Euclidean Jordan algebra. If $L \in Aut(V)$, then*

$$\sigma(L) \cap R \subseteq \{-1, 1\}.$$

In particular, $\sigma(L) \cap R = \{1\}$ if and only if $-1 \notin \sigma(L)$.

Proof. As $L(x \circ y) = L(x) \circ L(y)$ for all $x, y \in V$, we have $L(e) = e$ where e is the unit element in V. Thus $1 \in \sigma(L) \cap R$. Now suppose that λ is a real eigenvalue of L (which is nonzero since L is invertible), so that for some nonzero $x \in V$, $L(x) = \lambda x$. It follows that $L(x^2) = \lambda^2 x^2$, and

more generally, $L(x^k) = \lambda^k x^k$ for all natural numbers k. Since $x \neq 0 \Rightarrow x^k \neq 0$ (this can be seen by considering the spectral decomposition of x), λ^k is an eigenvalue of L for all k. As V is finite dimensional, $\sigma(L)$ is finite, and so two distinct powers of λ are equal, that is, $\lambda^m = 1$ for some natural number m. As λ is real, we must have $\lambda = \pm 1$. Thus we have $\sigma(L) \cap R \subseteq \{-1, 1\}$. The second statement in the proposition is obvious. $\qquad\square$

Remark. The above proposition can also be seen as follows. Given any Euclidean Jordan algebra $(V, \langle \cdot, \cdot \rangle, \circ)$, it is well known that $[x, y] := trace(x \circ y)$ induces another inner product on V that is compatible with the Jordan product. With respect to this inner product, any algebra automorphism is an orthogonal (i.e., $L^T L = I$). Working with the complexifications of $(V, \langle \cdot, \cdot \rangle)$, $(V, [\cdot, \cdot])$, and L, we see that the spectrum of L (which is independent of the inner product on V) is contained in the unit circle. The above proposition follows immediately from this.

THEOREM 4.2. *Let V be a Euclidean Jordan algebra, $L \in Aut(V)$, and $L \in \mathbf{R}_0$. Then $-1 \notin \sigma(L)$.*

Proof. Suppose $-1 \in \sigma(L)$. Then there exists a vector $0 \neq x \in V$ such that $L(x) = -x$. By Theorem 2.1, there exist unique real numbers $\mu_1, \dots, \mu_k$, all distinct, and a unique complete system of orthogonal idempotents $f_1, \dots, f_k$ such that

$$x = \mu_1 f_1 + \cdots + \mu_k f_k.$$

Without loss of generality assume that $\mu_1 \neq 0$. Then

$$-(\mu_1 f_1 + \cdots + \mu_k f_k) = -x = L(x) = \mu_1 L(f_1) + \cdots + \mu_k L(f_k).$$

Because $L \in Aut(V)$, $\{L(f_1), \dots, L(f_k)\}$ is also a complete system of orthogonal idempotents. Since $\mu_1 \neq -\mu_1$, by the uniqueness property, $-\mu_1 = \mu_i$ and $f_1 = L(f_i)$ for some $1 < i \leq k$. Then f_i is a solution of LCP$(L, K, 0)$, contradicting the $\mathbf{R}_0$-property of L. This completes the proof. $\qquad\square$

COROLLARY 4.3. *Let $L \in Aut(V)$, $L \in \mathbf{R}_0$, and $L = L^T$. Then $L = I$.*

Proof. Since $L = L^T$, $\sigma(L) \subseteq R$. As $\{1\} \subseteq \sigma(L) \cap R \subseteq \{-1, 1\}$ and $-1 \notin \sigma(L)$, we have $\sigma(L) = \{1\}$. By the spectral theorem (for operators), $L = I$. $\qquad\square$

The following examples show that the converse in the above theorem need not hold.

Example 7. Consider the Euclidean Jordan algebra R^3 with the usual inner product and Jordan product. Define $L : R^3 \to R^3$ by the matrix

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Certainly, $L \in Aut(R^3)$ and $\sigma(L) = \{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$, so $\sigma(L) \cap R = \{1\}$. For $x = (1, 0, 0)^T$, $L(x) = (0, 0, 1)^T$, and $\langle L(x), x \rangle = 0$, thus $L \notin \mathbf{R}_0$. Note that R^3 is not a simple Jordan algebra.

Example 8. On the simple algebra $\mathcal{S}^3$, let

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and define the transformation L on $\mathcal{S}^3$ by $L(X) = UXU^T$. As U is orthogonal, a simple argument shows that $UX + XU = 0 \Rightarrow X = 0$. Hence -1 is not an eigenvalue of L. By Proposition 4.1, $\sigma(L) \cap R = \{1\}$. As $\pm U$ is not positive definite, a result of Bhimasankaram et al. [2] (see also [11]) shows that L cannot have the $\mathbf{R}_0$-property.

5. A characterization of the global uniqueness property of an algebra automorphism on $\mathcal{L}^n$. In this section, we establish the equivalence of the global uniqueness property and the $\mathbf{R}_0$-property for an automorphism on $\mathcal{L}^n$.

THEOREM 5.1. *For $L \in Aut(\mathcal{L}^n)$, the following are equivalent:*

(i) L has the $\mathbf{GUS}$-property.

(ii) L has the $\mathbf{P}$-property.

(iii) L has the $\mathbf{R}_0$-property.

(iv) L has the $\mathbf{Q}$-property.

(v) $-1 \notin \sigma(L)$ (or equivalently, $-1 \notin \sigma(D)$ with D given in (4.1)).

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow from Theorem 14 in [12] and the definitions. The implication $(iii) \Rightarrow (iv)$ follows from Proposition 3.1. We show that $(iv) \Rightarrow (v)$. Now write L as in (4.1). Suppose L has the $\mathbf{Q}$-property and -1 is an eigenvalue of L (as well as that of D) so that for some nonzero $u \in R^{n-1}$,

$$Du = -u.$$

Now let x be a solution of $\mathrm{LCP}(L, \mathcal{L}^n_+, q)$ where

$$q = \begin{bmatrix} 0 \\ u \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}.$$

Then

$$x \in \mathcal{L}^n_+, \ L(x) + q \in \mathcal{L}^n_+, \ \text{and} \ \langle x, L(x) + q \rangle = 0$$

and so

$$x_0 \geq ||\overline{x}||, \ x_0 \geq ||D\overline{x} + u||, \ \text{and} \ x_0^2 + \langle \overline{x}, D\overline{x} + u \rangle = 0.$$

Now, in view of Cauchy-Schwarz inequality, we have

$$x_0^2 = \langle -\overline{x}, D\overline{x} + u \rangle \leq ||\overline{x}|| \, ||D\overline{x} + u|| \leq x_0^2.$$

As $q \notin \mathcal{L}^n_+$, x cannot be zero; hence x_0, $\overline{x}$, and $D\overline{x} + u$ are all nonzero. Consequently,

$$D\overline{x} + u = -\theta\overline{x}, \ x_0 = ||\overline{x}||, \ \text{and} \ x_0 = ||D\overline{x} + u||$$

for some positive θ. From these, we get $\theta = 1$ and

$$D\overline{x} + u = -\overline{x}.$$

As D is orthogonal, $Du = -u \Rightarrow D^T u = -u$. Thus,

$$-\langle \overline{x}, u \rangle = \langle D\overline{x} + u, u \rangle = \langle D\overline{x}, u \rangle + ||u||^2 = \langle \overline{x}, D^T u \rangle + ||u||^2 = -\langle \overline{x}, u \rangle + ||u||^2.$$

This leads to $||u||^2 = 0$, which is a contradiction. Hence L satisfies (v).

Now for the last implication $(v) \Rightarrow (i)$.

First we show that (v) implies (ii). So assume (v).

Let x be a vector such that x and $L(x)$ operator commute, with $x \circ L(x) \leq 0$. For

$$x = \left[\begin{array}{c} x_0 \\ \bar{x} \end{array} \right] \quad \text{and} \quad L(x) = \left[\begin{array}{c} x_0 \\ D\bar{x} \end{array} \right]$$

we have

$$x \circ L(x) = \left[\begin{array}{c} x_0^2 + \langle D\bar{x}, \bar{x} \rangle \\ x_0 D\bar{x} + x_0\bar{x} \end{array} \right] \leq 0.$$

Case (i): $D\bar{x} = 0$ or equivalently, $\bar{x} = 0$ (as D is nonsingular). Then $\left[\begin{array}{c} x_0^2 \\ 0 \end{array} \right] \leq 0$, so $x = 0$.

Case (ii): $D\bar{x} \neq 0$. Hence, $\bar{x} \neq 0$ and by the operator commutativity, $D\bar{x} = \mu\bar{x}$ for some μ. Since $\mu \in \sigma(L) \cap R = \{1\}$, $\mu = 1$, thus $D\bar{x} = \bar{x}$. Consequently, $x_0^2 + ||\bar{x}||^2 \leq 0$, so $x = 0$.
Hence, L has the **P**-property.

Now to show (i). Take any $q \in \mathcal{L}^n$ and let x and u be two solutions of $\mathrm{LCP}(L, \mathcal{L}_+^n, q)$ so that

$$x \geq 0, \quad y = L(x) + q \geq 0, \quad \text{and} \quad \langle x, y \rangle = 0$$

and

$$u \geq 0, \quad v = L(u) + q \geq 0, \quad \text{and} \quad \langle u, v \rangle = 0.$$

We know that x and y operator commute, and u and v operator commute (see Proposition 2.2). If we can show that x and v operator commute, and u and y operator commute, then $x - u$ operator commutes with $y - v$. In this situation,

$$(x - u) \circ L(x - u) = (x - u) \circ (y - v) = -(x \circ v + u \circ y) \leq 0$$

where the last inequality follows from the fact that if two vectors in K operator commute, then their Jordan product is also in K. At this stage, we can apply the **P**-property (item (ii)) and get $x = u$. Thus (i) is proved provided the following claim can be proved:
Claim: u and y, as well as x and v, operator commute. (Equivalently, $\bar{u}$ and $\bar{y}$ are proportional and $\bar{x}$ and $\bar{v}$ are proportional.) We now proceed to prove the claim. Let

$$x = \left[\begin{array}{c} x_0 \\ \bar{x} \end{array} \right], \ u = \left[\begin{array}{c} u_0 \\ \bar{u} \end{array} \right], \ L(x) + q = \left[\begin{array}{c} x_0 + q_0 \\ D\bar{x} + \bar{q} \end{array} \right], \ \text{and} \ L(u) + q = \left[\begin{array}{c} u_0 + q_0 \\ D\bar{u} + \bar{q} \end{array} \right].$$

We have

$$x_0 \geq ||\bar{x}||, \ u_0 \geq ||\bar{u}||, \ x_0 + q_0 \geq ||D\bar{x} + \bar{q}||, \ \text{and} \ u_0 + q_0 \geq ||D\bar{u} + \bar{q}||$$

and moreover,

$$x_0(x_0 + q_0) + \langle D\bar{x} + \bar{q}, \bar{x} \rangle = 0 = u_0(u_0 + q_0) + \langle D\bar{u} + \bar{q}, \bar{u} \rangle.$$

If both x and u are positive, then $L(x) + q = 0 = L(u) + q$, and by invertibility of L, $x = u$. In this case, the claim is true (because x and y operator commute).

Suppose that (exactly) one of x or u is on the boundary of $\mathcal{L}_+^n$, say, $x > 0$ and $u \in \partial\mathcal{L}_+^n$. Then $x_0 > 0$ and $0 = y = L(x) + q$. If $v > 0$, then $u = 0$ in which case, $q > 0$. But then,

$$\begin{bmatrix} x_0 \\ D\bar{x} \end{bmatrix} = L(x) = -q < 0$$

implies that $x_0 < 0$ which is a contradiction.
On the other hand, if $v \in \partial\mathcal{L}_+^n$, then

$$u_0 + q_0 = ||D\bar{u} + \bar{q}|| = ||D(\bar{u} - \bar{x})|| = ||\bar{u} - \bar{x}|| \geq ||\bar{u}|| - ||\bar{x}||.$$

As, $x_0 = -q_0$ we have $u_0 - x_0 \geq ||\bar{u}|| - ||\bar{x}||$. Thus, $u_0 - ||\bar{u}|| \geq x_0 - ||\bar{x}|| > 0$, so $u > 0$ contradicting our assumption that $u \in \partial\mathcal{L}_+^n$.

Hence we may assume that both x and u are on the boundary of $\mathcal{L}_+^n$. Similarly, by considering L^{-1}, we may assume that both y and v are on the boundary of $\mathcal{L}_+^n$. (Note that $L^{-1} \in Aut(\mathcal{L}^n)$, y and v are solutions of $\text{LCP}(L^{-1}, K, L^{-1}q)$.)

So at this stage, $x, u, y, v \in \partial\mathcal{L}_+^n$. Thus,

$$x_0 = ||\bar{x}||, \ \ x_0 + q_0 = ||D\bar{x} + \bar{q}||, \ \ u_0 = ||\bar{u}||, \ \ \text{and} \ \ u_0 + q_0 = ||D\bar{u} + \bar{q}||.$$

Case 1: $\bar{x} = 0$. Then $x = 0$ and $q_0 = ||\bar{q}||$.
Subcase 1.1: $\bar{u} = 0$. Then $u = 0$, so $x = u = 0$ and the claim holds.
Subcase 1.2: $\bar{u} \neq 0$. Since u and v operator commute,

$$(5.1) \qquad\qquad\qquad \bar{v} = D\bar{u} + \bar{q} = \beta\bar{u}, \ \ \text{for some} \ \ \beta \in R.$$

Also, $u_0 + q_0 = ||D\bar{u} + \bar{q}|| \leq ||\bar{u}|| + q_0 \leq u_0 + q_0$. The equality in the triangle inequality gives, along with $\overline{u} \neq 0$,

$$(5.2) \qquad\qquad\qquad D\bar{u} = \theta\bar{q}, \ \ \text{for some} \ \ \theta > 0.$$

Now, by (5.2),

$$v = L(u) + q = \begin{bmatrix} u_0 + q_0 \\ D\bar{u} + \bar{q} \end{bmatrix} = \begin{bmatrix} u_0 + ||q|| \\ \theta\bar{q} + \bar{q} \end{bmatrix}$$

and

$$y = L(x) + q = 0 + q = \begin{bmatrix} q_0 \\ \bar{q} \end{bmatrix}.$$

Combining (5.1) and (5.2), we get $\theta\bar{q} + \bar{q} = \beta\bar{u}$. Hence, $\bar{q} = \dfrac{1}{\theta + 1}\beta\bar{u}$ $(\theta > 0)$, so $\bar{q}$ and $\bar{u}$ are proportional, and u and q operator commute. Thus, u and y $(= q)$ operator commute. Also, $x = 0$ and v operator commute. Therefore, in Case 1, the claim holds.
Case 2: $\bar{x} \neq 0$.
Subcase 2.1: $\bar{u} = 0$. This is analogous to Subcase 1.2.
Subcase 2.2: $\bar{u} \neq 0$. Now we have, $x, u, y, v \in \partial\mathcal{L}_+^n$, and $\bar{x} \neq 0$, $\bar{u} \neq 0$. Again, complementarity property, hence operator commutativity, gives

$$D\bar{x} + \bar{q} = \alpha\bar{x} \ \ \text{and} \ \ D\bar{u} + \bar{q} = \gamma\bar{u}$$

for some $\alpha, \gamma \in R$. Also,

$$x_0 + q_0 = ||D\bar{x} + \bar{q}|| = |\alpha| \cdot ||\bar{x}|| \quad \text{and} \quad u_0 + q_0 = ||D\bar{u} + \bar{q}|| = |\gamma| \cdot ||\bar{u}||.$$

Since $y = L(x) + q = \begin{bmatrix} x_0 + q_0 \\ \alpha \bar{x} \end{bmatrix}$ and $0 = \langle x, y \rangle = x_0(x_0 + q_0) + \alpha||\bar{x}||^2$, we have $\alpha \le 0$. Similarly, $\gamma \le 0$. Since $x_0 + q_0 = |\alpha| \cdot ||\bar{x}||$ and $x \in \partial\mathcal{L}^n_+$, $q_0 = |\alpha| \cdot ||\bar{x}|| - x_0 = (|\alpha| - 1)||\bar{x}||$; likewise, $q_0 = (|\gamma| - 1)||\bar{u}||$. Also, $\bar{q} = \alpha\bar{x} - D\bar{x} = \gamma\bar{u} - D\bar{u}$. Then,

$$D(\bar{x} - \bar{u}) = \alpha\bar{x} - \gamma\bar{u}.$$

Since D is orthogonal, $||\bar{x} - \bar{u}|| = ||\alpha\bar{x} - \gamma\bar{u}||$ and $q_0 = (|\alpha| - 1)||\bar{x}|| = (|\gamma| - 1)||\bar{u}||$, $\alpha, \gamma \le 0$. Put $\lambda := -\alpha$ and $\mu := -\gamma$, so $\lambda, \mu \ge 0$,

$$D(\overline{x} - \overline{u}) = \mu\overline{u} - \lambda\overline{x}, \quad ||\bar{x} - \bar{u}|| = ||\mu\bar{u} - \lambda\bar{x}|| \quad \text{and} \quad (\lambda - 1)||\bar{x}|| = (\mu - 1)||\bar{u}||.$$

If $\overline{x}$ and $\overline{u}$ are proportional, then so are $\overline{y}\,(= \alpha\overline{x})$ and $\overline{u}$, and $\overline{x}$ and $\overline{v}\,(= \gamma\overline{u})$. In this setting, the claim holds. Suppose $\overline{x}$ and $\overline{u}$ are not proportional. Then, as the signs of $\lambda - 1$ and $\mu - 1$ are the same, two-dimensional (Euclidean) geometric considerations show that the quadrilateral with vertices $\bar{x}, \lambda\bar{x}, \mu\bar{u},$ and $\bar{u}$ (which is part of a triangle) can be a parallelogram only when $\lambda = 1$ and $\mu = 1$ (see the Appendix for an algebraic proof). In this situation,

$$D(\bar{x} - \bar{u}) = -(\bar{x} - \bar{u})$$

violating the condition (v) that $-1 \notin \sigma(D)$. Thus we have the claim in Subcase 2.2. This shows that the claim is proved for Case 2. Hence $(v) \Rightarrow (i)$. $\qquad\square$

Remark. One may wonder if the above result (Theorem 5.1) is true for algebra automorphisms on a direct sum of $\mathcal{L}^n$s. To address this, let $V := \mathcal{L}^{n_1} \oplus \mathcal{L}^{n_2} \oplus \cdots \oplus \mathcal{L}^{n_k}$ and $L \in Aut(V)$. If L is "diagonal", that is, if $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ where $L_i : \mathcal{L}^{n_i} \to \mathcal{L}^{n_i}$, then $L_i \in Aut(\mathcal{L}^{n_i})$ for all i. In this situation, Theorem 5.1 extends to L on V. On the other hand, if L is not "diagonal", Theorem 5.1 may not extend to L. This can be seen by modifying Example 7:

Put $V = \mathcal{L}^n \oplus \mathcal{L}^n \oplus \mathcal{L}^n$ and $L(x, y, z) = (y, z, x)$. It is easily seen that $L \in Aut(V)$ and $-1 \notin \sigma(L)$, yet $L \notin \mathbf{R}_0$ (because $(e, 0, 0)$ is a solution of $\mathrm{LCP}(L, K, 0)$).

6. Quadratic representations. The algebra automorphisms of a Euclidean Jordan algebra, studied in the previous section, form an important subclass of $Aut(K)$. In this section, we consider another important subclass of $\overline{Aut(K)}$, namely, quadratic representations. Given any element a in the Euclidean Jordan algebra V, the quadratic representation of a is defined by

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

It is known that P_a is a self-adjoint linear transformation on V, and belongs to $Aut(K)$ when a is invertible. Our main result below establishes the equivalence of global uniqueness, global solvability, and the $\mathbf{R}_0$ properties for such transformations. Our motivation comes from the following.

Consider $V = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$. Then a linear transformation L on V belongs to $\mathrm{Aut}(\mathcal{S}^n_+)$ if and only if there exists an invertible matrix $A \in R^{n \times n}$ such that

$$L(X) = AXA^T \quad (X \in \mathcal{S}^n),$$

see [24]. For such transformations, it has been shown in [2] (see also [11]), that global uniqueness and $\mathbf{R}_0$ properties are equivalent, and that these properties hold if and only if $\pm A$ is positive definite. Now when $A \in \mathcal{S}^n$, the above transformation coincides with P_A given by:

$$P_A(X) = AXA.$$

In [22], Sampangi Raman considers P_A on $\mathcal{S}^n$ and shows that the global solvability and the $\mathbf{R}_0$ properties are equivalent, and that these properties hold if and only if $\pm A$ is positive definite. Working on $\mathcal{L}^n$, Malik and Mohan [18] prove a similar result for quadratic representations on $\mathcal{L}^n$.

In our main result below, we extend these two results to arbitrary Euclidean Jordan algebras. We remark that the crucial idea of our analysis comes from [22].

We recall the following from [6] (see Propositions II.3.1, II.3.2, and III.2.2.)

PROPOSITION 6.1. *Let $a \in V$. Then the following statements hold.*

(i) a is invertible if and only if P_a is invertible.

(ii) $P_{P_a(x)} = P_a P_x P_a$.

(iii) If a is invertible, then $P_a(K) = K$. Hence $P_a(K) \subseteq K$ for all $a \in V$.

(iv) If for some x, $P_a(x)$ is invertible, then a is invertible.

The following lemmas are needed to prove our main theorem. These lemmas and their proofs are somewhat similar, except for technical details, to those in [22] and [18].

LEMMA 6.2. *Let V be any Euclidean Jordan algebra. Let a be invertible in V with spectral decomposition given by*

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r.$$

Define $|a| := |a_1| e_1 + |a_2| e_2 + \cdots + |a_r| e_r$ and $s = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_r e_r$, where $\varepsilon_i = sign\,(a_i)$. If P_a has the $\mathbf{Q}$-property, then so does P_s.

Proof. Let $b := \sqrt{|a|}$. Then $P_b(s) = a$ (by using the definition) and $P_a = P_b P_s P_b$ using item (ii) in the previous lemma. Assume that P_a has the $\mathbf{Q}$-property and let $q \in V$. Let x be a solution of LCP(P_a, K, r) where $r = P_b(q)$. Then $x \geq 0$, $y := P_a(x) + r \geq 0$ and $\langle x, y \rangle = 0$. From $P_a = P_{P_b(s)} = P_b P_s P_b$, we have $P_b^{-1} y = P_s(P_b(x)) + q$. Using item (iii) in the above lemma and the self-adjointness of P_b, we have $u := P_b(x) \geq 0$, $v := P_b^{-1} y \geq 0$ and $\langle u, v \rangle = \langle P_b(x), P_b^{-1} y \rangle = \langle x, y \rangle = 0$. This means that LCP$(P_s, K, q)$ has a solution, proving the result. $\quad\square$

LEMMA 6.3. *Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V and $x = \sum_1^r x_i e_i + \sum_{i<j} x_{ij}$ be the Peirce decomposition of an element $x \in V$ with respect to this Jordan frame. Let*

$$s = e_1 + e_2 + \cdots + e_k - (e_{k+1} + \cdots + e_r)$$

for some k with $1 \leq k < r$ and $0 \neq q_{k\,k+1} \in V_{k\,k+1}$. Then the following hold:

(a) $P_s(x) = \sum_1^r x_i e_i + \sum_\beta x_{ij} - \sum_\alpha x_{ij}$, where
    $\alpha := \{(i,j) : 1 \leq i \leq k, k+1 \leq j \leq r\}$ and $\beta := \{(i,j) : 1 \leq i < j \leq r\} \setminus \alpha$.

(b) The $(k\,k+1)$-term in the Peirce decomposition of $x \circ P_s(x)$ is zero.

(c) the $(k\,k+1)$-term in the Peirce decomposition of $x \circ q_{k\,k+1}$ is $\frac{1}{2}(x_k + x_{k+1}) q_{k\,k+1}$.

(d) If $x \geq 0$, then $x_k e_k + x_{k+1} e_{k+1} + x_{k\,k+1} \geq 0$.

Proof. Let $f = e_1 + e_2 + \cdots + e_k$ so that $s = 2f - e$ where e is the unit element in V. Then $P_s(x) = 2s \circ (s \circ x) - s^2 \circ x$. Since $s^2 = e$ and $s = 2f - e$, simplification leads to $P_s(x) = 8f \circ (f \circ x) - 8f \circ x + x$. Using the properties

$$e_l \circ e_i = \delta_{il} e_l \quad \text{and} \quad e_l \circ x_{ij} = \frac{1}{2} x_{ij} \text{ if } l \in \{i, j\}, \text{ or } 0 \text{ if } l \notin \{i, j\},$$

we get

$$
\begin{aligned}
f \circ x &= x_1 e_1 + \tfrac{1}{2}(x_{12} + x_{13} + \cdots + x_{1r}) \\
&+ x_2 e_2 + \tfrac{1}{2}(x_{12} + x_{23} + x_{24} + \cdots + x_{2r}) \\
&+ x_3 e_3 + \tfrac{1}{2}(x_{13} + x_{23} + x_{34} + \cdots + x_{3r}) \\
&+ \cdots\cdots\cdots + \\
&+ x_k e_k + \tfrac{1}{2}(x_{1k} + x_{2k} + \cdots + x_{k-1\,k} + x_{k\,k+1} \cdots + x_{kr}).
\end{aligned}
$$

We note that in the above expression, the term x_{ij} for $1 \le i \le k$ and $k+1 \le j \le r$ appears only once and the term x_{ij} for $1 \le i < j \le k$ appears twice. Hence

$$
f \circ x = \sum_1^k x_i e_i + \sum_{1 \le i < j \le k} x_{ij} + \frac{1}{2} \sum_{1 \le i \le k,\, k+1 \le j \le r} x_{ij}.
$$

From this, we get $f \circ (f \circ x) = \sum_1^k x_i e_i + \sum_{1 \le i < j \le k} x_{ij} + \frac{1}{4} \sum_\alpha x_{ij}$. Using $P_s(x) = 8 f \circ (f \circ x) - 8 f \circ x + x$, a simple calculation leads to item (a).

We now prove item (b). Consider any element y with its Peirce decomposition:

$$
y = \sum_1^r y_i e_i + \sum_{m<n} y_{mn}.
$$

Then, in view of the properties of the spaces V_{ij}, the $(k\,k+1)$ term in the Peirce decomposition of $x \circ y$ is obtained by adding all terms of the form $x_{ij} \circ y_{mn}$ and $x_{mn} \circ y_{ij}$ where

$$
k \in \{i,j\},\ k+1 \in \{m,n\},\ \text{and}\ |\{i,j\} \cap \{m,n\}| = 1.
$$

This sum reduces to

$$
\begin{aligned}
&\sum_{1 \le i < k} x_{ik} \circ y_{i\,k+1} + \sum_{1 \le i < k} y_{ik} \circ x_{i\,k+1} \\
&+ x_{kk} \circ y_{k\,k+1} + y_{kk} \circ x_{k\,k+1} \\
&+ x_{k\,k+1} \circ y_{k+1\,k+1} + y_{k\,k+1} \circ x_{k+1\,k+1} \\
&+ \sum_{k+1 < i \le r} x_{ki} \circ y_{k+1\,i} + \sum_{k+1 < i \le r} y_{ki} \circ x_{k+1\,i}.
\end{aligned}
$$

Now, when $y = P_s(x)$, we have $y_{ij} = -x_{ij}$ for $(i,j) \in \alpha$ and $y_{ij} = x_{ij}$ for $(i,j) \notin \alpha$. Putting these in the above sum and simplifying, we get the item (b).

Upon putting $y = q_{k\,k+1}$, we see that the $(k\,k+1)$ term in the Peirce decomposition of $x \circ y$ is

$$
x_{kk} \circ q_{k\,k+1} + x_{k+1\,k+1} \circ q_{k\,k+1} = \frac{1}{2}(x_k + x_{k+1}) q_{k\,k+1}
$$

which is item (c).

Now we prove item (d). Suppose $x \ge 0$. Let

$$
V_{\{e_k, e_{k+1}\}} = \{x \in V : x \circ (e_k + e_{k+1}) = x\}.
$$

It is known (see Proposition IV.1.1 in [6]) that this is a Euclidean Jordan algebra and its corresponding symmetric cone is given by (Theorem 3.1, [13])

$$
V^+_{\{e_k, e_{k+1}\}} := \{x \in K : x \circ (e_k + e_{k+1}) = x\}.
$$

Let y belong to this (sub)cone. Then its Peirce decomposition in V with respect to $\{e_1, e_2, \ldots, e_r\}$ ■ is given by (see Lemma 20, [12])

$$y = y_k e_k + y_{k+1} e_{k+1} + y_{k\,k+1}.$$

Then, using the orthogonality properties of spaces V_{ij}, we have

$$0 \leq \langle x, y \rangle = \langle x_k e_k + x_{k+1} e_{k+1} + x_{k\,k+1}, y \rangle.$$

As y is arbitrary, we see from the self-duality of $V^{+}_{\{e_k, e_{k+1}\}}$ that $x_k e_k + x_{k+1} e_{k+1} + x_{k\,k+1} \in V^{+}_{\{e_k, e_{k+1}\}}$, and in particular belongs to K. □

PROPOSITION 6.4. *Suppose V is simple. Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V. Let $s = e_1 + e_2 + \cdots + e_k - (e_{k+1} + \cdots + e_r)$ for some k with $1 \leq k < r$. Then P_s does not have the* **Q**-*property.*

Proof. As V is simple, $V_{k\,k+1}$ is nontrivial (see Proposition IV.2.3, [6]). Let $0 \neq q_{k\,k+1} \in V_{k\,k+1}$. We claim that $\text{LCP}(P_s, K, q_{k\,k+1})$ has no solution. If possible, let x be a solution of this problem. Then $x \geq 0$, $y := P_s(x) + q_{k\,k+1} \geq 0$ and $x \circ y = 0$. Applying the previous lemma to this setting, we get

$$0 = (x \circ y)_{k\,k+1} = (x \circ P_s(x))_{k\,k+1} + (x \circ q_{k\,k+1})_{k\,k+1} = \frac{1}{2}(x_k + x_{k+1})q_{k\,k+1}.$$

As $q_{k\,k+1} \neq 0$, we must have $x_k + x_{k+1} = 0$. But $x \geq 0$ implies that $x_k e_k + x_{k+1} e_{k+1} + x_{k\,k+1} \geq 0$. So x_k and x_{k+1} are both nonnegative and hence $x_k = x_{k+1} = 0$. By Proposition 3.2 in [13], $x_{k\,k+1} = 0$. Now $y \geq 0$ implies that $y_k e_k + y_{k+1} e_{k+1} + y_{k\,k+1} \geq 0$. From $y = P_s(x) + q_{k\,k+1}$ and the above lemma, we get $x_k e_k + x_{k+1} e_{k+1} - x_{k\,k+1} + q_{k\,k+1} \geq 0$. As $0 = x_k = x_{k+1}$ and $x_{k\,k+1} = 0$, we have (by Proposition 3.2 in [13]), $q_{k\,k+1} = 0$ which is a contradiction. Hence $\text{LCP}(P_s, K, q_{k\,k+1})$ has no solution. □

THEOREM 6.5. *Let V be any Euclidean Jordan algebra and $a \in V$. Then the following are equivalent:*

(1) P_a is positive definite on V.
(2) P_a has the **GUS**-*property.*
(3) P_a has the **P**-*property.*
(4) P_a has the **R**$_0$-*property.*
(5) P_a has the **Q**-*property.*

If, in addition, V is simple, then the above conditions are further equivalent to

(6) $\pm a \in K^{\circ}$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold for any linear transformation on V, see [12]. Now suppose that (4) holds. Since $P_a(K) \subseteq K$, by Proposition 3.1, P_a has the **Q**-property. Hence (5) holds. Now suppose that (5) holds. If x is a solution of $\text{LCP}(P_a, K, -e)$, then $P_a(x) - e \geq 0$ and hence $P_a(x) > 0$. By Item (iv) in Proposition 6.1, a is invertible. Let $a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r$ be the spectral decomposition of a. Note that each a_i is nonzero. By Lemma 6.2, P_s has the **Q**-property where $s = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_r e_r$ with $s_i = sign\,(a_i)$. First suppose that V is simple. Then by Proposition 6.4, $s_i = 1$ for all i or $s_i = -1$ for all i. This means that $\pm a \in K^{\circ}$. Since $P_a = P_{-a}$, we may assume that $a > 0$. Then $P_a = P_{P_{\sqrt{a}}e} = P_{\sqrt{a}} P_e P_{\sqrt{a}} = P_{\sqrt{a}}^2$ and so P_a is positive semidefinite on V. As P_a is invertible and symmetric (recall that a is invertible), P_a must be positive definite on V. Thus in the case of simple V, conditions $(1) - (6)$ are equivalent. When V is not simple, we show that (5) is equivalent to (1) by decomposing V as a product of simple Euclidean Jordan algebras (Cf. Theorem 2.4). Write $V = V^{(1)} \times V^{(2)} \times \cdots \times V^{(m)}$ where each $V^{(k)}$ is simple. Let

$a = (a^{(1)}, a^{(2)}, \dots, a^{(m)})$ in this product. As $P_a = P_{a^{(1)}} \times P_{a^{(2)}} \times \cdots \times P_{a^{(m)}}$, we see that each $P_{a^{(k)}}$ has the **Q**-property in $V^{(k)}$. By what has been proved, $\pm a^{(k)} > 0$ in $V^{(k)}$ and $P_{a^{(k)}}$ is positive definite on $V^{(k)}$. It follows that P_a is positive definite on V. This completes the proof. $\qquad\square$

7. Lyapunov-like transformations. A real square matrix A is said to be a **Z**-matrix if all its off-diagonal entries are nonpositive. If $A \in R^{n \times n}$, then this property is equivalent to

$$x, y \in R^n_+, \ \langle x, y \rangle = 0 \Rightarrow \langle Ax, y \rangle \leq 0.$$

This concept can be extended to symmetric cones: Following [14], we say a linear transformation $L : V \to V$ has the **Z**-property if

$$x, y \in K, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle \leq 0.$$

It has been shown (see [26], [5]) that this property is equivalent to: $e^{-tL} \in \Pi(K)$ for all $t \geq 0$.

The recent article [14] contains properties of such transformations; in particular, it is shown in that paper that L has the global solvability property (item (β) of the Introduction) if and only if L is positive stable. Examples of **Z**-transformations include both Lyapunov and Stein transformations on $\mathcal{S}^n$. We now say that a linear transformation L on V is **Lyapunov-like** transformation if both L and $-L$ have the **Z**-property, that is,

$$x, y \in K, \ \langle x, y \rangle = 0 \Rightarrow \langle L(x), y \rangle = 0.$$

Recently, Damm [4] has shown that for $\mathcal{S}^n$ and $\mathcal{H}^n$ (the space of all $n \times n$ Hermitian matrices over complex numbers) L has the above property if and only if it is a Lyapunov transformation (that is, it is of the form L_A for some square matrix A). While the form of a Lyapunov-like transformation on a general Euclidean Jordan algebera is not known, it can be easily shown [27] that on $\mathcal{L}^n$, a matrix is Lyapunov-like if and only if it is of the form

$$\begin{bmatrix} a & b^T \\ b & D \end{bmatrix}$$

where $a \in R$, $D \in R^{(n-1) \times (n-1)}$ with $D + D^T = 2aI$. For any Euclidean Jordan algebra V and $a \in V$, the transformation L_a (called a Lyapunov transformation) defined by

$$L_a(x) = a \circ x$$

is also a Lyapunov-like transformation.

Extending the result in item (1) of the Introduction, we present the following global uniqueness result.

THEOREM 7.1. *Let $L : V \to V$ be a Lyapunov-like transformation. Then L has the **GUS**-property if and only if L is positive stable (that is, all its eigenvalues have positive real parts) and positive semidefinite.*

Proof. Suppose L has the **GUS**-property. Then it has the global solvability property and so by Theorem 7 in [14], L is positive stable. Also, by Theorem 4.1 in [28],

$$\langle L(c), c \rangle \geq 0$$

for any primitive idempotent c in V. Now for any $x \in V$, we have the spectral decomposition

$$x = \sum_1^r \lambda_i e_i$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame. Since L is a Lyapunov-like transformation, we have $\langle L(e_i), e_j \rangle = 0$ for all $i \neq j$, and so

$$\langle L(x), x \rangle = \sum_{i,j} \lambda_i \lambda_j \langle L(e_i), e_j \rangle = \sum_i \lambda_i^2 \langle L(e_i), e_i \rangle \geq 0.$$

This proves that L is positive semidefinite.

Now assume that L is positive stable and positive semidefinite. Because of positive stability, for every q, $\mathrm{LCP}(L, K, q)$ has a solution, see Theorems 6 and 7 in [14]. We now prove uniqueness. Fix q and suppose that x and u are two solutions of $\mathrm{LCP}(L, K, q)$ so that

$$x \geq 0, \ y := L(x) + q \geq 0, \ \text{and} \ \langle x, y \rangle = 0$$

and

$$u \geq 0, \ v := L(u) + q \geq 0, \ \text{and} \ \langle u, v \rangle = 0.$$

Now, as L is positive semidefinite, the solution set of $\mathrm{LCP}(L, K, q)$ is convex (Theorem 2.3.5, [7]). So for any $t \in [0, 1]$, $tx + (1-t)u$ is also a solution of $\mathrm{LCP}(L, K, q)$. Writing out the complementarity conditions, we see that

$$\langle x, v \rangle = 0 = \langle u, y \rangle.$$

We conclude (by Proposition 2.2) that x and u operator commute with both y and v, and $x \circ v = 0 = u \circ y$; hence $z := x - u$ operator commutes with $y - v = L(z)$, and $z \circ L(z) = 0$. In this situation, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and scalars μ_i such that

$$z = \mu_1 e_1 + \mu_2 e_2 + \cdots + \mu_l e_l \quad \text{and} \quad L(z) = \mu_{l+1} e_{l+1} + \mu_{l+2} e_{l+2} + \cdots + \mu_r e_r$$

for some l between 1 and r. We then have

$$(7.1) \qquad L(z) = \mu_1 L(e_1) + \mu_2 L(e_2) + \cdots + \mu_l L(e_l) = \mu_{l+1} e_{l+1} + \mu_{l+2} e_{l+2} + \cdots + \mu_r e_r.$$

Now for any i between $l+1$ and r, and k between 1 and l, we have $\langle e_k, e_i \rangle = 0$ and $\langle L(e_k), e_i \rangle = 0$. From (7.1), we get

$$\mu_i \| e_i \|^2 = 0.$$

This implies that $L(z) = 0$. Since L is positive stable, it is invertible and so $z = 0$, thus proving the the uniqueness of solution for $\mathrm{LCP}(L, K, q)$. Hence L has the **GUS**-property.   □

Concluding Remarks. In this article, we have proved that global uniqueness, global solvability, and the $\mathbf{R}_0$ properties are equivalent for algebra automorphisms over $\mathcal{L}^n$ and for quadratic representations over any Euclidean Jordan algebra. We have given a characterization of the global uniqueness property for Lyapunov-like transformations. All the transformations considered in this paper are related to symmetric-cone-invariant transformations. Motivated by our results, we pose the following problems:

(1) Do global uniqueness and $\mathbf{R}_0$ properties coincide for algebra automorphisms on general Euclidean Jordan algebras?
(2) Do global solvability (i.e., the $\mathbf{Q}$-property) and $\mathbf{R}_0$ properties coincide for a cone automorphism? For an element of $\Pi(K)$?

8. Appendix. Here we justify an assertion made in the proof of Theorem 5.1.

Lemma 8.1. *Let $\bar{x}$ and $\bar{u}$ be two nonzero vectors in a real inner product space that are not proportional. Assume that for some nonnegative scalars λ and μ,*

$$||\bar{x} - \bar{u}|| = ||\mu\bar{u} - \lambda\bar{x}|| \quad \text{and} \quad (\lambda - 1)||\bar{x}|| = (\mu - 1)||\bar{u}||.$$

Then $\lambda = \mu = 1$.

Proof. The second equality shows that $(\lambda - 1)(\mu - 1) \geq 0$. Assume now that $(\lambda - 1)(\mu - 1) > 0$.

Case 1: $\lambda > 1$ and $\mu > 1$.

Expanding the first equality above, applying the Cauchy-Schwarz inequality, and using the second equality, we get

$$
\begin{aligned}
(8.1) \qquad 0 &= (\mu^2 - 1)||\bar{u}||^2 + (\lambda^2 - 1)||\bar{x}||^2 + 2(\lambda\mu - 1)\langle -\bar{x}, \bar{u}\rangle \\
&\geq (\mu^2 - 1)||\bar{u}||^2 + (\lambda^2 - 1)||\bar{x}||^2 - 2(\lambda\mu - 1)||\bar{u}|| \cdot ||\bar{x}|| \\
&= (\mu^2 - 1)||\bar{u}||^2 + (\lambda^2 - 1)\frac{(\mu-1)^2}{(\lambda-1)^2}||\bar{u}||^2 - 2\frac{(\lambda\mu-1)}{(\lambda-1)}(\mu - 1)||\bar{u}||^2.
\end{aligned}
$$

Simplifying (8.1), we get

$$0 \geq (\mu^2 - 1)(\lambda - 1)^2 + (\lambda^2 - 1)(\mu - 1)^2 - 2(\lambda\mu - 1)(\mu - 1)(\lambda - 1) = 0.$$

Since $\lambda\mu > 1$, we have equality in the Cauchy-Schwarz inequality $\langle \bar{x}, \bar{u}\rangle \leq ||\bar{x}|| \cdot ||\bar{u}||$. This means that the vectors $\bar{x}$ and $\bar{u}$ are proportional, which is a contradiction.

Case 2: $0 \leq \lambda < 1$ and $0 \leq \mu < 1$.

We omit the proof as it is similar to case 1.

Contradictions obtained in both cases imply that $\lambda = \mu = 1$. $\qquad \square$

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