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Some P-Properties for Nonlinear Transformations on Euclidean Jordan Algebras

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In this article, we introduce the concepts of **P** and $\mathbf{P}_0$ properties for a nonlinear transformation defined on a Euclidean Jordan algebra and study existence of solution in the associated complementarity problems. In particular, we show, in this general setting, that if a transformation has the $\mathbf{P}_0$ and $\mathbf{R}_0$ properties, then all associated complementarity problems have solutions. We also describe a necessary condition for a transformation to have the (global) uniqueness of solution property.

Key words: Euclidean Jordan algebra; **P**-property; complementarity problem; globally uniquely solvable property
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1. Introduction. A real $n \times n$ matrix M is said to be a **P**-matrix if all its principal minors are positive. Introduced by Fiedler and Pták [7] in 1962, **P**-matrices have found many applications in various fields, particularly in optimization; see, e.g., Facchinei and Pang [5]. It is well known (Cottle et al. [4]) that the **P**-matrix property can be equivalently described by the following condition:

$$x \in R^n, \ x * Mx \leq 0 \quad \Rightarrow \quad x = 0 \tag{1.1}$$

where "$*$" denotes the componentwise product and $z \leq 0$ means that all components of z are nonpositive. Equally well known is the unique solvability of linear complementarity problem LCP(M, q) corresponding to M and any $q \in R^n$: Find $x \in R^n$ such that

$$x \geq 0, \qquad Mx + q \geq 0 \qquad \text{and} \qquad \langle x, Mx + q\rangle = 0.$$

In the complementarity literature (Facchinei and Pang [5]), the nonlinear version of (1.1) has been extensively studied: A continuous function ϕ: $R^n \to R^n$ is said to be a **P**-function if the following condition holds:

$$(x - y) * (\phi(x) - \phi(y)) \leq 0 \quad \Rightarrow \quad x = y.$$

Similar to a linear complementarity problem, we have a nonlinear complementarity problem NCP(ϕ, q) corresponding to ϕ and $q \in R^n$: Find $x \in R^n$ such that

$$x \geq 0, \quad \phi(x) + q \geq 0 \qquad \text{and} \qquad \langle x, \phi(x) + q\rangle = 0.$$

When ϕ is a **P**-function, NCP(ϕ, q) will have at most one solution. Under the assumption that ϕ is a $\mathbf{P}_0$-function (which means that $\phi + \epsilon I$ is a **P**-function for all $\epsilon > 0$) and a so-called $\mathbf{R}_0$-condition, it can be shown (see e.g., Facchinei and Pang [5, Corollary 9.1.31]) that NCP(ϕ, q) has a solution for every $q \in R^n$. The NCP is a special case of a variational inequality problem that has been extensively studied in the literature (see, e.g., Facchinei and Pang [5]). Facchinei and Pang [5] introduce **P** and $\mathbf{P}_0$ functions relative to a Cartesian product of sets in R^n and study some of their properties. Going in a different direction, Gowda and Song [11] extended the **P**-property (1.1) to a linear transformation L defined on $\mathcal{S}^n$ (the space of all $n \times n$ real symmetric matrices):

$$X \in \mathcal{S}^n, \ XL(X) = L(X)X \preceq 0 \quad \Rightarrow \quad X = 0, \tag{1.2}$$

where $X \preceq 0$ means that X is negative semi-definite. Some of the properties of **P** matrices continue to hold in this setting. For example, given L satisfying (1.2) and any $Q \in \mathcal{S}^n$, the following semidefinite linear complementarity problem, SDLCP(L, Q), has a solution: Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0, \qquad L(X) + Q \succeq 0, \qquad \text{and} \qquad \langle L(X) + Q, X\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ refers to the trace inner product between two matrices. However, because of the nonpolyhedrality of the semidefinite cone $\mathcal{S}^n_+$, not all **P**-matrix properties—including the uniqueness in LCP and the positive principal minor property—extend to this setting.

In Gowda and Song [11] and Gowda and Parthasarathy [10], the property (1.2) was specialized to the Lyapunov and Stein transformations, defined respectively by

$$L_A(X) := AX + XA^T \qquad \text{and} \qquad S_A(X) := X - AXA^T,$$

where A is a given real $n \times n$ matrix and $X \in \mathcal{S}^n$. It was shown in these papers that L_A has **P**-property (1.2) if and only if A is positive stable (that is, all eigenvalues of A lie in the open right-half plane of the complex plane), and S_A has the **P**-property if and only if A is Schur stable (that is, all eigenvalues of A lie in the open unit disk) thereby connecting the above **P**-property to the theorems of Lyapunov and Stein on continuous and discrete linear dynamical systems.

The space R^n with componentwise product and $\mathcal{S}^n$ with Jordan product $X \circ Y := \frac{1}{2}(XY + YX)$ are two examples of Euclidean Jordan algebras. Partly motivated by the recent interest in the study of conic optimization problems, Gowda et al. [14] extended this notion of **P**-property to a linear transformation defined on a Euclidean Jordan algebra. A Euclidean Jordan algebra is a finite-dimensional real inner product space along with a Jordan product $x \circ y$ satisfying certain properties; see §2 for the definition. In such an algebra, the so-called cone of squares forms a "symmetric" cone. Along with R^n and $\mathcal{S}^n$, other examples of such algebras include the space of $n \times n$ Hermitian matrices over complex numbers, $n \times n$ Hermitian matrices over quaternions, and 3×3 Hermitian matrices over octonions. There is another algebra (denoted by $\mathcal{L}^n$) defined on R^n ($n > 1$) that induces a cone called the Lorentz cone (also known as the second-order cone); see §2 for definitions. In this paper, we further extend the notion of **P**-property to nonlinear transformations defined on a Euclidean Jordan algebra V: A continuous transformation F: $V \to V$ is said to have the **P**-*property* if

$$\left.\begin{aligned} x - y \text{ and } F(x) - F(y) \text{ operator commute} \\ (x-y) \circ (F(x) - F(y)) \le 0 \end{aligned}\right\} \Rightarrow \ x = y.$$

(Here, "operator commutativity" refers to the commutativity of two corresponding Lyapunov transformations; see §2. In the context of $\mathcal{S}^n$, this reduces to the ordinary matrix product commutativity.) Along with this **P**-property, we introduce other generalizations of the **P**-matrix property. When F: $V \to V$ *is such that* $F + \varepsilon I$ *has the* **P**-*property for all* $\varepsilon > 0$, *we say that* F *has the* $\mathbf{P}_0$-*property*. Given a Euclidean Jordan algebra V with the corresponding symmetric cone K, $q \in V$, and a continuous transformation F: $V \to V$, we can define the complementarity problem CP(F, q): Find $x \in V$ such that

$$x \in K, \qquad F(x) + q \in K \qquad \text{and} \qquad \langle F(x) + q, x\rangle = 0.$$

We note that the extra structure available in Euclidean Jordan algebras allows us to go beyond the general study of cone complementarity problems (see, e.g., Facchinei and Pang [5]) of which the above symmetric cone complementarity problem is a special case. Assuming that V is either $\mathcal{S}^n$ or $\mathcal{L}^n$ (and monotonicity of F in some cases), a number of authors, such as Chen and Tseng [3], Chen et al. [2], and Fukushima et al. [8], have discussed this problem. By going beyond monotonicity and $\mathcal{S}^n$ ($\mathcal{L}^n$) we show in this paper that when F has the $\mathbf{P}_0$-property along with a certain $\mathbf{R}_0$-property, all associated complementarity problems have solutions. In this way, we extend the classical result valid for nonlinear complementarity problems (defined on R^n) to the setting of Euclidean Jordan algebras.

In §4, we address the uniqueness issue in the complementarity problems associated with a continuous transformation defined on a Euclidean Jordan algebra. By adopting a terminology coined by Megiddo and Kojima [18] in the context of nonlinear complementarity problems, we say that $F: V \to V$ has the *globally uniquely solvable* (**GUS**) *property* if for all $q \in V$, $\mathrm{CP}(F, q)$ has a unique solution. In the setting of linear complementarity problems, there is no difference between **P** and **GUS** properties. In the setting of nonlinear complementarity problems, extending Karamardian's strong monotonicity condition, Moré's uniform **P**-condition, and Cottle's positively bounded Jacobians condition, Megiddo and Kojima [18] formulate necessary and/or sufficient conditions for the **GUS** property to hold. They also point out that in this setting, the **GUS** property does not imply the **P**-property. In the setting of Euclidean Jordan algebras two results are known: When F is strongly monotone on V this **GUS** property holds (Facchinei and Pang [5, Theorem 2.3.3]). When F is linear, the **GUS** property holds if and only if F has the **P**-property and the so-called cross commutativity property (Gowda et al. [14, Theorem 14]). Because this cross-commutative property is not easily verifiable and depends somewhat on the solution sets of complementarity problems, we seek other necessary conditions for the **GUS** property to hold. In §4 we describe one such necessary condition. The condition says that when F has the **GUS** property, $\langle F(c) - F(0), c\rangle \geq 0$ for all primitive idempotents c in V.

Finally, in §5, we introduce the so-called relaxation transformation on a general Euclidean Jordan algebra that is induced by a vector valued function, and study its **P** and **GUS** properties.

2. Preliminaries.

2.1. Euclidean Jordan algebras.

In this subsection, we recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in Faraut and Korányi [6], Schmieta and Alizadeh [19], and Gowda et al. [14].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$, where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y: V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in V$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(iii) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

Henceforth, *we assume that V is a Euclidean Jordan algebra* and call $x \circ y$ the Jordan product of x and y. We may assume (see Faraut and Korányi [6, p. 146]) that there is an element $e \in V$ (called the *unit* element) such that $x \circ e = x$ for all $x \in V$.

In V, the set of squares

$$K := \{x \circ x: x \in V\}$$

is a *symmetric cone* (see Faraut and Korányi [6, p. 46]). This means that K is a self-dual closed convex cone and for any two elements $x, y \in \text{interior } (K)$, there exists an invertible linear transformation $\Gamma: V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$.
For an element $z \in V$, we write

$$z \geq 0 \quad \text{if and only if} \quad z \in K,$$

and $z \leq 0$ when $-z \geq 0$. We also define

$$z^+ := \Pi_K(z) \qquad \text{and} \qquad z^- := z^+ - z$$

where $\Pi_K(z)$ denotes the (orthogonal) projection of z onto K. Finally, for any two elements $x, y \in V$, we let

$$x \sqcap y := x - (x-y)^+ \qquad \text{and} \qquad x \sqcup y := y + (x-y)^+.$$

For $x \in V$, we define $m(x) := \min\{k > 0\colon \{e, x, \ldots, x^k\}$ is linearly dependent$\}$ and *rank* of V by $r = \max\{m(x)\colon x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{if} \quad i \neq j \text{ and } \sum_1^m e_i = e.$$

Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$.

THEOREM 2.1 (THE SPECTRAL DECOMPOSITION THEOREM) (FARAUT AND KORÁNYI [6]). *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2.1}$$

The numbers λ_i are called the eigenvalues of x.

The expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the *spectral decomposition* (or the *spectral expansion)* of x. Given (2.1), we have

$$x = \sum_{i=1}^r \lambda_i^+ e_i - \sum_{i=1}^r \lambda_i^- e_i \qquad \text{and} \qquad \left\langle \sum_{i=1}^r \lambda_i^+ e_i, \sum_{i=1}^r \lambda_i^- e_i \right\rangle = 0.$$

From this we easily verify that

$$x^+ = \sum_{i=1}^r \lambda_i^+ e_i \qquad \text{and} \qquad x^- = \sum_{i=1}^r \lambda_i^- e_i,$$

and so

$$x = x^+ - x^- \quad \text{with} \quad \langle x^+, x^- \rangle = 0.$$

Corresponding to any $x \in V$, let $\lambda_i(x) (i = 1, 2 \ldots, r)$ denote the eigenvalues of x. We let

$$\omega(x) := \max_{1 \leq i \leq r} \lambda_i(x) \qquad \text{and} \qquad \nu(x) := \min_{1 \leq i \leq r} \lambda_i(x).$$

We note that $x \leq 0$ if and only if $\omega(x) \leq 0$.

PROPOSITION 2.1. *There exists a positive number θ such that for any $x, y \in V$ and any nonzero idempotent c, the following statements hold*:

(i) $\langle x, c \rangle \leq \omega(x) \|c\|^2$.
(ii) $\langle x, y \rangle \leq \omega(x \circ y) \|e\|^2$.
(iii) $\theta \leq \|c\| \leq \|e\|$.
(iv) $|\omega(x+y) - \omega(x)| \leq (1/\theta)\|y\|$ *and* $|\nu(x+y) - \nu(x)| \leq (1/\theta)\|y\|$.
(v) *If* $x^{(k)} \in V$ $(k = 1, 2, \ldots)$ *and* $y^{(k)} \to 0$, *then* $\liminf \omega(x^{(k)} + y^{(k)}) = \liminf \omega(x^{(k)})$ *and* $\liminf \nu(x^{(k)} + y^{(k)}) = \liminf \nu(x^{(k)})$.

PROOF. (i) By using the spectral decomposition of $x = \sum \lambda_i(x) e_i$, we have $\langle x, c \rangle = \sum \lambda_i(x) \langle e_i, c \rangle$. Because $c, e_i \in K$ and $\langle e, c \rangle = \langle e, c^2 \rangle = \langle c, c \rangle = \|c\|^2$, we have $\langle e_i, c \rangle \geq 0$ and hence $\langle x, c \rangle \leq \omega(x) \sum \langle e_i, c \rangle = \omega(x) \langle \sum e_i, c \rangle = \omega(x) \langle e, c \rangle = \omega(x) \|c\|^2$.

(ii) We have $\langle x, y \rangle = \langle x \circ y, e \rangle \leq \omega(x \circ y) \|e\|^2$ from Item (i).

(iii) The second inequality follows from $\|c\|^2 = \langle c, e\rangle \le \|c\|\|e\|$. To see the first inequality, suppose there is a sequence of nonzero idempotents $c^{(k)} \to 0$. Assuming $x^{(k)} := c^{(k)}/\|c^{(k)}\| \to x$ and taking the limit in $c^{(k)} \circ x^{(k)} = x^{(k)}$, we see that $0 \circ x = x$. This is a contradiction because x has unit norm.

(iv) Let $x = \sum \lambda_i(x)e_i$ be the spectral decomposition of x. By considering the spectral decomposition of $x+y$, we see that $\omega(x+y) = \langle x+y, c\rangle/\|c\|^2 = \langle x, c\rangle/\|c\|^2 + \langle y, c\rangle/\|c\|^2$ for some primitive idempotent c. The first term $\langle x, c\rangle/\|c\|^2$ is less than or equal to $\omega(x)$ from Item (i). The second term is less than or equal to $(1/\theta)\|y\|$ in view of Cauchy-Schwarz inequality and Item (iii). It follows that $\omega(x+y) \le \omega(x) + (1/\theta)\|y\|$. Similarly, $\omega(x) \le \omega(x+y) + (1/\theta)\|y\|$. Now the first part of Item (iv) follows. The second part follows from the first part and the observation $\omega(-x) = -\nu(x)$.

(v) is an easy consequence of Item (iv). □

Remark 2.1. (i) While the above proof of item (iv) is elementary, the inequalities in (iv) are not new. As noted by a referee, they follow from a result of Gårding ([9, Theorem 2.1]) on hyperbolic polynomials.

(ii) Item (iv) shows that ω is a continuous function. (This also follows from the fact that eigenvalues are continuous functions of the argument.)

Example 2.1. Consider R^n with the (usual) inner product and Jordan product defined respectively by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \qquad \text{and} \qquad x \circ y = x * y,$$

where x_i denotes the ith component of x, etc., and $x * y$ denotes the componentwise product of vectors x and y. Then R^n is a Euclidean Jordan algebra with R^n_+ as its cone of squares. In this setting, for $x, y \in R^n$,

$$x \sqcap y := x - (x-y)^+ = \min\{x, y\} \qquad \text{and} \qquad x \sqcup y := y + (x-y)^+ = \max\{x, y\}.$$

Example 2.2. Let $\mathcal{S}^n$ be the set of all $n \times n$ real symmetric matrices with the inner and Jordan product given by

$$\langle X, Y\rangle := \text{trace}(XY) \qquad \text{and} \qquad X \circ Y := \frac{1}{2}(XY + YX).$$

In this setting, the cone of squares $\mathcal{S}^n_+$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$. The identity matrix is the unit element. The set $\{E_1, E_2, \ldots, E_n\}$ is a Jordan frame in $\mathcal{S}^n$ where E_i is the diagonal matrix with 1 in the (i, i)-slot and zeros elsewhere. Note that the rank of $\mathcal{S}^n$ is n. Given any $X \in \mathcal{S}^n$, there exists an orthogonal matrix U with columns $u_1, u_2, \ldots, u_n$ and a real diagonal matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $X = UDU^T$. Clearly,

$$X = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

is the spectral decomposition of X; In particular, $\{u_1u_1^T, u_2u_2^T, \ldots, u_nu_n^T\}$ is a Jordan frame. Note that we may think of R^n (of Example 2.1) as the product of n copies of $\mathcal{S}^1$.

Example 2.3. Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix},$$

with $x_0 \in R$ and $\bar{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}.$$

We denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by

$$\mathcal{L}^n_+ = \{x\colon\ \|\bar{x}\| \le x_0\}.$$

The unit element in $\mathcal{L}^n$ is $e = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We note the spectral decomposition of any x with $\bar{x} \neq 0$:

$$x = \lambda_1 e_1 + \lambda_2 e_2,$$

where

$$\lambda_1 := x_0 + \|\bar{x}\|, \qquad \lambda_2 := x_0 - \|\bar{x}\|,$$

and

$$e_1 := \frac{1}{2}\begin{bmatrix} 1 \\ \dfrac{\bar{x}}{\|\bar{x}\|} \end{bmatrix} \quad \text{and} \quad e_2 := \frac{1}{2}\begin{bmatrix} 1 \\ -\dfrac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}.$$

In a Euclidean Jordan algebra V, for a given $x \in V$, we define the corresponding *Lyapunov transformation* L_x: $V \to V$ by

$$L_x(z) = x \circ z.$$

(Traditionally, the notation $L(x)$ has been used to denote the Lyapunov transformation; see Faraut and Korányi [6]. In this paper, we reserve the notation L_x for the Lyapunov transformation and write $L(x)$ to denote the image of an element $x \in V$ under a linear transformation L: $V \to V$. We also note that our previous notation used to describe the Lyapunov transformation L_A defined in the introduction is a commonly used notation in various literature; it differs slightly from the above.)

We say that elements x and y *operator commute* if L_x and L_y commute, i.e.,

$$L_x L_y = L_y L_x.$$

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Faraut and Korányi [6, Lemma X.2.2] or Schmieta and Alizadeh [19, Theorem 27]). In the case of $\mathcal{S}^n$, matrices X and Y operator commute if and only if $XY = YX$. In the case of $\mathcal{L}^n$, vectors x and y (see Example 2.3) operator commute if and only if either $\bar{y}$ is a multiple of $\bar{x}$ or $\bar{x}$ is a multiple of $\bar{y}$.

We recall the following propositions from Gowda et al. [14].

Proposition 2.2. *For $x, y \in V$, the following conditions are equivalent*:
(i) $x \sqcap y = 0$.
(ii) $x \ge 0$, $y \ge 0$, *and* $\langle x, y \rangle = 0$.
(iii) $x \ge 0$, $y \ge 0$, *and* $x \circ y = 0$.
In each case, elements x and y operator commute.

Proposition 2.3. *For x, $y \in V$, consider the following statements*:
(i) *x and y operator commute, and* $x \circ y \le 0$.
(ii) $x \circ y \le 0$.
(iii) $x \sqcap y \le 0 \le x \sqcup y$.
(iv) $\langle x, y \rangle \le 0$.
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

The Peirce decomposition. Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V\colon\ x \circ e_i = x\} = Re_i$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V\colon x \circ e_i = \tfrac{1}{2}x = x \circ e_j\right\}.$$

Then we have the following theorem.

THEOREM 2.2 (FARAUT AND KORÁNYI [6], THEOREM IV.2.1). *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{aligned} &V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj} \\ &V_{ij} \circ V_{jk} \subset V_{ik} \text{ if } i \neq k \\ &V_{ij} \circ V_{kl} = \{0\} \text{ if } \{i, j\} \cap \{k, l\} = \varnothing. \end{aligned}$$

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$.

Simple Jordan algebras and the structure theorem. A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. The classification theorem (Faraut and Korányi [6, Chapter V]) says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(1) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices (Example 2.2).

(2) The algebra $\mathcal{L}^n$ (Example 2.3).

(3) The algebra $\mathcal{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

(4) The algebra $\mathcal{Q}_n$ of all $n \times n$ quaternion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

(5) The algebra $\mathcal{O}_3$ of all 3×3 octonion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

THEOREM 2.3 (FARAUT AND KORÁNYI [6], PROPOSITIONS III.4.4 AND III.4.5, AND THEOREM V.3.7). *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras.*

2.2. Complementarity problems. Given a Euclidean Jordan algebra V with the associated cone K, a continuous transformation $F\colon V \to V$, and a $q \in V$, we define the **complementarity problem** $\text{CP}(F, q)$ as follows: Find $x \in V$ such that

$$x \in K, \;\; F(x) + q \in K \qquad \text{and} \qquad \langle x, F(x) + q \rangle = 0.$$

In the above condition, in view of Proposition 2.2, we can replace $\langle x, F(x) + q\rangle = 0$ by $x \circ (F(x) + q) = 0$. Furthermore, finding a solution to $\text{CP}(F, q)$ is equivalent to solving the equation

$$x \sqcap (F(x) + q)) = 0.$$

We say that $F\colon V \to V$ has the **globally uniquely solvable** (**GUS**) property if for all $q \in V$, $\text{CP}(F, q)$ has a unique solution.

3. Some monotone and P-properties. We recall that for any $x \in V$, $\lambda_i(x)$ $(i = 1, 2, \ldots, r)$ denote the eigenvalues of x and

$$\omega(x) := \max_{1\leq i\leq r} \lambda_i(x).$$

Definition 3.1. Let V be an Euclidean Jordan algebra. A continuous transformation $F\colon V \to V$ is said to be

(i) **monotone** if $\langle x-y, F(x)-F(y)\rangle \geq 0\ \forall\, x, y \in V$;

(ii) **strictly monotone** if $\langle x-y, F(x)-F(y)\rangle > 0\ \forall\, x \neq y \in V$;

(iii) **strongly monotone** if there is an $\alpha > 0$ such that

$$\langle x-y, F(x)-F(y)\rangle \geq \alpha\|x-y\|^2 \quad \forall\, x, y \in V.$$

It is said to have the

(a) **order P**-property if $(x-y) \sqcap (F(x)-F(y)) \leq 0 \leq (x-y) \sqcup (F(x)-F(y)) \Rightarrow x = y$;

(b) **Jordan P**-property if $(x-y) \circ (F(x)-F(y)) \leq 0 \Rightarrow x = y$, or equivalently,

$$x \neq y \ \Rightarrow\ \omega[(x-y) \circ (F(x)-F(y))] > 0;$$

(c) **P**-property if

$$\left.\begin{array}{r} x-y \text{ and } F(x)-F(y) \text{ operator commute} \\ (x-y)\circ(F(x)-F(y)) \leq 0 \end{array}\right\} \Rightarrow x = y;$$

(d) **uniform Jordan P**-property if there is an $\alpha > 0$ such that for all x and y in V, we have

$$\omega[(x-y) \circ (F(x)-F(y))] \geq \alpha\|x-y\|^2;$$

(e) **uniform P**-property if there is an $\alpha > 0$ such that for all x and y in V with $x-y$ operator commuting with $F(x)-F(y)$, we have

$$\omega[(x-y) \circ (F(x)-F(y))] \geq \alpha\|x-y\|^2;$$

(f) $\mathbf{P}_0$-property if $F(x) + \varepsilon x$ has the **P**-property for all $\varepsilon > 0$.

Remark 3.1. (i) It is easily seen that when $V = R^n$ with componentwise product (see Example 2.1), order **P** = Jordan **P** = **P** and uniform Jordan **P** = uniform **P**.

(ii) When F is linear, (i) strong monotonicity and strict monotonicity concepts coincide, and (ii) uniform (Jordan) **P** and (Jordan) **P** properties coincide. In this setting, the above properties have been introduced in Gowda et al. [14].

(iii) Consider the Lyapunov and Stein transformations L_A and S_A defined on $\mathcal{S}^n$ (see Introduction). It is known that L_A has the **P**-property if and only if A has all eigenvalues in the open right-half plane and S_A has the **P**-property if and only if all eigenvalues of A lie in the open unit disk (Gowda and Song [11] and Gowda and Parthasarathy [10]). Because

$$(L_A + \varepsilon I)(X) = AX + XA^T + \varepsilon X = L_{A+\frac{\varepsilon}{2}I}(X),$$

and

$$(S_A + \varepsilon I)(X) = X - AXA^T + \varepsilon X = (1+\varepsilon) S_{\frac{1}{\sqrt{1+\varepsilon}}A}(X),$$

we see that L_A has the $\mathbf{P}_0$-property if and only if all eigenvalues of A lie in the closed right half-plane, and S_A has the $\mathbf{P}_0$-property if and only if all eigenvalues of A lie in the closed unit disk.

In what follows, we establish various interconnections between the above concepts.

Proposition 3.1. *For a continuous $F\colon V \to V$, the following implications hold*:

Strong monotonicity ⇒ strict monotonicity ⇒ Order **P** ⇒ *Jordan* **P** ⇒ **P** ⇒ $\mathbf{P}_0$,

strong monotonicity ⇒ uniform Jordan **P** ⇒ *uniform* **P** ⇒ **P**, *and monotonicity* ⇒ $\mathbf{P}_0$.

PROOF. The implications *strong monotonicity* $\Rightarrow$ *strict monotonicity*, uniform Jordan **P** $\Rightarrow$ Jordan **P**, and *Jordan* **P** $\Rightarrow$ **P** are obvious. That strict monotonicity implies order **P** follows immediately from the implication

$$x \sqcap y \leq 0 \leq x \sqcup y \quad \Rightarrow \quad \langle x, y\rangle \leq 0;$$

see Proposition 2.3. That order **P** $\Rightarrow$ Jordan **P** follows from the implication

$$x \circ y \leq 0 \quad \Rightarrow \quad x \sqcap y \leq 0 \leq x \sqcup y;$$

see Proposition 2.3.

To see that $\mathbf{P} \Rightarrow \mathbf{P}_0$, assume that F has the **P**-property, let $\varepsilon > 0$, $G(x) := F(x) + \varepsilon x$, and suppose $(x-y)\circ(G(x)-G(y)) \leq 0$, where the objects in this product operator commute. We have

$$\begin{aligned}(x-y)\circ(G(x)-G(y)) \leq 0 &\Rightarrow (x-y)\circ[(F(x)-F(y))+\varepsilon(x-y)] \leq 0\\ &\Rightarrow (x-y)\circ(F(x)-F(y))+\varepsilon(x-y)^2 \leq 0\\ &\Rightarrow (x-y)\circ(F(x)-F(y)) \leq -\varepsilon(x-y)^2 \leq 0.\end{aligned}$$

As $x-y$ and $F(x)-F(y)$ operator commute and F has the **P**-property, we have $x = y$. Thus F has the $\mathbf{P}_0$-property.

Now to prove the second set of implications, suppose that F is strongly monotone so that for some positive α, $\langle x-y, F(x)-F(y)\rangle \geq \alpha\|x-y\|^2$ for all $x, y \in V$. Using Item (b) in Proposition 2.1, we have

$$\alpha\|x-y\|^2 \leq \omega[(x-y)\circ(F(x)-F(y))]\|e\|^2.$$

This implies that F has the uniform Jordan **P**-property. That uniform **P** implies **P** follows from the fact that $x \leq 0$ if and only if $\omega(x) \leq 0$.

Finally, to show that monotonicity implies $\mathbf{P}_0$, let F be monotone and $G(x) = F(x) + \varepsilon x$ for $\varepsilon > 0$ and suppose that $(x-y)\circ(G(x)-G(y)) \leq 0$. Because $(x-y)\circ(G(x)-G(y)) \leq 0$ $\Rightarrow \langle x-y, G(x)-G(y)\rangle \leq 0$ (by Proposition 2.1) and

$$\begin{aligned}\langle x-y, G(x)-G(y)\rangle &= \langle x-y, F(x)-F(y)+\varepsilon(x-y)\rangle\\ &= \langle x-y, F(x)-F(y)\rangle + \varepsilon\|x-y\|^2,\end{aligned}$$

we have that $\langle x-y, F(x)-F(y)\rangle \leq -\varepsilon\|x-y\|^2 \leq 0$. Because F is monotone, we have $x = y$. Thus, G has the Jordan **P**-property, which implies the **P**-property. Hence, F has the $\mathbf{P}_0$-property. $\square$

Our next result deals with complementarity problems. When $F = L$ is linear with the **P**-property, one can use a result of Karamardian [16] to show that for all $q \in V$, CP(F, q) has a solution (Gowda et al. [14, Theorem 12]). In the (general) nonlinear case, Karamardian's result cannot be used. In what follows, we use degree-theoretic arguments to show that under a certain $\mathbf{R}_0$-type condition, every $\mathbf{P}_0$ complementarity problem has a solution. The usage of degree theory to prove existence results is standard; see, for example, §2.6 in Facchinei and Pang [5]. Given a bounded open set Ω in V (which is isomorphic to some R^k), a continuous function $f\colon \overline{\Omega} \to V$ such that $0 \notin f(\partial\Omega)$, we can define the (topological) degree of f with respect to Ω at 0; see Lloyd [17]. We denote this degree by $\deg(f, \Omega, 0)$.

THEOREM 3.1. *Suppose that the continuous transformation* $F\colon V \to V$ *has the* $\mathbf{P}_0$*-property, and for any* $\Delta > 0$ *in* R*, the set*

$$\{x\colon x \textit{ solves } CP(F, q),\ \|q\| \leq \Delta\} \tag{3.1}$$

is bounded. Then for any $q \in V$*,* $CP(F, q)$ *has a nonempty bounded solution set.*

Proof. We fix $q \in V$ and define $q_1 = q + F(0)$. Consider the function

$$\Phi(x) := x \sqcap [F(x) + q].$$

Define the homotopy

$$H_1(x, t) = x \sqcap [F(x) - F(0) + tq_1], \quad t \in [0, 1].$$

We have $H_1(x, 0) = x \sqcap [F(x) - F(0)]$ and $H_1(x, 1) = \Phi(x)$ for all x. Because F satisfies (3.1), the zero sets of $H_1(\cdot, t)$ (as t varies over $[0, 1]$) are uniformly bounded. Now let Ω be a bounded open set in V containing all these zero sets. Because 0 is a zero of $H_1(x, 0)$, we see that $0 \in \Omega$. Then, by the homotopy invariance of degree (Lloyd [17, Theorem 2.1.2]),

$$\deg(H_1(\cdot, 0), \Omega, 0) = \deg(H_1(\cdot, 1), \Omega, 0) = \deg(\Phi, \Omega, 0).$$

As $0 \in \Omega$, $0 \notin H_1(\partial\Omega, 0)$ and so $\operatorname{dist}(0, H_1(\partial\Omega, 0)) > 0$. Let

$$\Psi_\varepsilon(x) := x \sqcap [F(x) + \varepsilon x - F(0)]$$

for any $\varepsilon > 0$. Because $\|u \sqcap v - u \sqcap z\| \leq \|v - z\|$ by the nonexpansiveness of the projection map, we choose a small $\varepsilon > 0$ such that

$$\sup_{x \in \bar{\Omega}} \|\Psi_\varepsilon(x) - H_1(x, 0)\| < \operatorname{dist}(0, H_1(\partial\Omega, 0)).$$

We have $\deg(\Psi_\varepsilon, \Omega, 0) = \deg(H_1(\cdot, 0), \Omega, 0)$ by Lloyd [17, Theorem 2.1.2]. Thus

$$\deg(\Psi_\varepsilon, \Omega, 0) = \deg(\Phi, \Omega, 0).$$

Now define the homotopy

$$H_2(x, t) = x \sqcap [t(F(x) - F(0) + \varepsilon x) + (1 - t)x], \quad t \in [0, 1].$$

We have $H_2(x, 0) = x \sqcap x = x$ and $H_2(x, 1) = \Psi_\varepsilon(x)$ for all x.

We claim that $0 \notin H_2(\partial\Omega, t)$ for any $t \in [0, 1]$. If possible, suppose $H_2(x, t) = 0$ for some $t \in [0, 1]$ and $x \in \partial\Omega$. If $t = 0$, then $H_2(x, 0) = 0$ implies that $x = 0$, which is a contradiction (because $0 \in \Omega$). If $t \neq 0$, then from $H_2(x, t) = 0$, we have

$$x \geq 0, \ (F(x) - F(0) + \varepsilon x) + \left(\frac{1}{t} - 1\right)x \geq 0 \quad \text{and} \quad x \circ \left[(F(x) - F(0) + \varepsilon x) + \left(\frac{1}{t} - 1\right)x\right] = 0.$$

Now because F has the $\mathbf{P}_0$-property, the function $G(x) := F(x) + (\varepsilon + 1/t - 1)x$ has the $\mathbf{P}$-property. Now x and $G(x) - G(0)$ operator commute (see Proposition 2.2) and $(x - 0) \circ (G(x) - G(0)) = 0$. Hence $x = 0$, which leads to a contradiction. Hence the claim.

Now by the homotopy invariance of degree, we have

$$\deg(H_2(\cdot, 0), \Omega, 0) = \deg(H_2(\cdot, 1), \Omega, 0) = \deg(\Psi_\varepsilon, \Omega, 0).$$

Because $\deg(H_2(\cdot, 0), \Omega, 0) = 1$, we have $\deg(\Phi, \Omega, 0) = \deg(\Psi_\varepsilon, \Omega, 0) = 1$ which implies that the equation $\Phi(x) = 0$ has a solution (Lloyd [17, Theorem 2.1.1]). This solution solves $CP(F, q)$. By the imposed condition (3.1), CP(F, q) has a bounded solution set. □

In the setting of linear complementarity problems, a matrix M is said to have the $\mathbf{R}_0$-property if the homogeneous problem LCP$(M, 0)$ has only the trivial solution. This condition is equivalent to saying that for any $\Delta > 0$, the set $\{x\colon x \text{ solves } LCP(M, q), \|q\| \leq \Delta\}$ is a bounded set. Our condition (3.1) is a nonlinear analog of this boundedness assumption. In Definition 3.2, we formulate an $\mathbf{R}_0$ condition on F that implies (3.1). We recall that

$$\omega(z) = \max_{1 \leq i \leq r} \lambda_i(z) \qquad \text{and} \qquad \nu(z) = \min_{1 \leq i \leq r} \lambda_i(z).$$

Definition 3.2. A continuous transformation $F: V \to V$ is said to have the $\mathbf{R}_0$-property if the following condition holds: For any sequence $x^{(k)}$ in V with

$$\|x^{(k)}\| \to \infty, \qquad \liminf \frac{\nu(x^{(k)})}{\|x^{(k)}\|} \geq 0 \qquad \text{and} \qquad \liminf \frac{\nu(F(x^{(k)}))}{\|x^{(k)}\|} \geq 0, \tag{3.2}$$

we have $\liminf \omega((x^{(k)} \circ F(x^{(k)}))/\|x^{(k)}\|^2) > 0$.

The above condition is a variation of a condition used in Chen and Harker [1] for nonlinear complementarity problems; see also §5 in Gowda and Tawhid [15]. In the case of linear F, it is easily seen that the above condition reduces to the statement that CP$(L, 0)$ has only one solution, namely, zero.

In what follows, we describe two conditions under which the $\mathbf{R}_0$-property holds.

Proposition 3.2. *Suppose F satisfies either the uniform Jordan* **P***-property or the following: For any sequence $x^{(k)}$ in V with*

$$\|x^{(k)}\| \to \infty, \qquad \liminf \frac{\nu(x^{(k)})}{\|x^{(k)}\|} \geq 0 \qquad \textit{and} \qquad \liminf \frac{\nu(F(x^{(k)}))}{\|x^{(k)}\|} \geq 0,$$

we have $\liminf \langle x^{(k)}, F(x^{(k)})\rangle/\|x^{(k)}\|^2 > 0$. *Then F has the $\mathbf{R}_0$-property.*

Proof. Let $x^{(k)}$ be a sequence in V such that

$$\|x^{(k)}\| \to \infty, \qquad \liminf \frac{\nu(x^{(k)})}{\|x^{(k)}\|} \geq 0 \qquad \text{and} \qquad \liminf \frac{\nu(F(x^{(k)}))}{\|x^{(k)}\|} \geq 0.$$

Now suppose that F has the uniform Jordan **P**-property. Then for all large k we have

$$0 < \alpha \leq \omega\left(\frac{(x^{(k)} - 0) \circ (F(x^{(k)}) - F(0))}{\|x^{(k)}\|^2}\right) = \omega\left(\frac{x^{(k)} \circ F(x^{(k)})}{\|x^{(k)}\|^2} - \frac{x^{(k)}}{\|x^{(k)}\|} \circ \frac{F(0)}{\|x^{(k)}\|}\right).$$

Letting $k \to \infty$ and using Proposition 2.1, we have

$$\liminf \omega\left(\frac{x^{(k)} \circ F(x^{(k)})}{\|x^{(k)}\|^2}\right) > 0,$$

proving the $\mathbf{R}_0$-property. If F satisfies the other condition, then the $\mathbf{R}_0$-property follows from Item (b) in Proposition 2.1. □

Remark 3.2. It is not clear if uniform **P**-property implies the $\mathbf{R}_0$-property.

Proposition 3.3. *If F has the $\mathbf{R}_0$-property, then for any $\Delta > 0$, the set*

$$\{x\colon x \textit{ solves } CP(F, q), \; \|q\| \leq \Delta\}$$

is bounded.

Proof. Suppose the described set is not bounded. Then there exists a sequence $q^{(k)}$ with $\|q^{(k)}\| \leq \Delta$ and a sequence $x^{(k)}$ with $\|x^{(k)}\| \to \infty$ such that

$$x^{(k)} \geq 0, \quad y^{(k)} = F(x^{(k)}) + q^{(k)} \geq 0 \quad \text{and} \quad x^{(k)} \circ y^{(k)} = 0, \qquad \forall\, k.$$

Because $x^{(k)} \geq 0$, we have $\nu(x^{(k)}) \geq 0$ for all k and hence $\liminf \nu(x^{(k)})/\|x^{(k)}\| \geq 0$. Also, because $y^{(k)} \geq 0$ and $q^{(k)}$ is bounded, we have from Proposition 2.1,

$$\liminf \frac{\nu(F(x^{(k)}))}{\|x^{(k)}\|} = \liminf \frac{\nu(F(x^{(k)}) + q^{(k)})}{\|x^{(k)}\|} = \liminf \frac{\nu(y^{(k)})}{\|x^{(k)}\|} \geq 0.$$

By the imposed $\mathbf{R}_0$-condition, we have

$$\liminf \omega\left(\frac{x^{(k)} \circ F(x^{(k)})}{\|x^{(k)}\|^2}\right) > 0.$$

However, $(x^{(k)} \circ F(x^{(k)}))/\|x^{(k)}\|^2 = (x^{(k)} \circ y^{(k)})/\|x^{(k)}\|^2 - (x^{(k)} \circ q^{(k)})/\|x^{(k)}\|^2 \to 0$ as $x^{(k)} \circ y^{(k)} = 0$ and $q^{(k)}$ is bounded. From Proposition 2.1, this yields

$$\liminf \omega\left(\frac{x^{(k)} \circ F(x^{(k)})}{\|x^{(k)}\|^2}\right) = 0,$$

which is a contradiction. Hence the given set is bounded. □

COROLLARY 3.1. *Suppose F has $\mathbf{P}_0$ and $\mathbf{R}_0$ properties. Then for all $q \in V$, the solution set of $CP(F, q)$ is nonempty and bounded. Moreover, there exists an $\bar{x} \in V$ such that*

$$\bar{x} > 0 \qquad \text{and} \qquad F(\bar{x}) > 0.$$

PROOF. In view of the previous proposition and Theorem 3.1, CP(F, q) has a nonempty bounded solution set for all q. In particular, CP$(F, -e)$ has a solution, say x. Then $x \geq 0$ and $F(x) - e \geq 0$, yielding $F(x) \geq e > 0$. By continuity, there exists $\bar{x} \in V$ such that $\bar{x} > 0$ and $F(\bar{x}) > 0$. □

REMARK 3.3. Suppose $F = L$ is linear. The above corollary implies that when L has the **P**-property, there exists $\bar{x} \in V$ such that $\bar{x} > 0$ and $F(\bar{x}) > 0$.

It is interesting to note that the converse of the above statement holds in the case of Lyapunov and Stein transformations L_A and S_A defined on $\mathcal{S}^n$; see Gowda and Song [11] and Gowda and Parthasarathy [10].

4. A necessary condition for the GUS-property. Recall that a continuous transformation F: $V \to V$ is said to have the **GUS**-property if for all $q \in V$, CP(F, q) has a unique solution. In what follows, we will provide a necessary condition for the **GUS** property. To motivate the next result, consider the linear case. In this setting (see Gowda et al. [14]),

$$\mathbf{GUS} \Rightarrow \mathbf{P}.$$

In the context of R^n with componentwise product, the diagonal of a **P**-matrix has positive entries. In the context of $V = \mathcal{S}^n$, if a linear transformation L has the **GUS**-property, then the (i, i) entry of $L(E_i)$ is nonnegative for all $i = 1, 2. \ldots, n$ where E_i is an $n \times n$ matrix with one in the (i, i) slot and zeros elsewhere; see Theorem 8 in Gowda and Song [11] and its corrected version in Gowda and Song [12]. This statement is false if L has only the **P**-property. (To see an example, let A be a 2×2 real positive stable matrix with $(1, 1)$-entry negative. Then the Lyapunov transformation L_A (see Introduction) has the **P**-property, yet $(L_A(E_1))_{11}$ is negative.) The above result on $\mathcal{S}^n$ was used to characterize Lyapunov transformations L_A that have the **GUS**-property: L_A has the **GUS**-property on $\mathcal{S}^n$ if and only if A is positive stable and positive semidefinite (Gowda and Song [11, Theorem 9]).

The following result is a generalization of the abovementioned results for a nonlinear transformation on a Euclidean Jordan algebra.

THEOREM 4.1. *If F: $V \mapsto V$ has the **GUS**-property, then for any primitive idempotent $c \in V$, $\langle F(c) - F(0), c\rangle \geq 0$.*

We begin with two lemmas.

LEMMA 4.1. *Let V be a Euclidean Jordan algebra with rank $r > 1$. Let a Jordan frame $\{e_1, \ldots, e_r\}$ and an element $p_{12} \in V_{12}$ be given. Then for all large positive λ*

$$e_1 + \lambda e_2 + p_{12} \geq 0.$$

PROOF. For $\lambda \in R$, let $p = e_1 + \lambda e_2 + p_{12}$. Note that $p \geq 0$ if and only if $\langle x, x \circ p\rangle = \langle x^2, p\rangle \geq 0$ for all $x \in V$. Consider any $x \in V$ with the corresponding Peirce decomposition:

$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$. Using the properties of V_{ij} (see Theorem 2.2), we have

$$\begin{aligned} x \circ p &= \left(\sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij} \right) \circ (e_1 + \lambda e_2 + p_{12}) \\ &= x_1 e_1 + \sum_{1<j} \frac{1}{2} x_{1j} + \lambda \left(x_2 e_2 + \frac{1}{2} x_{12} + \sum_{2<j} \frac{1}{2} x_{2j} \right) + \frac{1}{2} x_1 p_{12} + \frac{1}{2} x_2 p_{12} \\ &\quad + p_{12} \circ x_{12} + \sum_{2<j} p_{12} \circ x_{1j} + \sum_{2<j} p_{12} \circ x_{2j}. \end{aligned}$$

Once again using the properties of V_{ij} (particularly the orthogonality of these spaces), we have

$$\begin{aligned} \langle x, x \circ p \rangle &= x_1^2 \|e_1\|^2 + \frac{1}{2} \sum_{1<j} \|x_{1j}\|^2 + \lambda \left(x_2^2 \|e_2\|^2 + \frac{1}{2} \|x_{12}\|^2 + \frac{1}{2} \sum_{2<j} \|x_{2j}\|^2 \right) \\ &\quad + \frac{1}{2} x_1 \langle p_{12}, x_{12} \rangle + \frac{1}{2} x_2 \langle p_{12}, x_{12} \rangle + \sum_{2<j} \langle p_{12} \circ x_{1j}, x_{2j} \rangle \\ &\quad + \sum_{2<j} \langle p_{12} \circ x_{2j}, x_{1j} \rangle + x_1 \langle e_1, p_{12} \circ x_{12} \rangle + x_2 \langle e_2, p_{12} \circ x_{12} \rangle. \end{aligned} \tag{4.1}$$

Because $\langle e_1, p_{12} \circ x_{12} \rangle = \langle e_1 \circ p_{12}, x_{12} \rangle = \frac{1}{2} \langle p_{12}, x_{12} \rangle$, $\langle e_2, p_{12} \circ x_{12} \rangle = \langle e_2 \circ p_{12}, x_{12} \rangle = \frac{1}{2} \langle p_{12}, x_{12} \rangle$, and $\langle p_{12} \circ x_{2j}, x_{1j} \rangle = \langle p_{12} \circ x_{1j}, x_{2j} \rangle$, we have

$$\begin{aligned} \langle x, x \circ p \rangle &= x_1^2 \|e_1\|^2 + x_1 \langle p_{12}, x_{12} \rangle + x_2 \langle p_{12}, x_{12} \rangle + \lambda x_2^2 \|e_2\|^2 + \frac{1}{2} \sum_{1<j} \|x_{1j}\|^2 \\ &\quad + \frac{\lambda}{2} \|x_{12}\|^2 + \frac{\lambda}{2} \sum_{2<j} \|x_{2j}\|^2 + \sum_{2<j} \langle p_{12} \circ x_{1j}, x_{2j} \rangle + \sum_{2<j} \langle p_{12} \circ x_{1j}, x_{2j} \rangle \\ &\geq \left[x_1^2 \|e_1\|^2 - |x_1| \|p_{12}\| \|x_{12}\| + \frac{\lambda - 2}{2} \|x_{12}\|^2 \right] \\ &\quad + \left[\lambda x_2^2 \|e_2\|^2 - |x_2| \|p_{12}\| \|x_{12}\| + \|x_{12}\|^2 \right] \\ &\quad + \frac{1}{2} \left[\sum_{2<j} \left(\|x_{1j}\|^2 - 4\delta \|x_{1j}\| \|x_{2j}\| + \lambda \|x_{2j}\|^2 \right) \right]. \end{aligned} \tag{4.2}$$

In the derivation of the above, we have used the following inequalities:

$$\begin{aligned} x_1 \langle p_{12}, x_{12} \rangle &\geq -|x_1| \|p_{12}\| \|x_{12}\|, \\ x_2 \langle p_{12}, x_{12} \rangle &\geq -|x_2| \|p_{12}\| \|x_{12}\|, \\ \langle p_{1j} \circ x_{1j}, x_{2j} \rangle &= \langle L_{p_{1j}} x_{1j}, x_{2j} \rangle \geq -\|L_{p_{1j}}\| \|x_{1j}\| \|x_{2j}\| \\ &= -\delta \|x_{1j}\| \|x_{2j}\|, \end{aligned}$$

where $\delta = \|L_{p_{1j}}\|$ is the norm of the bounded linear transformation $L_{p_{1j}}$.

The three terms on the right-hand side of (4.2) involve quadratic expressions; they are nonnegative if $\|p_{12}\|^2 - 2(\lambda - 2)\|e_1\|^2 \leq 0$, $\|p_{12}\|^2 - 4\lambda \|e_2\|^2 \leq 0$ and $16\delta^2 - 4\lambda \leq 0$. So when

$$\lambda \geq \max \left\{ \frac{\|p_{12}\|^2}{2\|e_1\|^2} + 2, \frac{\|p_{12}\|^2}{4\|e_2\|^2}, 4\delta^2 \right\},$$

we see that $\langle x^2, p \rangle \geq 0$ for all $x \in V$. In this situation, $p \geq 0$. $\square$

Lemma 4.2. *Let V be a Euclidean Jordan algebra with rank $r > 1$. Let a Jordan frame $\{e_1, \ldots, e_r\}$ and elements $p_{1j} \in V_{1j}$ $(1 < j)$ be given. Then for all large positive λ we have*

$$e_1 + \lambda \sum_{i=2}^{r} e_i + \sum_{1<j} p_{1j} \geq 0.$$

Proof. By Lemma 4.1, we can find a positive number $\hat{\lambda}$ such that

$$e_1 + \hat{\lambda} e_{i+1} + (r-1) p_{1i+1} \geq 0, \quad \forall\, i = 1, \ldots, r-1.$$

Adding these inequalities, we get

$$(r-1)e_1 + \hat{\lambda} \sum_{i=2}^{r} e_i + (r-1) \sum_{1<j} p_{1j} \geq 0. \tag{4.3}$$

This yields

$$e_1 + \frac{\hat{\lambda}}{r-1} \sum_{i=2}^{r} e_i + \sum_{1<j} p_{1j} \geq 0.$$

Putting $\lambda = \hat{\lambda}/(r-1)$, we get the desired result. □

Proof of Theorem 4.1. Let c be a primitive idempotent in V and suppose $\langle F(c) - F(0), c\rangle < 0$. First assume that $r > 1$. Corresponding to $e_1 := c$, there exists a Jordan frame $\{e_1, e_2 \ldots, e_r\}$ in V, the existence of which can be seen by considering the spectral decomposition of $e - e_1$. Now consider the Peirce decomposition

$$F(e_1) - F(0) = \theta_1 e_1 + \theta_2 e_2 + \cdots + \theta_r e_r + \sum_{i<j} \theta_{ij},$$

where $\theta_1 = \langle F(e_1) - F(0), e_1\rangle / \|e_1\|^2 < 0$. Define

$$q = -\theta_1 e_1 + \lambda(e_2 + \cdots + e_r) - \sum_{1<j} \theta_{1j}.$$

Then we have $F(e_1) - F(0) + q = \sum_{i=2}^{r} (\lambda + \theta_i) e_i + \sum_{2 \leq i < j} \theta_{ij}$. Now consider the eigenspace

$$V_{\{e_2, e_3, \ldots, e_r\}} = \{x \in V\colon x \circ (e_2 + e_3 + \cdots + e_r) = x\}.$$

It is known that this space is actually a Euclidean Jordan subalgebra of V (Faraut and Korányi [6, p. 72]). Also, every e_i and θ_{ij} for $2 \leq i < j$ belongs to this algebra; same goes for $\sum_{i=2}^{r} \theta_i e_i + \sum_{2 \leq i < j} \theta_{ij}$. Because $e_2 + e_3 + \cdots + e_r$ is the unit element in this subalgebra (hence, belongs to the interior of the symmetric cone in this subalgebra), we can take a large λ so that $F(e_1) - F(0) + q \geq 0$. In view of Lemma 4.1, we can also assume that $q \geq 0$. However, then it is easy to verify that e_1 and 0 are two solutions of $\mathrm{CP}(F, -F(0) + q)$, contradicting the **GUS**-property of F. Hence the result.

When $r = 1$, V is isomorphic to R. In this case, let $F(e_1) - F(0) = \theta_1 e_1$. Put $q = -\theta_1 e_1$ and proceed as before. □

Corollary 4.1. *If $L\colon V \to V$ is linear with the* **GUS***-property, then $\langle L(c), c\rangle \geq 0$ for all primitive idempotents. In the case of $V = \mathcal{L}^n$, this necessary condition reduces to: $\langle L(z), z\rangle \geq 0$ for all z on the boundary of $\mathcal{L}^n_+$.*

Proof. The first statement follows immediately from Theorem 4.1. Now suppose that $V = \mathcal{L}^n$. In this case, every nonzero element z on the boundary of $\mathcal{L}^n_+$ is a multiple of $c := \frac{1}{2}\begin{bmatrix} 1 \\ u \end{bmatrix}$ for some unit vector $u \in R^{(n-1)}$. Now c is a primitive idempotent, so $\langle L(c), c\rangle \geq 0$. From this we get $\langle L(z), z\rangle \geq 0$. □

5. The relaxation transformation. In this section, we apply the ideas of the previous sections to study a transformation $F = R_\phi$: $V \to V$ that arises from a vector function ϕ: $R^n \to R^n$.

Suppose we are given a Jordan frame $\{e_1, \ldots, e_r\}$ in V and a continuous function ϕ: $R^r \to R^r$. We define R_ϕ: $V \to V$ as follows. For any $x \in V$, write the Peirce decomposition

$$x = \sum_1^r x_i e_i + \sum_{i<j} x_{ij}.$$

Then

$$R_\phi(x) := \sum_1^r \tilde{x}_i e_i + \sum_{i<j} x_{ij},$$

where

$$[\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_r]^T = \phi\big([x_1, x_2, \ldots, x_r]^T\big).$$

This is a generalization of a concept introduced in Gowda and Song [13] for $V = \mathcal{S}^n$ and $\phi = A \in R^{n\times n}$. Our objective in this section is to study some interconnections between properties of ϕ and the properties of R_ϕ. Such a study has found to be quite interesting and useful in the context of matrix-based linear transformations on $V = \mathcal{S}^n$, particularly the Lyapunov and Stein transformations L_A and S_A; see Introduction. It should be noted that while the definition of R_ϕ depends on a specific Jordan frame, the results below (Proposition 5.1 and Theorem 5.1) do not.

Proposition 5.1. *The following are equivalent*:
(i) *ϕ is a* **P**-*function.*
(ii) *R_ϕ has the order* **P**-*property.*
(iii) *R_ϕ has the Jordan* **P**-*property.*
(iv) *R_ϕ has the* **P**-*property.*

Proof. (i) ⇒ (ii): Assume that ϕ is a **P**-function and let

$$(u-v) \sqcap (R_\phi(u) - R_\phi(v)) \le 0 \le (u-v) \sqcup (R_\phi(u) - R_\phi(v)).$$

Let

$$u = \sum_1^r u_i e_i + \sum_{i<j} u_{ij} \qquad \text{and} \qquad v = \sum_1^r v_i e_i + \sum_{i<j} v_{ij}.$$

We have

$$R_\phi(u) = \sum_1^r \tilde{u}_i e_i + \sum_{i<j} u_{ij} \qquad \text{and} \qquad R_\phi(v) = \sum_1^r \tilde{v}_i e_i + \sum_{i<j} v_{ij}.$$

Letting $x_i = u_i - v_i$, $y_i = \tilde{u}_i - \tilde{v}_i$ and $x_{ij} = u_{ij} - v_{ij}$, we have

$$\begin{aligned} 0 \ge (u-v) \sqcap (R_\phi(u) - R_\phi(v)) &= \sum_1^r x_i e_i + \sum_{i<j} x_{ij} - \left[\sum_1^r (x_i - y_i) e_i\right]^+ \\ &= \sum_1^r x_i e_i + \sum_{i<j} x_{ij} - \sum_1^r (x_i - y_i)^+ e_i. \end{aligned} \tag{5.1}$$

Taking the inner product of the above quantity with e_i and using Theorem 2.2, we get

$$x_i - (x_i - y_i)^+ \le 0 \quad \Rightarrow \quad \min\{x_i, y_i\} \le 0. \tag{5.2}$$

Similarly,

$$0 \le (u-v) \sqcup (R_\phi(u) - R_\phi(v)) = \sum_1^r y_i e_i + \sum_{i<j} x_{ij} + \sum_1^r (x_i - y_i)^+ e_i \tag{5.3}$$

yields

$$y_i + (x_i - y_i)^+ \geq 0 \quad \Rightarrow \quad \max\{x_i, y_i\} \geq 0. \tag{5.4}$$

From $\min\{x_i, y_i\} \leq 0$ and $\max\{x_i, y_i\} \geq 0$, we have $x_i y_i \leq 0$. Because this is true for all $i = 1, 2, \ldots, r$ we have

$$\begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_r - v_r \end{bmatrix} * \left(\phi \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - \phi \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \right) \leq 0. \tag{5.5}$$

Because ϕ is a **P**-function, we have $u_i = v_i$, hence $\tilde{u}_i = \tilde{v}_i$ for all $i = 1, 2, \ldots, r$.

Now putting $u_i = v_i$, $\tilde{u}_i = \tilde{v}_i$, that is $x_i = 0$ and $y_i = 0$ in (5.1) and (5.3), we get

$$\sum_{i<j} x_{ij} \leq 0 \qquad \text{and} \qquad \sum_{i<j} x_{ij} \geq 0.$$

Thus we have $\sum_{i<j} x_{ij} = 0$ and, hence, $u = v$. So $R_\phi(x)$ has the order **P**-property.

The implications (ii) ⇒ (iii) and (iii) ⇒ (iv) follow from Proposition 3.1.

To see (iv) ⇒ (i): Let

$$\begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_r - v_r \end{bmatrix} * \left(\phi \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - \phi \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \right) \leq 0, \tag{5.6}$$

and define $u - v = \sum_{i=1}^r (u_i - v_i) e_i$ and $R_\phi(u) - R_\phi(v) = \sum_{i=1}^r (\tilde{u}_i - \tilde{v}_i) e_i$. We then have

$$(u - v) \circ (R_\phi(u) - R_\phi(v)) = \sum_{i=1}^r (u_i - v_i)(\tilde{u}_i - \tilde{v}_i) e_i.$$

From (5.6) it follows that

$$(u_i - v_i)(\tilde{u}_i - \tilde{v}_i) \leq 0,$$

and so we have $(u - v) \circ (R_\phi(u) - R_\phi(v)) \leq 0$. Because $u - v$ and $R_\phi(u) - R_\phi(v)$ operator commute (as they share the same Jordan frame), by condition (iv), $u = v$. Thus, ϕ is a **P**-function. □

It is easy to verify that the monotonicity properties of ϕ are carried over to R_ϕ. However, it is not clear if the uniform **P**-properties are carried over. Yet, as far as the complementarity problems are concerned, we have the following proposition.

PROPOSITION 5.2. *Suppose that ϕ: $R^n \to R^n$ has the $\mathbf{P}_0$- and the $\mathbf{R}_0$-properties. Then R_ϕ satisfies condition (3.1) and has the $\mathbf{P}_0$-property. Hence, for every $q \in V$, $CP(R_\phi, q)$ has a solution.*

PROOF. Suppose that condition (3.1) fails for $F = R_\phi$. Then there exists a sequence $q^{(k)}$ with $\|q^{(k)}\| \leq \Delta$ and a sequence $\{x^{(k)}\}$ with $\|x^{(k)}\| \to \infty$ such that

$$x^{(k)} \geq 0, \qquad y^{(k)} = R_\phi(x^{(k)}) + q^{(k)} \geq 0 \qquad \text{and} \qquad x^{(k)} \circ y^{(k)} = 0, \quad \forall\, k. \tag{5.7}$$

Because $u = \sum u_i e_i + \sum_{i<j} u_{ij} \geq 0$ implies that $u_i \geq 0$ for all $i = 1, 2 \ldots, r$, we have

$$x_i^{(k)} \geq 0 \qquad \text{and} \qquad y_i^{(k)} \geq 0$$

for all i where $x^{(k)} = \sum x_i^{(k)} e_i + \sum_{i<j} x_{ij}^{(k)}$ and $y^{(k)} = \sum y_i^{(k)} e_i + \sum_{i<j} y_{ij}^{(k)}$ are the Peirce decompositions of $x^{(k)}$ and $y^{(k)}$. Let

$$\widehat{x^{(k)}} = \left[x_1^{(k)}, x_2^{(k)}, \ldots, x_r^{(k)}\right]^T,$$

and ϕ_i denote the ith component of ϕ. Now $\langle x^{(k)}, y^{(k)}\rangle = 0$ implies that

$$\sum_1^r x_i^{(k)}\Big[\phi_i(\widehat{x^{(k)}}) + q_i^{(k)}\Big]\|e_i\|^2 + \sum_{i<j}\Big[\|x_{ij}^{(k)}\|^2 + \langle x_{ij}^{(k)}, q_{ij}^{(k)}\rangle\Big] = 0.$$

Because the first term in the above expression is nonnegative and $q_{ij}^{(k)}$ is bounded by $\|q^{(k)}\|$ (which is bounded by Δ), we see that the sequence $\|x_{ij}^{(k)}\|$ is bounded for every pair (i, j) with $i < j$. From $\|x^{(k)}\| \to \infty$, we conclude that $\|\widehat{x^{(k)}}\|_2$ (the 2-norm of $\widehat{x^{(k)}}$) goes to ∞. Because $x_i^{(k)}$ and $y_i^{(k)}$ are nonnegative for all $i = 1$ to r, we have $\liminf(\min_i x_i^{(k)}/\|\widehat{x^{(k)}}\|_2) \geq 0$ and $\liminf(\min_i \phi_i(\widehat{x^{(k)}})/\|\widehat{x^{(k)}}\|_2) \geq 0$ where we have used the boundedness of $q^{(k)}$. By the $\mathbf{R}_0$-property of ϕ (as defined on the algebra R^r of Example 1), we have

$$\liminf \frac{\max_i x_i^{(k)}\phi_i(\widehat{x^{(k)}})}{\|\widehat{x^{(k)}}\|_2^2} > 0.$$

We may assume by going through a subsequence, if necessary, that for some index i, say $i = 1$, we have

$$\lim \frac{x_1^{(k)}\phi_1(\widehat{x^{(k)}})}{\|\widehat{x^{(k)}}\|_2^2} > 0.$$

Now using the properties of V_{ij}, we see that

$$e_1 \circ x^{(k)} = x_1^{(k)} + \frac{1}{2}\sum_{1<j} x_{1j}^{(k)},$$

so

$$0 = \langle e_1, x^{(k)} \circ y^{(k)}\rangle = \langle e_1 \circ x^{(k)}, y^{(k)}\rangle = x_1^{(k)}(\phi_1(\widehat{x^{(k)}}) + q_1^{(k)}) + \frac{1}{2}\sum_{1<j}\big\langle x_{1j}^{(k)}, x_{1j}^{(k)} + q_{1j}^{(k)}\big\rangle.$$

Dividing this expression by $\|\widehat{x^{(k)}}\|_2^2$ and taking the limit, we see that $\lim x_1^{(k)}\phi_1(\widehat{x^{(k)}})/\|\widehat{x^{(k)}}\|_2^2 = 0$ which is a contradiction. This shows that the condition (3.1) holds.

We now show that R_ϕ has the $\mathbf{P}_0$-property. For any $\varepsilon > 0$, let $G(u) = R_\phi(u) + \varepsilon u$, $G(v) = R_\phi(v) + \varepsilon v$, and suppose that

$$(u - v) \circ (G(u) - G(v)) \leq 0.$$

Upon writing $R_\phi(u) = \sum_1^r \tilde{u}_i e_i + \sum_{i<j} u_{ij}$, $R_\phi(v) = \sum_1^r \tilde{v}_i e_i + \sum_{i<j} v_{ij}$, $x_i = u_i - v_i$ and $y_i = \tilde{u}_i - \tilde{v}_i$ and $x_{ij} = u_{ij} - v_{ij}$, we have

$$\langle (u-v)\circ(G(u)-G(v)), e_i\rangle \leq 0 \;\;\Rightarrow\;\; x_i(y_i+\varepsilon x_i)\|e_i\|^2 + \frac{1+\varepsilon}{2}\sum_{i<j}\|x_{ij}\|^2 \leq 0,\;\; i = 1, 2, \ldots, r.$$

Thus we have

$$x_i(y_i + \varepsilon x_i) \leq 0, \quad i = 1, 2, \ldots, r,$$

which implies that

$$\begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_r - v_r \end{bmatrix} * \left(\bar{\phi}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - \bar{\phi}\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix}\right) \leq 0,$$

where

$$\bar{\phi}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = \phi\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + \varepsilon\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \quad \bar{\phi}\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} = \phi\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} + \varepsilon\begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix}.$$

Because ϕ has the $\mathbf{P}_0$-property, $\bar{\phi}$ is $\mathbf{P}$-function. Hence $x_i = 0$ for all $i = 1, 2, \ldots, r$, i.e., $u_i = v_i$ for all i. It follows that

$$\frac{1+\varepsilon}{2} \sum_{i<j} \|x_{ij}\|^2 \le 0, \quad i = 1, 2, \ldots, r.$$

Thus we have $x_{ij} = 0$, $\forall\, i < j$ proving $u = v$. Therefore R_ϕ has the $\mathbf{P}_0$-property. Consequently, for all $q \in V$, CP(ϕ, q) has a solution by Theorem 3.1. □

Now suppose that $\phi(x) = Ax$ where A is an $r \times r$ real matrix. We write R_A for R_ϕ. From Proposition 5.1, we see that A is a $\mathbf{P}$-matrix if and only if R_A has the $\mathbf{P}$-property. The following question naturally arises: When A is a $\mathbf{P}$-matrix, every LCP(M, q) for $q \in R^n$ has a unique solution; how about the corresponding R_A? Will it have the **GUS**-property? Below, we will provide an answer to this question in the negative.

We recall that a matrix A is *copositive* on R^n if $\langle Ax, x\rangle \ge 0$ for all $x \in R^n_+$. In what follows, E denotes a square matrix with zero diagonal entries and ones elsewhere.

THEOREM 5.1. *The following statements hold*:

(i) *When* $V = \mathcal{L}^2$, R_A *has the* **GUS**-*property if and only if* A *is a* $\mathbf{P}$-*matrix.*

(ii) *If* $V = \mathcal{L}^n$ $(n > 2)$ *and* R_A *has the* **GUS**-*property, then* A *is a* $\mathbf{P}$-*matrix and* $A + E$ *is copositive on* R^2.

(iii) *If* $V = \mathcal{S}^n$ *and* R_A *has the* **GUS**-*property, then* A *is a* $\mathbf{P}$-*matrix and* $A + E$ *is copositive on* R^n.

PROOF. (i) When $V = \mathcal{L}^2$, the cone $\mathcal{L}^2_+$ is polyhedral. In this setting, the $\mathbf{P}$- and **GUS**-properties coincide for a linear transformation; see Theorem 23 in Gowda et al. [14]. The result now follows from Proposition 5.1. (This can also be seen by considering the Jordan frame $\{e_1, e_2\}$ in $\mathcal{L}^n$, where $e_1 = \frac{1}{2}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $e_2 = \frac{1}{2}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ and showing that $\mathbf{P} \Rightarrow$ **GUS** by using the definitions.)

(ii) We now suppose $V = \mathcal{L}^n (n > 2)$ and R_A has the **GUS**-property. As the $\mathbf{P}$-property is clear, we verify that $A + E$ is copositive on R^2. Consider a Jordan frame $\{e_1, e_2\}$ in $\mathcal{L}^n$ with respect to which R_A is defined. We may write $e_1 = \frac{1}{2}\left[\begin{smallmatrix}1\\u\end{smallmatrix}\right]$, $e_2 = \frac{1}{2}\left[\begin{smallmatrix}1\\-u\end{smallmatrix}\right]$ where $\|u\| = 1$. Let z_1 and z_2 be two nonnegative numbers. As $n > 2$, we can pick a vector v in R^{n-1} such that $\langle u, v\rangle = 0$ and $\|v\|^2 = z_1 z_2$ and define

$$z := z_1 e_1 + z_2 e_2 + \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Then

$$R_A(z) = w_1 e_1 + w_2 e_2 + \begin{bmatrix} 0 \\ v \end{bmatrix},$$

where $[w_1, w_2]^T = A([z_1, z_2]^T)$. We easily verify that $z_{12} := \left[\begin{smallmatrix}0\\v\end{smallmatrix}\right]$ belongs to V_{12} and $z \in \partial\mathcal{L}^n_+$ by direct computation. By Corollary 4.1, we have

$$\begin{aligned}
0 \le \langle R_A(z), z\rangle &= \frac{z_1 w_1}{2} + \frac{z_2 w_2}{2} + \|z_{12}\|^2 \\
&= \frac{1}{2}\left\langle A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\rangle + z_1 z_2 \\
&= \frac{1}{2}\left\langle A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\rangle + \frac{1}{2}\left\langle E\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\rangle \\
&= \frac{1}{2}\left\langle (A+E)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\rangle. \qquad (5.8)
\end{aligned}$$

Therefore, $A+E$ is copositive on R_2.

(iii) Now suppose that $V = S^n$ and R_A has the **GUS**-property. For this case, we modify the proof presented in Gowda and Song [13].

First we prove the result for the Jordan frame $\{E_1, E_2, \ldots, E_n\}$ in S^n, where E_i is the diagonal matrix with 1 in the (i,i)-slot and zeros elsewhere.

Let $[x_1, x_2, \ldots, x_r]^T$ be a vector in R^r with $\sum_1^r x_i^2 = 1$, and U be an orthogonal matrix with this vector in its first column. It is easy to verify that the transformation

$$\widetilde{R}_A(X) := U^T R_A(UXU^T)U$$

has the **GUS**-property. Because

$$\begin{aligned}\langle \widetilde{R}_A E_1, E_1\rangle &= \langle U^T R_A(UE_1U^T)U, E_1\rangle \\ &= \langle R_A(UE_1U^T), UE_1U^T\rangle,\end{aligned}$$

and

$$UE_1U^T = \begin{bmatrix} {x_1}^2 & x_1x_2 & \cdots & x_1x_r \\ x_1x_2 & {x_2}^2 & \cdots & x_2x_r \\ \vdots & \ddots & \ddots & \vdots \\ x_1x_r & \cdots & x_{r-1}x_r & {x_r}^2 \end{bmatrix},$$

we have

$$\left\langle (A+E)\begin{bmatrix} {x_1}^2 \\ \vdots \\ {x_r}^2 \end{bmatrix}, \begin{bmatrix} {x_1}^2 \\ \vdots \\ {x_r}^2 \end{bmatrix} \right\rangle = \langle \widetilde{R}_A E_1, E_1\rangle \geq 0,$$

where the last inequality follows from Corollary 4.1. From this we easily deduce the copositivity property of $A+E$ on R^n.

Now consider a general Jordan frame $\{C_1, C_2, \ldots, C_n\}$ in $\mathcal{S}^n$. Define the relaxation transformation R_A^* with respect to this frame: For $X = \sum X_i C_i + \sum_{i<j} X_{ij}$, we have $R_A^*(X) = \sum \widetilde{X}_i C_i + \sum_{i<j} X_{ij}$, where

$$[\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_r]^T = A\big([X_1, X_2, \ldots, X_r]^T\big).$$

Because $\{C_1, C_2, \ldots, C_n\}$ is a Jordan frame and $\mathcal{S}^n$ is a simple Euclidean Jordan algebra, there exists an automorphism of $\mathcal{S}^n$ (i.e., an invertible linear transformation Λ on $\mathcal{S}^n$ such that $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all $x, y \in \mathcal{S}^n$) taking this Jordan frame to the Jordan frame $\{E_1, E_2, \ldots, E_n\}$ (Faraut and Korányi [6, Theorem IV.2.5]). The automorphisms of $\mathcal{S}^n$ are described by $X \mapsto QXQ^T$ where Q is an orthogonal matrix (Gowda et al. [14, Example 1.1]). We conclude that for some orthogonal matrix Q, $C_i = Q^T E_i Q$ for all $i = 1, 2, \ldots, n$. (This can also be seen as follows: Because the matrices in $\{C_1, C_2, \ldots, C_n\}$ commute pairwise, they are simultaneously diagonalizable by means of a (single) orthogonal matrix Q. Writing $C_i = QD_iQ^T$ where D_i is a diagonal matrix, and using the idempotent property of C_i, we see that every diagonal entry in D_i is either zero or one. Because the sum of C_is is the identity matrix, it follows that in the diagonal of every D_i exactly one entry is nonzero. We may then write, without loss of generality, $D_i = E_i$ for all i.)

Then,

$$QXQ^T = \sum X_i QC_iQ^T + \sum_{i<j} QX_{ij}Q^T = \sum X_i E_i + \sum_{i<j} Y_{ij}$$

where $Y_{ij} = QX_{ij}Q^T$. Because the transformation $Z \mapsto QZQ^T$ preserves inner/Jordan products, the last expression in the above statement is nothing but the Peirce decomposition of $Y = QXQ^T$ with respect to the Jordan frame $\{E_1, E_2, \ldots, E_n\}$. Using the definition of R_A

defined with respect to $\{E_1, E_2, \ldots, E_n\}$, we have $R_A(QXQ^T) = \sum \widetilde{X}_i E_i + \sum_{i<j} QX_{ij}Q^T$. This leads to

$$Q^T R_A(QXQ^T)Q = \sum \widetilde{X}_i C_i + \sum_{i<j} X_{ij} = R_A^*(X).$$

Now if R_A^* has the **GUS**-property, then $Q^T R_A(QXQ^T)Q$ has the **GUS**-property, or equivalently R_A (defined with respect to the Jordan frame $\{E_1, E_2, \ldots, E_n\}$) has the **GUS**-property. From the first part of the proof, we get the stated properties of A and $A+E$. $\square$

Example 5.1 (Gowda and Song [13]). Let $A = \begin{bmatrix} 1 & -10 \\ 1 & 1 \end{bmatrix}$. Then A is a P-matrix, but $A+E$ is not copositive. This means that R_A defined with respect to $\{E_1, E_2\}$ in $\mathcal{S}^2$ has the **P**-property but not the **GUS**-property.

References

[1] Chen, B., P. T. Harker. 1997. Smooth approximations to nonlinear complementarity problems. *SIAM J. Optim.* **7** 403–420.

[2] Chen, X. D., D. Sun, J. Sun. 2003. Complementarity functions and numerical experiments on some smoothing Newton methods for second-order-cone complementarity problems. *Comput. Optim. Appl.* **25** 39–56.

[3] Chen, X., P. Tseng. 2003. Non-interior continuation methods for solving semidefinite complementarity problems. *Math. Prog. Series A.* **95** 431–474.

[4] Cottle, R. W., J.-S. Pang, R. E. Stone. 1992. *The Linear Complementarity Problem*. Academic Press, Boston, MA.

[5] Facchinei, F., J.-S. Pang. 2003. *Finite Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York.

[6] Faraut, J., A. Korányi. 1994. *Analysis on Symmetric Cones*, Oxford University Press, Oxford, UK.

[7] Fiedler, M., V. Pták. 1962. On matrices with non-positive off-diagonal elements and positive principle minors. *Czechoslovak Math. J.* **12** 382–400.

[8] Fukushima, M., Z.-Q. Luo, P. Tseng. 2001. Smoothing functions for second-order-cone complementarity problems. *SIAM J. Optim.* **12** 436–460.

[9] Gårding, L. 1959. An inequality for hyperbolic polynomials. *J. Math. Mech.* **8** 957–965.

[10] Gowda, M. S., T. Parthasarathy. 2000. Complementarity forms of theorems of Lyapunov and Stein, and related results. *Linear Algebra Appl.* **320** 131–144.

[11] Gowda, M. S., Y. Song. 2000. On semidefinite linear complementarity problems. *Math. Programming Ser. A* **88** 575–587.

[12] Gowda, M. S., Y. Song. 2001. Errata: On semidefinite linear complementarity problems. *Math. Programming Ser. A* **91** 199–200.

[13] Gowda, M. S., Y. Song. 2002. *Semidefinite Relaxations of Linear Complementarity Problems*. Technical Report TRGOW02-01, Department of Mathematics & Statistics, UMBC. Baltimore, MD.

[14] Gowda, M. S., R. Sznajder, J. Tao. 2004. Some **P**-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra Appl.* **393** 203–232.

[15] Gowda, M. S., M. Tawhid. 1999. Existence and limiting behavior of trajectories associated with $\mathbf{P}_0$ equations. *Comput. Optim. Appl.* **12** 229–251.

[16] Karamardian, S. 1976. An existence theorem for the complementarity problem. *J. Optim. Theory Appl.* **19** 227–232.

[17] Lloyd, N. G. 1978. *Degree Theory*. Cambridge University Press, Cambridge, UK.

[18] Megiddo, N., M. Kojima. 1977. On the existence and uniqueness of solutions in nonlinear complementarity theory. *Math. Programming* **12** 110–130.

[19] Schmieta, H. H., F. Alizadeh. 2003. Extension of primal-dual interior point algorithms to symmetric cones. *Math. Programming Ser. A* **96** 409–438.