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Completely mixed linear games and irreducibility concepts for Z-transformations over self-dual cones

M. Seetharama Gowda[1]

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Abstract

In the setting of a self-dual cone in a finite-dimensional real inner product space, we consider (zero-sum) linear games. In our previous work, we showed that a Z-transformation with positive value is completely mixed. The present paper considers the case when the value is zero. Motivated by the matrix game result that a Z-matrix with value zero is completely mixed if and only if it is irreducible, we formulate our general results based on the concepts of cone-irreducibility and space-irreducibility. While the concept of cone-irreducibility for a positive linear transformation is well-known, we introduce space-irreducibility for a general linear transformation by reformulating the irreducibility concept of Elsner. Our main result is that for a Z-transformation with value zero, space-irreducibility is necessary and sufficient for the completely mixed property. We also extend a recent result of Parthasarathy et al. on matrix games with value zero to the setting of a symmetric cone (in a Euclidean Jordan algebra). Additionally, we present a refined cone/space-irreducibility result for positive transformations on symmetric cones.

Keywords Value of a zero-sum linear game · Completely mixed game · Z-transformation · Positive transformation · Cone and space irreducibility · Euclidean Jordan algebra · Symmetric cone

Mathematics Subject Classification 91A05 · 15B48 · 46N10 · 17C20 · 17C27

This paper is dedicated to the memory of Professor T. Parthasarathy, who passed away in 2023. A distinguished Indian mathematician, Professor Parthasarathy made substantial contributions to the areas of game theory and linear complementarity problems.

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✉ M. Seetharama Gowda
gowda@umbc.edu

1 University of Maryland, Baltimore County, Baltimore, Maryland 21250, USA

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1 Introduction

In the classical (zero-sum) matrix game setting, Kaplansky's well-known result ([16], Theorem 5) provides a necessary and sufficient condition for a matrix game with value zero to be completely mixed. About 50 years later, Kaplansky [17] gave another such result for skew-symmetric matrices of odd order in terms of the so-called Pfaffians. Considering Z-matrices (which are square real matrices with nonpositive off-diagonal entries), Raghavan [23] showed that if the corresponding matrix game has positive value, then it is completely mixed and provided several characterizations of non-singular M-matrices. In [7] and [9], the concepts of value of a matrix game and completely mixed property were generalized to the setting of a linear transformation over a self-dual cone in a finite dimensional real inner product space; for related work on symmetric games and proper cones, see [5], [6], and [20]. In [7] and [9], the results of Kaplansky [16] and Raghavan [23] were generalized and, in particular, it was shown that for a Z-transformation (which is a generalization of a Z-matrix), positive value implies the completely mixed property. In the present paper, we focus on Z-transformations with value zero and provide a necessary and sufficient condition for the completely mixed property in terms of an irreducibility condition.

To elaborate, we start with a brief description of relevant game-theoretic ideas. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be a finite dimensional real inner product space. Consider a *self-dual cone* K in $\mathcal{V}$, so K is a closed convex cone in $\mathcal{V}$ and $K = K^* := \{x \in \mathcal{V} : \langle x, y\rangle \geq 0,\ \forall\, y \in K\}$. We fix an element e in K° (the interior of K) and let

$$\Delta(e) := \{x \in K : \langle x, e\rangle = 1\},$$

the elements of which will be called 'strategies'. Given a linear transformation L on $\mathcal{V}$, the zero-sum linear game — denoted by (L, e) — is played by two players I and II in the following way: If player I chooses strategy $x \in \Delta(e)$ and player II chooses strategy $y \in \Delta(e)$, then the pay-off for player I is $\langle L(x), y\rangle$ and the pay-off for player II is $-\langle L(x), y\rangle$. Both players try to maximize their pay-offs. Since $\Delta(e)$ is a compact convex set and L is linear, by the min–max Theorem of von Neumann ([18], Theorems 1.5.1 and 1.3.1), there exist strategies $\overline{x}$ (for player I) and $\overline{y}$ (for player II) satisfying the inequalities

$$\langle L(x), \overline{y}\rangle \leq \langle L(\overline{x}), \overline{y}\rangle \leq \langle L(\overline{x}), y\rangle \quad \text{for all } x, y \in \Delta(e).$$

(So the players I and II do not gain by unilaterally changing their strategies from $\overline{x}$ and $\overline{y}$.) The real number

$$v(L, e) := \langle L(\overline{x}), \overline{y}\rangle$$

is called the *value of the game* and the pair $(\overline{x}, \overline{y})$ is called an *optimal strategy pair*. This value is also given by ([18], Theorems 1.5.1 and 1.3.1)

$$v(L, e) = \max_{x\in\Delta(e)} \min_{y\in\Delta(e)} \langle L(x), y\rangle = \min_{y\in\Delta(e)} \max_{x\in\Delta(e)} \langle L(x), y\rangle.$$

We say that the game (L, e) *is completely mixed if for every optimal strategy pair* $(\overline{x}, \overline{y})$, *both* $\overline{x}$ *and* $\overline{y}$ *belong to* K°. *(As we see below, in this case, there is a unique optimal strategy pair.)*

We note that with $\mathcal{V} = \mathcal{R}^n$ (under the usual inner product), $K = \mathcal{R}^n_+$ (nonnegative orthant), and e denoting the vector of ones, we get back to the classical *matrix-game setting*.

For ease of reference, we recall some results from [7] and [9]; here we use the notation $x \geq 0$ ($x > 0,\ x \leq 0,\ x \leq y$) to mean $x \in K$ (respectively, $x \in K^\circ$, $-x \geq 0,\ x - y \leq 0$) and write L^T for the transpose of L.

Theorem 1.1 ([7], [9]) *For the linear game* (L, e), *the following statements hold:*

1. *If* (x, y) *is an optimal strategy pair, then* $L^T(y) \leq v(L, e)\, e \leq L(x)$. *Conversely, if there exist* $x, y \in \Delta(e)$ *and a real number* v *such that* $L^T(y) \leq v\, e \leq L(x)$, *then* $v = v(L, e)$ *and* (x, y) *is an optimal strategy pair for* (L, e)*; additionally, if there exists an optimal strategy pair* $(\overline{x}, \overline{y})$ *with* $\overline{x}, \overline{y} > 0$, *then* $L^T(y) = v(L, e)\, e = L(x)$.
2. *If* (x, y) *is an optimal strategy pair for* (L, e), *then* (y, x) *is an optimal strategy pair for* $(-L^T, e)$*; moreover,* $v(-L^T, e) = -v(L, e)$.
3. $v(L, e) > 0$ *if and only if there exists* $u > 0$ *such that* $L(u) > 0$.
4. *The sign of* $v(L, e')$ *is a constant as* e' *varies over* K°.
5. *If for every optimal strategy pair* (x, y) *of* (L, e), $y > 0$ *(equivalently,* $x > 0$ *for every optimal strategy pair* (x, y)*), then* (L, e) *is completely mixed; in this case, there is a unique optimal strategy pair.*
6. *When* $v(L, e) = 0$, (L, e) *is completely mixed if and only if* $dim(ker(L)) = 1$ *and there exist* $\overline{x},\ \overline{y} > 0$ *such that for the transformation* $S := \overline{x}\,\overline{y}^T$ *(which takes any* x *to* $\langle \overline{y}, x\rangle\, \overline{x}$*), we have* $LS = SL = 0$.
7. *When* (L, e) *is completely mixed,* $v(L, e) \neq 0$ *if and only if* L *is invertible.*
8. *If* (L, e) *has value zero and is completely mixed, then for every* $e' \in K^\circ$, *the game* (L, e') *has value zero and is completely mixed.*

Motivated by matrix theory and dynamical systems considerations, some special types of linear transformations were studied in [7] and [9]. We recall that a linear transformation L on $\mathcal{V}$ (relative to a proper cone K with dual K^*) is said to be a

(*a*) *Z-transformation if* $\big[x \in K,\, y \in K^*,\, \langle x, y\rangle = 0\big] \Rightarrow \langle L(x), y\rangle \leq 0$;
(*b*) *Lyapunov-like transformation if* L *and* $-L$ *are Z-transformations, i.e.,* $\big[x \in K,\, y \in K^*,\, \langle x, y\rangle = 0\big] \Rightarrow \langle L(x), y\rangle = 0$;
(*c*) *Stein-like transformation if* $L = I - S$, *where* $S \in \overline{Aut(K)}$ *with* $Aut(K)$ *denoting the set of all bijective linear transformations from* $\mathcal{V}$ to $\mathcal{V}$ *such that* $S(K) = K$;
(*d*) *positive transformation if* $L(K) \subseteq K$.

We note that in the setting of $\mathcal{V} = \mathcal{R}^n$ with $K = \mathcal{R}^n_+$, a Z-transformation is just a Z-matrix. Also, in the setting of $\mathcal{V} = \mathcal{S}^n$ (the space of all $n \times n$ real symmetric matrices) with $K = \mathcal{S}^n_+$ (the cone of all positive semidefinite matrices in $\mathcal{S}^n$), corresponding to $A \in \mathcal{R}^{n\times n}$, the transformations defined by $L_A(X) := AX + XA^T$ and $S_A(X) := X - AXA^T$ are, respectively, examples of Lyapunov-like and Stein-like transformations. As is well-known, these two transformations, respectively called Lyapunov and Stein transformations, appear in the study of continuous and discrete dynamical systems.

We recall the following results from [9]; note that in these results, K is assumed to be self-dual.

Theorem 1.2 *The game (L, e) is completely mixed in each of the following instances:*

(1) *L is a Z-transformation with $v(L, e) > 0$.*
(2) *L is Lyapunov-like with $v(L, e) \neq 0$.*
(3) *L is Stein-like with $v(L, e) \neq 0$.*

What happens if the value is zero? Specifically, we ask:

For a Z-transformation with value zero, when is the game completely mixed?

In the matrix game setting, we have the result (see Theorem 3.1 below) that a *Z-matrix with value zero is completely mixed if and only if it is irreducible.* Motivated by this, we formulate our results based on the irreducibility concept(s). In the setting of a proper cone K, there is the known concept of K-irreducibility (to be called cone-irreducibility in this paper) for a positive transformation [1]: The only faces of K that are invariant under the transformation are $\{0\}$ and K. For a general linear transformation over a finite dimensional real inner product space (still relative to a proper cone), we define space-irreducibility by stipulating that the transformation leaves no non-trivial subspace of the form $F - F$ (where F is a face of K) invariant, see Definition 2.2 below. It turns out that this space-irreducibility concept is equivalent to the irreducibility concept introduced by Elsner [2]. Then, based on the exponential characterization of a cross-positive transformation (which is the negative of a Z-transformation) of Schneider and Vidyasagar [25] and an irreducibility result of Elsner [2], we prove our main result (in Theorem 3.2):

– *Over a self-dual cone, space-irreducibility is necessary and sufficient for a Z-transformation with value zero to be completely mixed.*

By way of an application, we show in Corollary 3.1 that a space-irreducible Z-transformation with value zero has 'almost monotone', 'trivially range monotone,' and group inverse properties; these latter properties were studied recently by Encinas, Mondal, and Sivakumar [3] for Lyapunov and Stein transformations over the space of all $n \times n$ real symmetric matrices.

Next, in Section 4 we describe several results and examples in the setting of symmetric cones (which are self-dual and homogeneous) in Euclidean Jordan algebras. In particular, we

– *generalize the following result of Parthasarathy, Ravindran, and Sunil Kumar* [22] *on matrix games: For a real $n \times n$ matrix A with $v(A) = 0$, A is completely mixed if and only if $v(A + D_i) > 0$ for all $i = 1, 2, \ldots, n$, where D_i denotes the diagonal matrix with 1 in the (i, i) slot and zeros elsewhere,*
– *provide a finer characterization of cone-irreducibility of a positive transformation T: For each $0 \neq x \geq 0$, $(I + T)^{n-1}x > 0$, where n denotes the rank of $\mathcal{V}$,*
– *characterize the completely mixed property of a symmetric/skew-symmetric Lyapunov-like transformation, and*
– *provide a necessary and sufficient condition for the space-irreducibility of a Lyapunov transformation L_A on $\mathcal{S}^n$.*

While the focus of the paper is on game-theoretic ideas/results, one may consider studying Z-transformations with value zero for their dynamical systems and complementarity properties. In the case of a Z-transformation with value positive (equivalently, positive stable, see Proposition 2.2), the *dynamical system*

$$\frac{dx}{dt} + L(x) = 0$$

is (globally) asymptotically stable, that is, starting from any initial point, the trajectory of the above system converges to zero as (time) $t \to \infty$. The question of what happens when the value is zero is left for a future study.

Given a linear transformation L, a proper cone K, and a $q \in \mathcal{V}$, the *linear complementarity problem* $\mathrm{LCP}(L, K, q)$ is to find $x \in \mathcal{V}$ such that

$$x \in K, \;\; y := L(x) + q \in K^*, \text{ and } \langle x, y\rangle = 0.$$

When L is a Z-transformation (on a self-dual cone K) with value positive, this problem has a solution for all $q \in \mathcal{V}$ [12]; in the setting of $\mathcal{V} = \mathcal{R}^n$ and $K = \mathcal{R}^n_+$, this can be described in more than 50 equivalent ways, see [1]. What happens to such properties when the value is zero? Again, we relegate this to a future study.

2 Preliminaries

Throughout this paper, $\mathcal{V}$ *denotes a finite dimensional real inner product space and* K *is a proper cone in* $\mathcal{V}$ (that is, K is a pointed closed convex cone with nonempty interior). We define the *dual* of K by

$$K^* := \{x \in \mathcal{V} : \langle x, y\rangle \geq 0, \;\forall\, y \in K\}$$

and say that K is *self-dual* if $K = K^*$. *In all game-theoretic settings, we assume that* K *is self-dual. Moreover, when* $\mathcal{V}$ *is a Euclidean Jordan algebra, we assume that* K *is the corresponding symmetric cone.* The interior, closure, boundary, and span of a set X in $\mathcal{V}$ are denoted, respectively, by X°, $\overline{X}$, $\partial(X)$, and $\mathrm{span}(X)$. For a linear transformation L, $\ker(L)$ denotes its kernel (null space). Recall that a convex cone F within (the proper cone) K is a *face* if an element a in F is of the form $b + c$ with $b, c \in K$, then $b, c \in F$. Since K is proper, every face of K is a pointed closed convex cone in $\mathcal{V}$. If F is a face of K, then $\mathrm{span}(F) := F - F$ is the subspace generated by F. For further definitions and properties regarding cones, proper cones, faces, etc., we refer to [1].

Throughout, we use the following notation: Relative to the given proper cone K in $\mathcal{V}$, we write

$$x \geq 0 \text{ when } x \in K \text{ and } x > 0 \text{ when } x \in K^\circ \,(= \text{interior of } K).$$

With K^* denoting the dual of K, we frequently use the well-known fact

$$\left[x \in K^\circ, 0 \neq y \in K^*\right] \Rightarrow \langle y, x\rangle > 0. \tag{1}$$

We freely use the game-theoretic concepts/results formulated in the Introduction. Throughout this paper, we will be focusing on describing the completely mixed property for linear transformations with value zero relative to some $e \in K^\circ$. In view of Item (8) in Theorem 1.1, the element e can be replaced by any other $e' \in K^\circ$. So,

when $v(L, e) = 0$ and (L, e) is completely mixed, we suppress e and write $v(L) = 0$ and say that L is completely mixed.

2.0.1 Z-transformations

For the given proper cone K in $\mathcal{V}$, we consider the following sets of linear transformations:

$Z(K) :=$ Set of all Z-transformations on K,
$\mathrm{LL}(K) :=$ Set of all Lyapunov-like transformations on K,
$\pi(K) :=$ Set of all positive transformations on K.

When K is the nonnegative orthant in $\mathcal{R}^n$, these become, respectively, the set of Z-matrices, diagonal matrices, and nonnegative matrices.

We recall the following results of Schneider and Vidyasagar from [25]. In this reference, the authors deal with a 'cross-positive matrix' which is the negative of a Z-transformation.

Theorem 2.1 ([25], Theorems 3 and 6)

$$L \in Z(K) \Leftrightarrow exp(-tL) \in \pi(K) \text{ for all } t \geq 0, \tag{2}$$

where $exp(L) := \sum_{k=0}^{\infty} \frac{L^k}{k!}$. Moreover, When $L \in Z(K)$,

$$\alpha := \min\{Re(\lambda) : \lambda \text{ is an eigenvalue of } L\}$$

is an eigenvalue of L with an associated eigenvector $u \in K$.

We mention one interesting consequence:

Proposition 2.1 *Suppose K is self-dual, $e \in K^\circ$, and $L \in Z(K)$. If (L, e) is completely mixed, then $v(L, e)$ and α have the same sign; in particular, when $v(L, e)$ is zero, we have $\alpha = 0$ and so L is nonnegative stable.*

Proof Assume that (L, e) is completely mixed. Then, by Theorem 1.1, there exist $\overline{x}, \overline{y} > 0$ such that $L^T(\overline{y}) = v(L, e)\, e = L(\overline{x})$. From Theorem 2.1, there exists (an eigenvector) u such that $0 \neq u \geq 0$ and $L(u) = \alpha u$; without loss of generality, let $\langle e, u\rangle = 1$. Then, $v(L, e) = \langle v(L, e)\, e, u\rangle = \langle L^T(\overline{y}), u\rangle = \langle \overline{y}, L(u)\rangle = \alpha\langle \overline{y}, u\rangle$.

As $\langle \overline{y}, u \rangle > 0$ (from (1)), we conclude that $v(L, e)$ and α have the same sign. In particular, when the $v(L, e)$ is zero, $\alpha = 0$; so, the real part of any eigenvalue of L is nonnegative, that is, L is nonnegative stable. □

We also have the following from [12], Theorem 6 and [9], Theorem 6:

Proposition 2.2 *The following are equivalent for $L \in Z(K)$:*

- *L is positive stable (that is, the real part of any eigenvalue of L is positive).*
- *There exists $u > 0$ such that $L(u) > 0$.*
- *L^{-1} exists and belongs to $\pi(K)$.*

When K is self-dual and $e \in K^\circ$, these are further equivalent to: $v(L, e) > 0$. Moreover, in this case, (L, e) is completely mixed.

2.1 Irreducibility concepts for a linear transformation relative to a proper cone

Here we describe two irreducibility concepts, both relative to a proper cone K (with dual K^*) in a finite dimensional real inner product space $\mathcal{V}$. The first one, which we call cone-irreducibility, is known. It is defined (only) for those transformations in $\pi(K)$. Recall that $L \in \pi(K)$ if L is linear and $L(K) \subseteq K$.

The following definition and result(s) are from [1], where they are stated for $\mathcal{R}^n$; by isomorphism considerations, we state them here for $\mathcal{V}$.

Definition 2.1 *([1], Definition 3.14)* Let K be a proper cone in $\mathcal{V}$. A transformation T in $\pi(K)$ is said to be K-irreducible or cone-irreducible if the only faces of K that are invariant under T are $\{0\}$ and K.

Proposition 2.3 ([27] and [1], Chapter 1, Section 3) *Let K be a proper cone in $\mathcal{V}$ and $T \in \pi(K)$. Then the following are equivalent:*

(a) *T is K-irreducible.*
(b) *T has no eigenvectors on the boundary of K.*
(c) *Up to scalar multiples, T has exactly one eigenvector in K, and this eigenvector belongs to K°.*
(d) *The transpose of T is K^*-irreducible.*
(e) $(I + T)^{n-1}\big(K\backslash\{0\}\big) \subseteq K^\circ$, *where n is the dimension of $\mathcal{V}$.*

Our second irreducibility concept is motivated by the usual irreducibility concept of a matrix. Recall that a real square matrix is said to be irreducible if no (simultaneous) permutation of rows and columns produces a 2×2 block matrix in which the diagonal blocks are square and the southwest (equivalently, northeast) block is zero. We extend this to a general linear transformation (but still based on the proper cone K):

Definition 2.2 Let K be a proper cone in $\mathcal{V}$ and L be linear on $\mathcal{V}$. Then L is said to be space-irreducible if

$$\Big[F \text{ is a face of } K \text{ with } L(F - F) \subseteq F - F\Big] \Rightarrow F = \{0\} \text{ or } K.$$

We say that L is space-reducible if it is not space-irreducible. Trivially, when $\mathcal{V}$ has dimension one, every linear transformation is space-irreducible. A few simple observations are recorded below.

Proposition 2.4 *Let K be a proper cone in $\mathcal{V}$ and L be linear on $\mathcal{V}$. Then, the following hold:*

(a) *If L is space-irreducible, then $\{0\}$ and possibly K are the only faces of K that are invariant under L.*
(b) *When $L \in \pi(K)$, space-irreducibility is the same as cone-irreducibility.*
(c) *When $L = rI - S$, where $r \in \mathcal{R}$ and $S \in \pi(K)$, space-irreducibility of L is equivalent to the cone-irreducibility of S.*
(d) *When $\mathcal{V} = \mathcal{R}^n$ and $K = \mathcal{R}^n_+$, space-irreducibility reduces to the usual/classical one.*

Proof (a) Suppose L is space-irreducible and let F be a face of K with $L(F) \subseteq F$. Then, $L(F - F) = L(F) - L(F) \subseteq F - F$. By space-irreducibility, $F = \{0\}$ or K.

(b) Suppose $L \in \pi(K)$. If L is space-irreducible, then by (a), L is cone-irreducible. Now assume that L is cone-irreducible and let $L(F - F) \subseteq F - F$, where F is a face of K. Consider any $a \in F$. Then, $L(a)$ (which is in $L(F - F)$) is of the form $b - c$, where $b, c \in F$. So, $L(a) = b - c$. Since $L \in \pi(K)$ and $a \in F \subseteq K$, we have $L(a) \in K$. As $b = c + L(a)$ and F is a face of K, we see that $L(a) \in F$. Hence, $L(F) \subseteq F$. From cone-irreducibility, $F = \{0\}$ or K. Thus, L is space-irreducible.

(c) Clearly, $L = rI - S$ is space-irreducible if and only if S is space-irreducible; the stated assertion follows from (b).

(d) This statement is easy to see as every face of $\mathcal{R}^n_+$, up to a permutation, is of the form $\mathcal{R}^k_+ \times \{0\}$. □

In [2], Elsner introduced the following type of irreducibility (which we shall call E-irreducibility). Let K be a proper cone in $\mathcal{V}$. Recall the order relation relative to K: $x \leq y$ if $y - x \in K$. For any $y \in K$, let

$$[y] := \{x \in \mathcal{V} : 0 \leq x \leq y\} \quad \text{and} \quad S_y := \operatorname{span}([y]).$$

Definition 2.3 [2] Let K be a proper cone in $\mathcal{V}$ and L be linear on $\mathcal{V}$. Then L is said to be E-irreducible if

$$\big[y \in K,\ L(S_y) \subseteq S_y\big] \Rightarrow S_y = \{0\} \text{ or } S_y = \mathcal{V}.$$

We now have an important observation (due to Orlitzky [21]): *The space-irreducibility is the same as E-irreducibility*:
This is seen as follows. For any $y \in K$, $[y] = K \cap (y - K)$. Consider the face F generated by y (which is the smallest face containing y) so $y \in \operatorname{relint}(F)$, that is, y is in the relative interior of F. Then, by the definition of face, $K \cap (y - K) = F \cap (y - F)$. As $y \in \operatorname{relint}(F)$, $\operatorname{span}([y]) = \operatorname{span}(F \cap (y - F)) = F - F$. Hence $S_y = F - F$. On the other hand, suppose F is a face of K; take any y that is in the relative interior

of F. Then $K \cap (y - K) = F \cap (y - F)$ and so $S_y = \text{span}([y]) = F - F$. Thus, for each $y \in K$, there is a face F such that $S_y = F - F$ and conversely, for every face F, there is some $y \in K$ such that $S_y = F - F$. It is now easy to see that the two definitions are equivalent.

Now, based on the above equivalence and the results in [2] (where Elsner states his results for quasimonotone/cross-positive matrices, i.e., for the negatives of our Z-transformations), we have the following:

Theorem 2.2 *Let K be a proper cone in $\mathcal{V}$ and $L \in Z(K)$. Then, relative to K, the following are equivalent:*

- (i) *$exp(-tL)$ is cone-irreducible for some $t > 0$.*
- (ii) *L is space-irreducible (equivalently, E-irreducible).*
- (iii) *L has no eigenvector on the boundary of K.*
- (iv) *Except for a countable number of ts in the interval $(0, \infty)$, $exp(-tL)$ is cone-irreducible.*

Proof $(i) \Rightarrow (ii)$: Consider/fix a $t > 0$ for which $exp(-tL)$ is K-irreducible. As $L \in Z(K)$, we know from (2), $exp(-tL) \in \pi(K)$. From Proposition 2.4, $exp(-tL)$ is space-irreducible. Suppose, $L(F - F) \subseteq F - F$ for some face F of K. Let $W := F - F$. Then for every $k \in \{0, 1, 2 \ldots\}$, $(-tL)^k$ keeps the subspace W invariant. Therefore $exp(-tL)$ keeps W invariant, i.e., $exp(-tL)(F - F) \subseteq F - F$. By the space-irreducibility of $exp(-tL)$, we have $F = \{0\}$ or K. This proves that L is space-irreducible.

$(ii) \Rightarrow (iii)$: See [2], Satz 1 and Satz 3.

$(iii) \Rightarrow (iv)$: See [2], Satz 3.

$(iv) \Rightarrow (i)$: Obvious. □

Remarks. It is easy to see that $L \in Z(K)$ if and only if $L^T \in Z(K^*)$. Now suppose $L \in Z(K)$ and is space-irreducible relative to K. Then, by the above result, $exp(-tL)$ is K-irreducible for some $t > 0$. By the equivalence $(a) \Leftrightarrow (d)$ in Proposition 2.3, $exp(-tL^T)$ is K^*-irreducible. It follows that L^T is space-irreducible relative to K^*. Since $(L^T)^T = L$ and $(K^*)^* = K$, we see that

When K is a proper cone and $L \in Z(K)$,
L is space-irreducible relative to K if and only if L^T is space-irreducible relative to K^*.

We note that Item (i) in the above theorem deals with a scaled L. In particular, the cone-irreducibility of $exp(-L)$ implies the space-irreducibility of L. We may ask when the converse holds. Here is a partial answer.

Corollary 2.1 *Let K be a proper cone in $\mathcal{V}$ and $L = rI - S$, where $r \in \mathcal{R}$ and $S \in \pi(K)$. Then, L is space-irreducible if and only if $exp(-L)$ is cone-irreducible.*

Proof Clearly $L \in Z(K)$, so $exp(-L)(K) \subseteq K$. If $exp(-L)$ is cone-irreducible, then by Theorem 2.2, L is space-irreducible. Now suppose L is space-irreducible and let F ($\neq K$) be a face of K such that $exp(-L)(F) \subseteq F$. By Proposition 2.4(c), S is cone-irreducible and by Proposition 2.3,

$$(I+S)^{n-1}(K\backslash\{0\}) \in K^\circ, \tag{3}$$

where n is the dimension of $\mathcal{V}$. Now, as rI and S commute, $exp(-L)(F) \subseteq F \Rightarrow exp(S)(F) \subseteq F$. Fix $x \in F$ so that $exp(S)x \in F \subseteq \partial(K)$. By the supporting hyperplane theorem, there exists $0 \neq y \in K^*$ such that $\langle exp(S)x, y\rangle = 0$. Expanding $exp(S)$ as a series, we see that $\langle S^k(x), y\rangle = 0$ for all $k = 0, 1, 2 \ldots$. Then, $\langle (I + S)^{n-1}x, y\rangle = 0$. From (3), we see that $x = 0$. (Note that $0 \neq y \in K^*$, $u \in K^\circ \Rightarrow \langle u, y\rangle > 0$). Thus, $F = \{0\}$; so $exp(-L)$ is cone-irreducible. □

3 The completely mixed property

In this section, we present our main result describing the completely mixed property of a Z-transformation with value zero. This is motivated by the following result in the classical matrix game setting which, perhaps, is known. For lack of a precise reference (see Theorem 3.7 in [19] for an implicit formulation dealing with a singular Z-matrix), we state the result with its proof. Recall that in the matrix game setting, $\mathcal{V} = \mathcal{R}^n$, $K = \mathcal{R}^n_+$ and $e = \mathbf{1}$ (the vector of 1s).

Theorem 3.1 *Suppose A is a Z-matrix with value zero. Then A is completely mixed if and only if it is irreducible.*

Proof We are given that A is a Z-matrix with value zero. Let $A \in \mathcal{R}^{n\times n}$. Suppose A is irreducible and consider an optimal strategy $\overline{x}$ so that $\overline{x} \geq 0$, $\langle \overline{x}, \mathbf{1}\rangle = 1$, and $A\overline{x} \geq v(A)\,\mathbf{1} = 0$. We claim that $\overline{x} > 0$. Suppose, if possible, $\overline{x}$ has a zero component. Modulo a permutation, we may assume that (only) the first k components of $\overline{x}$ are nonzero, where $1 \leq k < n$. Let $\overline{x}_1$ be the vector in $\mathcal{R}^k$ with these nonzero components. Correspondingly, we rewrite A as a 2×2 block matrix with square diagonal blocks and A_3 in the southwest corner. Then, $A\overline{x} \geq 0$ implies that $A_3\overline{x}_1 \geq 0$. Since $A_3 \leq 0$ (as A is a Z-matrix) and $\overline{x}_1 > 0$, we must have $A_3 = 0$. As A is assumed to be irreducible, this cannot happen. Hence, $\overline{x} > 0$. This proves that A is completely mixed.

Now suppose A is completely mixed. Let $(\overline{x}, \overline{y})$ be the unique strategy pair such that $A^T\overline{y} = 0 = A\overline{x}$. We also know from Kaplansky's result [16] that the $ker(A)$ has dimension 1. Assume if possible, A is reducible. Without loss of generality, we write (in the block form)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \quad \overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}, \quad \text{and} \quad \overline{y} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \end{bmatrix}.$$

From $A^T\overline{y} = 0 = A\overline{x}$, we get $A_1\overline{x}_1 + A_2\overline{x}_2 = 0 = A_1^T\overline{y}_1$. Since A is a Z-matrix, $A_2 \leq 0$; hence, $A_1\overline{x}_1 = -A_2\overline{x}_2 \geq 0$. Thus, $A_1^T\overline{y}_1 = 0 \leq A_1\overline{x}_1$. Since $0 = \langle A_1^T\overline{y}_1, \overline{x}_1\rangle = \langle \overline{y}_1, A_1\overline{x}_1\rangle$ and $\overline{y}_1$ is positive, we must have $A_1\overline{x}_1 = 0$. But then, $A_2\overline{x}_2 = 0$. Since $A_2 \leq 0$ and $\overline{x}_2 > 0$, we must have $A_2 = 0$. Thus,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$$

with $A_1\overline{x}_1 = 0$ and $A_4\overline{x}_2 = 0$. By considering scalar multiples of positive vectors $\overline{x}_1$ and $\overline{x}_2$, we see that $ker(A)$ has dimension at least 2. This yields a contradiction. Hence, A must be irreducible. □

Here is a simple example illustrating the above result.

Example 3.1 In the setting of a matrix game, consider the symmetric irreducible Z-matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

With $\mathbf{1}$ denoting the vector of ones, we see that $A\mathbf{1} = 0$. So, with $\overline{x} = \overline{y} = \frac{1}{2}\mathbf{1}$, we have $A^T(\overline{y}) = 0 = A\overline{x}$. Thus, the value of A is zero. Since the conditions $x \geq 0$, $Ax \geq 0$ imply that x is a multiple of $\mathbf{1}$, we see that A is completely mixed.

Remarks. Suppose A is an irreducible Z-matrix with value zero. Then, by the above result, there exists $\overline{x} > 0$ such that $A\overline{x} = 0$. Writing $A = rI - B$, where B is a nonnegative matrix, we have $B\overline{x} = r\overline{x}$. As B is nonnegative and irreducible (cf. Prop. 2.4), by the Perron-Frobenius theorem, $r = \rho(B)$. Thus, $A = \rho(B)\, I - B$ is now an M-matrix. So, *every irreducible Z-matrix with value zero is an irreducible singular M-matrix.*

We now present our main theorem.

Theorem 3.2 *Let $\mathcal{V}$ be a finite dimensional real inner product space, K be a self-dual cone in $\mathcal{V}$ and $e \in K^\circ$. Suppose $L \in Z(K)$ with $v(L, e) = 0$. Then (L, e) is completely mixed if and only if L is space-irreducible.*

Proof We are given that $L \in Z(K)$ with $v(L, e) = 0$.

'If' part: Assume that L is space-irreducible. By (2), $exp(-tL) \in \pi(K)$ for all $t \geq 0$; also, by Theorem 2.2, $exp(-t_0L)$ is K-irreducible for some $t_0 > 0$. Since $t_0L \in Z(K)$, $v(t_0L, e) = t_0v(L, e) = 0$, and the game (L, e) is completely mixed if and only if the game (t_0L, e) is completely mixed, it is enough to show that (t_0L, e) is completely mixed. We assume, without loss of generality, that $t_0 = 1$. Then, $L \in Z(K)$, $v(L, e) = 0$ and $exp(-L)$ is K-irreducible. We now show that (L, e) is completely mixed.

Since $L \in Z(K)$, by Theorem 2.1,

$$\alpha := \min\{Re(\lambda) : \lambda \text{ is an eigenvalue of } L\}$$

is an eigenvalue of L with an associated eigenvector $u \in K$. So, $L(u) = \alpha\, u$ with $0 \neq u \geq 0$. Then, $exp(-L)u = exp(-\alpha)\, u$. As $exp(-L)$ is K-irreducible, we must have $u > 0$ (by Proposition 2.3). Thus,

$$L(u) = \alpha\, u \quad \text{with} \quad u > 0.$$

Since K is self-dual, we see that L^T is a Z-transformation on K; moreover, $exp(-L^T)$, which is the transpose of $exp(-L)$, is also K-irreducible by Proposition 2.3. Since the eigenvalues of L and L^T are the same, we see that $\alpha = \min\{Re(\lambda) : \lambda \text{ is an eigenvalue of } L^T\}$. So, as in the previous argument, there exists $v > 0$ such that $L^T(v) = \alpha\, v$.

As $v(L, e) = 0$, let $(\overline{x}, \overline{y})$ be an (arbitrary) optimal strategy pair so that

$$L^T(\overline{y}) \leq 0 \leq L(\overline{x}).$$

We claim that $\overline{x} > 0$.

Now, from $L^T(\overline{y}) \leq 0$ and $u > 0$, we get

$$0 \geq \langle L^T(\overline{y}), u\rangle = \langle \overline{y}, L(u)\rangle = \alpha\langle \overline{y}, u\rangle.$$

Since $0 \neq \overline{y} \geq 0$ and $u > 0$, we have $\langle \overline{y}, u\rangle > 0$. Thus, $\alpha \leq 0$.

Similarly, by working with $L(\overline{x}) \geq 0$ and $L^T(v) = \alpha\, v$, we deduce that $\alpha \geq 0$. Hence, $\alpha = 0$. So, $L(u) = 0 = L^T(v)$ with $u, v > 0$; consequently, $exp(-L)(u) = u$. Now the inequalities $v > 0$, $L(\overline{x}) \geq 0$, and $\langle L(\overline{x}), v\rangle = \langle \overline{x}, L^T(v)\rangle = 0$ imply that $L(\overline{x}) = 0$. From this, we get $exp(-L)(\overline{x}) = \overline{x}$. So, $\overline{x}$ is an eigenvector of $exp(-L)$; as $\overline{x} \in K$, by the K-irreducibility of $exp(-L)$, $\overline{x} > 0$. So, we have proved that for any strategy pair $(\overline{x}, \overline{y})$, $\overline{x} > 0$. Thus, (L, e) is completely mixed.

'Only if' part: Now assume that (L, e) is completely mixed. As $v(L, e) = 0$, by Theorem 1.1, there exists a unique strategy pair $(\overline{x}, \overline{y})$ such that $\overline{x}, \overline{y} > 0$ and

$$L^T(\overline{y}) = 0 = L(\overline{x}).$$

Additionally, $ker(L)$ (being one-dimensional) consists of multiples of $\overline{x}$. We claim that L is space-irreducible. Suppose F is a nonzero face of K such that $L(F - F) \subseteq F - F$. Let $W := F - F$. Then, for all $t \geq 0$, $exp(-tL)(W) \subseteq W$. As $L \in Z(K)$, we have $exp(-tL)(K) \subseteq K$ for all $t \geq 0$ and so $exp(-tL)(W \cap K) \subseteq W \cap K$. However, $W \cap K = F$ as F is a face of K. Hence, $exp(-tL)(F) \subseteq F$ for all $t \geq 0$. As F is a proper cone in W, by Theorem 2.1, $L|_W$ (L restricted to W) is a Z-transformation on F and

$$\alpha := \min\{Re(\lambda) : \lambda \text{ is an eigenvalue of } L|_W\}$$

is an eigenvalue of L with an associated eigenvector $u \in F$. Explicitly, $L(u) = \alpha u$ with $0 \neq u \in F$. Then

$$0 = \langle L^T(\overline{y}), u\rangle = \langle \overline{y}, L(u)\rangle = \langle \overline{y}, \alpha\, u\rangle = \alpha\, \langle \overline{y}, u\rangle.$$

Since $\overline{y} > 0$ and $0 \neq u \in F \subseteq K$, we have $\langle \overline{y}, u\rangle > 0$. Hence, $\alpha = 0$. At this stage, we have $L(u) = 0$, that is, $u \in ker(L)$. Since $ker(L)$ is one-dimensional and $L(\overline{x}) = 0$, we see that u is a positive multiple of $\overline{x}$. This proves that $u > 0$, that is, $u \in K^\circ$. Since u belongs to the face F, we must have $F = K$. This shows that L is space-irreducible. □

Remarks. In an earlier version of the paper, see [8], the equivalence of completely mixed property and space-irreducibility was proved under certain restrictions. The implication that the completely mixed property implies space-irreducibility was proved under the additional assumption that each face of K is self-dual in its span, see [8], Theorem 3.4. Also, the implication that space-irreducibility implies completely mixed property was proved only for certain types of Z-transformations over Euclidean Jordan algebras, see [8], Theorem 4.2. The above theorem/proof removes these restrictions.

In their recent work [3], Encinas, Mondal, and Sivakumar study the concepts of range monotone and trivially range monotone properties of Lyapunov and Stein transformations on the space $\mathcal{S}^n$ of all real $n \times n$ symmetric matrices. Their motivation comes from the matrix theory result that every singular irreducible M-matrix is almost monotone and has a group inverse, see [1], Theorem 4.1 and [3], Theorem 1.3. We make three observations: First, a singular M-matrix is a Z-matrix with value zero; second, Theorem 4.16 in [1] appears to be related to Kaplansky's characterization of completely mixed property; and third, Lyapunov and Stein transformations on $\mathcal{S}^n$ are particular examples of Z-transformations. Motivated by these considerations, we now formulate the following broader result.

Corollary 3.1 *Let $\mathcal{V}$ be a finite dimensional real inner product space, K be a self-dual cone in $\mathcal{V}$ and $e \in K^\circ$. Suppose $L \in Z(K)$, $v(L, e) = 0$, and L is space-irreducible. Then, the following statements hold:*

(a) *L is almost monotone, that is, $L(x) \geq 0 \Rightarrow L(x) = 0$.*
(b) *L is trivially range monotone, that is, $L^2(x) \geq 0 \Rightarrow L(x) = 0$.*
(c) $\ker(L^2) = \ker(L)$.
(d) *the group inverse of L exists.*

Proof Under the specified assumptions, by the above theorem, space-irreducibility implies the completely mixed property. Now, from Theorem 1.1, there exist $\overline{x}, \overline{y} > 0$ such that

$$L^T(\overline{y}) = 0 = L(\overline{x}).$$

To see (a), suppose $L(x) \geq 0$. Then,

$$0 \leq \langle L(x), \overline{y} \rangle = \langle x, L^T(\overline{y}) \rangle = 0.$$

As $\overline{y} > 0$ and $L(x) \geq 0$, we must have $L(x) = 0$.

(b): Now suppose $L^2(x) \geq 0$, that is, $L(L(x)) \geq 0$ for some $x \in \mathcal{V}$. Then, from (a), $L(L(x)) = 0$, that is, $L(x) \in \ker(L)$. We know from Theorem 1.1, Item (6) that $\ker(L) = \{\lambda\overline{x} : \lambda \in \mathcal{R}\}$. So, $L(x) = \lambda\overline{x}$ for some $\lambda \in \mathcal{R}$. We claim that $\lambda = 0$. Suppose $\lambda > 0$ so that $L(x) = \lambda\overline{x} > 0$. But then, for any small $\varepsilon > 0$, $\overline{x} + \varepsilon x > 0$ and $L(\overline{x} + \varepsilon x) = \varepsilon L(x) > 0$. By Theorem 1.1, Item (3), the value of the game is positive, contradicting our assumption. Hence, λ cannot be positive. If $\lambda < 0$, we can work with $-x$ in place of x to get a contradiction. Hence, $\lambda = 0$, so $L(x) = 0$. Thus,

$$L^2(x) \geq 0 \Rightarrow L(x) = 0.$$

This proves (b).

When $L^2(x) = 0$, we have, from (b), $L(x) = 0$. Thus, $Ker(L^2) \subseteq \ker(L)$. As $\ker(L) \subseteq \ker(L^2)$ always, we get the equality in (c).

Because of (c), the group inverse of L exists, see [24], Theorem 5. □

Example 3.2 Let $\mathcal{V}$ be a finite dimensional real inner product space, K be a self-dual cone in $\mathcal{V}$ and $e \in K^\circ$. On $\mathcal{V}$, consider the transformation $L = \rho(S)\, I - S$, where $S \in \pi(K)$ with $\rho(S)$ denoting the spectral radius of S. Suppose L is space-irreducible (equivalently, S is K-irreducible, by Proposition 2.4). Then, by the Krein-Rutman Theorem (see Theorem 3.2 in [1]) and Proposition 2.3, there exists $\overline{x}, \overline{y} > 0$ such that $S^T(\overline{y}) = \rho(S)\,\overline{y}$ and $S(\overline{x}) = \rho(S)\,\overline{x}$. We see that $L^T(\overline{y}) = 0 = L(\overline{x})$ and so $v(L, e) = 0$. Moreover, by Theorem 3.2, (L, e) is completely mixed. In analogy with matrices, we may call L, a 'singular M-transformation'. So, as in the matrix case, *a singular space-irreducible M-transformation is completely mixed.* A simple example of such an L can be seen by defining $S(x) = \langle x, e\rangle e$. To get a general L, we may consider a 'strictly positive transformation' which satisfies the condition $0 \neq x \geq 0 \Rightarrow S(x) > 0$. Then, S is K-irreducible (by Proposition 2.3) and $\rho(S) > 0$. Thus the game (L, e), where $L = \rho(S)\, I - S$, has value zero and is completely mixed. In this way, we get a large collection of Z-transformations with value zero and having the completely mixed property.

4 Completely mixed and irreducibility properties on symmetric cones in Euclidean Jordan algebras

In this section, we specialize by assuming that the given proper cone K is a symmetric cone which, by definition, is self-dual and homogeneous. Since every symmetric cone appears as the cone of squares in a Euclidean Jordan algebra, we (can) use the Jordan algebraic machinery to describe our completely mixed and irreducibility properties and also to provide interesting examples. First, we cover some background material.

For basic definitions and results on Euclidean Jordan algebras (particularly, those that are not explicitly mentioned/referenced), we refer to [4, 11]. Let $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra with unit element e; here, for any two elements x, y, the Jordan product and inner product are denoted, respectively, by $x \circ y$ and $\langle x, y\rangle$. It is well-known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five (families of) algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra $\mathcal{L}^n$. In the algebras $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices) and $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices), the Jordan product and the inner product are given by $X \circ Y := \frac{XY+YX}{2}$ and $\langle X, Y\rangle := tr(XY)$. The algebra $\mathcal{R}^n$ carries the componentwise product (as the Jordan product) and the usual inner product.

In the Euclidean Jordan algebra $\mathcal{V}$, the set

$$K = \mathcal{V}_+ := \{x \circ x : x \in \mathcal{V}\}$$

is (called) the symmetric cone of $\mathcal{V}$. It is a *self-dual cone*, so $x \in K \Leftrightarrow \langle x, y\rangle \geq 0$ for all $y \in K$. Moreover, $u \in K^\circ \Leftrightarrow \langle u, y\rangle > 0$ for all $0 \neq y \in K$. In the case of $\mathcal{H}^n$ (or $\mathcal{S}^n$), K is the cone of positive semidefinite matrices.

In $\mathcal{V}$, the following holds [11]: When $x, y \in K$,

$$x \circ y = 0 \Leftrightarrow \langle x, y\rangle = 0. \tag{4}$$

An element c in $\mathcal{V}$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as sum of two other nonzero idempotents. A *Jordan frame* $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ consists of primitive idempotents that are mutually orthogonal and with sum equal to the unit element e. It is known that all Jordan frames in $\mathcal{V}$ have the same number of elements – called the rank of $\mathcal{V}$. *With n denoting the rank of $\mathcal{V}$*, we have the *spectral decomposition theorem* ([4], Theorem III.1.2): *Every x in $\mathcal{V}$ can be written as*

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

where the real numbers $x_1, x_2, \ldots, x_n$ are (called) the eigenvalues of x and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. Then, we define the rank of x – denoted by rank x – as the number of nonzero eigenvalues of x. Corresponding to the above decomposition, we define the trace of x as $\mathrm{tr}(x) := x_1 + x_2 + \cdots + x_n$. It is known that $(x, y) \mapsto \mathrm{tr}(x \circ y)$ defines another inner product on $\mathcal{V}$ that is compatible with the Jordan product. *Throughout this paper we assume that the inner product on $\mathcal{V}$ is this trace inner product, that is, $\langle x, y\rangle = \mathrm{tr}(x \circ y)$.*

Given an idempotent c and $\gamma \in \{1, \frac{1}{2}, 0\}$, we let

$$\mathcal{V}(c, \gamma) := \{x \in \mathcal{V} : x \circ c = \gamma\, x\}.$$

It is known, see [4], Prop. IV.1.1, that $\mathcal{V}(c, 1)$ and $\mathcal{V}(c, 0)$ are subalgebras of $\mathcal{V}$ and

$$\mathcal{V}(c, 1) \circ \mathcal{V}(c, 0) = \{0\}.$$

More importantly, the following *Peirce (orthogonal) decomposition* holds:

$$\mathcal{V} = \mathcal{V}(c, 1) + \mathcal{V}(c, \frac{1}{2}) + \mathcal{V}(c, 0). \tag{5}$$

Let $\mathcal{V}_+(c, 1)$ denote the symmetric cone of the (sub)algebra $\mathcal{V}(c, 1)$. Then,

$$\mathcal{V}(c, 1) = \mathcal{V}_+(c, 1) - \mathcal{V}_+(c, 1)$$

and, moreover, $\mathcal{V}_+(c, 1)$ is self-dual in $\mathcal{V}(c, 1)$. Regarding the faces of a symmetric cone, we have the following:

Proposition 4.1 ([10], Theorem 3.1) *Every face of the symmetric cone K is of the form $\mathcal{V}_+(c, 1)$ for some idempotent c.*

If $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$, then we have (another) *Peirce orthogonal decomposition* (Theorem IV.2.1 in [4]):

$$\mathcal{V} = \sum_{1\leq i\leq j\leq n} \mathcal{V}_{ij}, \tag{6}$$

where $\mathcal{V}_{ii} := \mathcal{V}(e_i, 1) = \mathcal{R}\, e_i$ for all i and for $i \neq j$, $\mathcal{V}_{ij} := \mathcal{V}(e_i, \frac{1}{2}) \cap \mathcal{V}(e_j, \frac{1}{2})$.

Given $a \in \mathcal{V}$, the corresponding *Lyapunov transformation* and *quadratic representation* are defined by:

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x \quad (x \in \mathcal{V}).$$

These two transformations are self-adjoint, $L_a \in Z(K)$, and $P_a \in \pi(K)$.

Given (6), consider $a, x \in \mathcal{V}$ with $a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$ and

$$x = \sum_{1\leq i\leq j\leq n} x_{ij} \quad (x_{ij} \in \mathcal{V}_{ij}).$$

Then we have the well-known formulae

$$L_a(x) = \sum_{1\leq i\leq j\leq n} \Big(\frac{a_i + a_j}{2}\Big) x_{ij} \quad \text{and} \quad P_a(x) = \sum_{1\leq i\leq j\leq n} a_i a_j\, x_{ij}. \tag{7}$$

Let L be a Lyapunov-like transformation on the symmetric cone K. Then, its symmetric part $\frac{L+L^T}{2}$ and skew-symmetric part $\frac{L-L^T}{2}$ are both Lyapunov-like. Moreover, it is known, see [26], that the symmetric part is of the form L_a for some $a \in \mathcal{V}$ and the skew-symmetric part is a derivation D (which means that $D(x \circ y) = Dx \circ y + x \circ Dy$ for all $x, y \in \mathcal{V}$). So, in the setting of a Euclidean Jordan algebra,

$$L \in \mathrm{LL}(K) \Leftrightarrow L = L_a + D \text{ for some } a \in \mathcal{V} \text{ and derivation } D. \tag{8}$$

It is known (a result due to Damm, see [26]) that in the Euclidean Jordan algebra $\mathcal{H}^n$, every Lyapunov-like transformation is of the form L_A for some complex matrix A, where

$$L_A(X) = AX + XA^* \quad (X \in \mathcal{H}^n).$$

A similar statement holds in $\mathcal{S}^n$ with A real and A^* replaced by A^T.

Henceforth, in all game-theoretic results/concepts dealing with Euclidean Jordan algebras, we assume that $\mathcal{V}$ carries the trace inner product, K is the symmetric cone of $\mathcal{V}$, and e is the unit element; We also write $v(L)$ in place of $v(L, e)$, etc.

4.1 A generalization of a result of Parthasarathy et al.

The following is a generalization of the recent result by Parthasarathy et al. [22], mentioned in the Introduction.

Theorem 4.1 *Suppose L is a linear transformation on the Euclidean Jordan algebra $\mathcal{V}$ with $v(L) = 0$. Then, L is completely mixed if and only if $v(L + P_c) > 0$ for every primitive idempotent c in V.*

Proof It is given that $v(L) = 0$. Take any primitive idempotent c in V. As $P_c(K) \subseteq K$, $\langle (L + P_c)x, y\rangle \geq \langle L(x), y\rangle$ for all strategies x and y; hence, by the min-max theorem mentioned in the Introduction, $v(L + P_c) \geq v(L) = 0$.

Now suppose L is completely mixed. Then there exist optimal strategies $\overline{x}, \overline{y} > 0$ such that $L^T(\overline{y}) = 0 = L(\overline{x})$. Additionally, for (arbitrary) strategies z, w,

$$L^T(z) \leq 0 \leq L(w) \Rightarrow (w, z) \text{ optimal} \Rightarrow z, w > 0 \Rightarrow L^T(z) = 0 = L(w).$$

Now, assume if possible, $v(L + P_c) = 0$ for some c. Then, there is a strategy z such that

$$(L + P_c)^T(z) \leq 0.$$

Since P_c is self-adjoint and $P_c(K) \subseteq K$, we have $L^T(z) \leq 0$. As $L(\overline{x}) = 0$, we now have

$$L^T(z) \leq 0 = L(\overline{x}).$$

From the above implications, $z > 0$ and $L^T(z) = 0$. But then, $P_c(z) \leq 0$ with $z > 0$. Again, since $P_c(K) \subseteq K$, we must have $P_c(z) = 0$. As $z > 0$ and $0 \neq c \in K$, we must have $\langle z, c\rangle > 0$. However, $0 < \langle z, c\rangle = \langle z, P_c(c)\rangle = \langle P_c(z), c\rangle = \langle 0, c\rangle = 0$. We reach a contradiction. Thus, the value of $L + P_c$ must be positive.

To see the reverse implication, suppose the value of $L + P_c$ is positive for all primitive idempotents c. Assume, if possible, that L is not completely mixed. Then, there are strategies x and y such that $L^T(y) \leq 0 \leq L(x)$, where some eigenvalue of y, say, y_1 is zero. we write the spectral decomposition $y = y_1e_1 + y_2e_2 + \cdots + y_ne_n = 0e_1 + y_2e_2 + \cdots + y_ne_n$ and let $c = e_1$ (which is a primitive idempotent). Then, by (7), $P_c(y) = 0$. As $P_c(x) \geq 0$, we have $(L + P_c)^T(y) \leq 0 \leq (L + P_c)(x)$. This implies that the value of $L + P_c$ is zero, leading to a contradiction. Hence, L must be completely mixed. ◻

4.2 Irreducibility of a linear transformation on a symmetric cone

Now consider a Euclidean Jordan algebra $\mathcal{V}$ with its symmetric cone K $(= \mathcal{V}_+)$ and unit element e. In this setting, as noted in Proposition 4.1, every face of K is of the form $\mathcal{V}_+(c, 1)$ (the symmetric cone of $\mathcal{V}(c, 1)$) for some idempotent c. Moreover, $\mathcal{V}_+(c, 1) = \{0\}$ or $\mathcal{V}_+$ if and only if $c = 0$ or e. Since $\mathcal{V}(c, 1) = \mathcal{V}_+(c, 1) - \mathcal{V}_+(c, 1)$, Definition 2.2 reduces to the following.

Definition 4.1 Let L be a linear transformation on a Euclidean Jordan algebra $\mathcal{V}$. Then, L is space-irreducible if

$$L\big(\mathcal{V}(c,1)\big) \subseteq \mathcal{V}(c,1) \Rightarrow c \in \{0, e\}.$$

So, space-irreducibility means that other than $\{0\}$ and $\mathcal{V}$, L does not have any invariant subalgebras of the form $\mathcal{V}(c,1)$. We note that when L is not space-irreducible, with $c \neq 0, e$, $W := \mathcal{V}(c,1)$, and $L(W) \subseteq W$, we can write $\mathcal{V} = W^{\perp} + W$ and represent L in the block form

$$\begin{bmatrix} L_1 & L_2 \\ 0 & L_3 \end{bmatrix},$$

where $L_1 : W^{\perp} \to W^{\perp}$, $L_2 : W \to W^{\perp}$, and $L_3 : W \to W$ are linear transformations.

The following two examples deal with the space-irreducibility of L_A on $\mathcal{S}^n$ and $\mathcal{H}^n$.

Example 4.1 Let $n > 1$. For any $A \in \mathcal{R}^{n\times n}$, consider the transformation L_A on $\mathcal{S}^n$ defined by $L_A(X) = AX + XA^T$. *We first claim that L_A is space-irreducible if and only if for every orthogonal matrix U, the matrix U^TAU is irreducible.* We sketch a proof. Suppose L_A is space-reducible, that is, there exists an idempotent C $(\neq 0, I)$ in $\mathcal{S}^n$ such that $L_A(\mathcal{V}(C,1)) \subseteq \mathcal{V}(C,1)$. As C is an idempotent, its eigenvalues are either 0 or 1; we can write $C = UPU^T$, where U is orthogonal and P is a diagonal matrix written in the block form as

$$P = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

with I_k denoting the identity matrix of size $k \times k$, $1 \leq k < n$. Since $X \in \mathcal{V}(C,1)$ if and only if $C \circ X = X$, that is, $\frac{CX+XC}{2} = X$, we see that $\mathcal{V}(C,1) = U\mathcal{V}(P,1)U^T$. Moreover, the condition $L_A(\mathcal{V}(C,1)) \subseteq \mathcal{V}(C,1)$ is equivalent to $L_B(\mathcal{V}(P,1)) \subseteq \mathcal{V}(P,1)$, where $B = U^TAU$. Now, every element in $\mathcal{V}(P,1)$ has the same block form as P, except that in place of I_k, we have a matrix from $\mathcal{S}^k$. So then $L_B(P) \in \mathcal{V}(P,1)$ implies that the southwest block of the matrix B is zero, implying that B is reducible. Now suppose there is an orthogonal U such that the matrix $B = U^TAU$ is reducible. By choosing an appropriate diagonal matrix P and reversing the argument given above, one can construct an idempotent matrix C $(\neq 0, I)$ for which $L_A(\mathcal{V}(C,1)) \subseteq \mathcal{V}(C,1)$ holds. Thus, we have shown that L_A is space-reducible if and only if for some orthogonal matrix U, U^TAU is reducible. This proves our claim.

Now, by the real version of Schur's upper triangularization theorem ([14], Theorem 2.3.4), A is orthogonally similar to a real block upper triangular matrix with each diagonal block either a 1×1-matrix or a 2×2-matrix with a non-real pair of complex conjugate eigenvalues. So, if L_A is space-irreducible (in which case, by our claim above, every matrix of the form UAU^T is irreducible), A must be a 2×2 matrix with a non-real pair of complex conjugate eigenvalues. Conversely, if A is a 2×2

matrix with this specific property, then every $U^T AU$ is irreducible; hence, L_A is space-irreducible. In summary, we have proved:

On $\mathcal{S}^n$ $(n > 1)$, L_A is space-irreducible if and only if $n = 2$ and A is a real matrix with a non-real pair of complex conjugate eigenvalues.

To see a specific example, consider the Lyapunov transformation L_A on $\mathcal{S}^2$ defined by

$$L_A(X) = AX + XA^T \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

As $A + A^T = 0$, we see that $L_A + (L_A)^T = 0$. We also have $L_A(I) = 0 = (L_A)^T(I)$ and so, $v(L_A, I) = 0$. Additionally, if X in $\mathcal{S}^2$ is positive semidefinite with $L_A(X) \geq 0$, then X is a multiple of I. This shows that L_A is completely mixed, hence space-irreducible. It is easy to verify that this L_A cannot be of the form $rI - S$ for any $S \in \pi(\mathcal{S}^2_+)$. *Thus, in contrast to the matrix game case (cf. Remarks made after Theorem 3.1), we have an example of a Lyapunov-like transformation with value zero that is completely mixed and which is not a singular M-transformation.*

Example 4.2 For any $n \times n$ complex matrix A, $n > 1$, consider L_A on $\mathcal{H}^n$ defined by $L_A(X) = AX + XA^*$, where A^* is the conjugate-transpose of A. By Schur's upper-triangularization theorem ([14], Theorem 2.3.1), we can write A as UQU^*, where Q is an upper-triangular matrix and U is unitary. We note that the matrix Q is reducible. We can mimic the argument given in the previous example to show that L_A keeps some nontrivial $\mathcal{V}(C, 1)$ invariant. This means that *over $\mathcal{H}^n$ $(n > 1)$, every L_A is space-reducible.* By Theorem 3.2, no such transformation can (simultaneously) have value zero and be completely mixed. Thus, because of Theorem 1.2, *Over $\mathcal{H}^n$ $(n > 1)$, L_A is completely mixed if and only if its value is nonzero.*

4.3 Irreducibility of a positive transformation on a symmetric cone

The following theorem is a modified form of Proposition 2.3 stated for Euclidean Jordan algebras. We emphasize that the natural number n appearing in Item (iv) below is the rank (and not the dimension) of the algebra. This result is motivated by a similar result in the setting of Hermitian matrices, see [28], Theorem 6.2 or [15], Proposition C.1.

Before stating the theorem, we present two lemmas. The first lemma is motivated by the result that when a (Hermitian) positive semidefinite matrix is written in a block form, if a diagonal block is zero, then the corresponding off-diagonal blocks are also zero. The second lemma is about the rank of a sum.

Lemma 4.1 *For an idempotent c, consider the Peirce orthogonal decomposition (5). Suppose an element $z \in \mathcal{V}$ has no component in $\mathcal{V}(c, 0)$, that is,*

$$z = x + y,$$

where $x \in \mathcal{V}(c, 1)$ and $y \in \mathcal{V}(c, \frac{1}{2})$. If $z \geq 0$, then, $y = 0$.

Proof Since $e - c \in \mathcal{V}(c, 0)$, by the orthogonality of the spaces involved, we have $\langle x + y, e - c\rangle = 0$. As $z = x + y \geq 0$ and $e - c \geq 0$, by (4), we have

$$(x + y) \circ (e - c) = 0.$$

This leads to $(x + y) \circ e - (x + y) \circ c = 0$. Now, $x \in \mathcal{V}(c, 1) \Rightarrow x \circ c = x$ and $y \in \mathcal{V}(c, \frac{1}{2}) \Rightarrow y \circ c = \frac{1}{2} y$. Hence,

$$(x + y) - (x + \frac{1}{2} y) = 0.$$

This simplifies to $y = 0$. ◻

Lemma 4.2 *Suppose* $0 \neq x \geq 0$ *and* rank $(x) = k$, *where* $1 \leq k < n$. *Consider the spectral decomposition* $x = x_1 e_1 + x_2 e_2 + \cdots + x_k e_k + 0 e_{k+1} + \cdots + 0 e_n$, *where* $\{e_1, e_2, \ldots, e_n\}$ *is a Jordan frame, and* $x_i > 0$ *for all* $i = 1, 2, \ldots, k$. *Let*

$$c := e_1 + e_2 + \cdots + e_k.$$

If $y \geq 0$ *and* $y \notin \mathcal{V}(c, 1)$, *then* rank $(x + y) >$ rank x.

Proof Let $z := x + y$ and suppose, if possible, $l :=$ rank $z \leq k$. Then, we write the spectral decomposition of z as $z = z_1 c_1 + z_2 c_2 + \cdots + z_l c_l + 0 c_{l+1} + \cdots + 0 c_n$, where $\{c_1, c_2, \ldots, c_n\}$ is a Jordan frame and $z_i \neq 0$ for $1 \leq i \leq l$. Noting that $l \leq k < n$, we consider the idempotent

$$d := c_{k+1} + c_{k+2} + \cdots + c_n.$$

Clearly, $\langle z, d\rangle = 0$. This implies (recall $x, y \geq 0$) that $\langle x, d\rangle = 0 = \langle y, d\rangle$. From $0 = \langle x, d\rangle = \sum_{i=1}^{k} x_i \langle e_i, d\rangle$ with $x_i > 0$ for all i appearing in the sum, we see that $\langle e_i, d\rangle = 0$ for $i = 1, 2 \ldots, k$. It follows that $\{e_1, e_2, \ldots, e_k\} \perp \{c_{k+1}, c_{k+2}, \ldots, c_n\}$; hence, by (4), the Jordan product between any two (different) elements in the set $\{e_1, e_2, \ldots, e_k, c_{k+1}, c_{k+2}, \ldots, c_n\}$ is zero. As the rank of $\mathcal{V}$ is n, $\{e_1, e_2, \ldots, e_k, c_{k+1}, c_{k+2}, \ldots, c_n\}$ forms a Jordan frame (else, this set together with the primitive idempotents corresponding to the nonzero terms in the spectral decomposition of the idempotent $e - (c + d)$ produces a Jordan frame containing more than n elements); in particular, $c + d = e_1 + e_2 + \cdots + e_k + c_{k+1} + \cdots + c_n = e$. From $\langle y, d\rangle = 0$ and (4), we have, $y \circ d = 0$. Consequently, as $d = e - c$, we have $y \in \mathcal{V}(d, 0) = \mathcal{V}(c, 1)$. We reach a contradiction to our assumption that $y \notin \mathcal{V}(c, 1)$. This proves that $l > k$, i.e., rank $(x + y) >$ rank x. ◻

Theorem 4.2 *Suppose* $\mathcal{V}$ *is a Euclidean Jordan algebra with unit* e, $K = \mathcal{V}_+$, *and* $T \in \pi(K)$. *Let* c *denote a generic idempotent. Then the following are equivalent:*

(i) $\langle T(c), e-c\rangle = 0 \Rightarrow c \in \{0, e\}$.
(ii) *T is space-irreducible, that is,* $T\big(\mathcal{V}(c,1)\big) \subseteq \mathcal{V}(c,1) \Rightarrow c \in \{0, e\}$.
(iii) *T is K-irreducible.*
(iv) *For each* $0 \neq x \geq 0$, *it holds that* $(I+T)^{n-1}x > 0$, *where* $n\,(\geq 1)$ *denotes the rank of* $\mathcal{V}$.
(v) *For each* $0 \neq x \geq 0$ *and every* $t > 0$, $exp(tT)x > 0$.

Proof $(i) \Leftrightarrow (ii)$: Suppose the negation of (i) holds so that there is an idempotent $c \notin \{0, e\}$ such that $\langle T(c), e-c\rangle = 0$. Let $d := e - c$, so that $d \notin \{0, e\}$ and $\langle T(c), d\rangle = 0$. Then, in the algebra $\mathcal{V}(c,1)$, c is the unit element and in $\mathcal{V}(c,0)$ (which is $\mathcal{V}(d,1)$), d is the unit element. Now take primitive idempotents c_1 in $\mathcal{V}(c,1)$ and d_1 in $\mathcal{V}(c,0)$. Since $0 \leq c_1 \leq c$ and $0 \leq d_1 \leq d$, we have, by the positivity of T, $0 \leq \langle T(c_1), d_1\rangle \leq \langle T(c), d\rangle = 0$. Thus, $\langle T(c_1), d_1\rangle = 0$. Since (from the spectral theorem) any element in $\mathcal{V}(c,1)$ is a linear combination of primitive idempotents in $\mathcal{V}(c,1)$ and any element in $\mathcal{V}(d,1)$ is a linear combination of primitive idempotents in $\mathcal{V}(d,1)$, we see that

$$T\big(\mathcal{V}(c,1)\big) \perp \mathcal{V}(c,0).$$

Now take any element $x \in \mathcal{V}_+(c,1)$ and consider the Peirce orthogonal decomposition of $T(x)$:

$$T(x) = u + v + w, \quad \text{where} \quad u \in \mathcal{V}(c,1), v \in \mathcal{V}(c,\frac{1}{2}), w \in \mathcal{V}(c,0).$$

Since $T(x) \perp w$, we must have $w = 0$. So now, $0 \leq T(x) = u + v$. By Lemma 4.1, we have $v = 0$. Thus, for any $x \in \mathcal{V}_+(c,1)$, $T(x) \in \mathcal{V}(c,1)$. Since any element in $\mathcal{V}(c,1)$ is a difference of two elements in $\mathcal{V}_+(c,1)$, we see that $T\big(\mathcal{V}(c,1)\big) \subseteq \mathcal{V}(c,1)$. Hence, the negation of (ii) holds.

Now suppose the negation of (ii) holds so that there exists $c\,(\neq 0, e)$ with $T(c) \in T\big(\mathcal{V}(c,1)\big) \subseteq \mathcal{V}(c,1)$. Then, with $0 \neq d := e - c$, we have $\mathcal{V}(c,1) \perp \mathcal{V}(d,1)$, hence $\langle T(c), d\rangle = 0$. This gives the negation of (i). Hence, (i) and (ii) are equivalent.

$(ii) \Leftrightarrow (iii)$: This follows from the facts that for $T \in \pi(K)$, $T(\mathcal{V}(c,1)) \subseteq \mathcal{V}(c,1)$ if and only if $T(\mathcal{V}_+(c,1)) \subseteq \mathcal{V}_+(c,1)$, and that every face of (the symmetric cone) K is of the form of $\mathcal{V}_+(c,1)$ for some idempotent c.

$(ii) \Rightarrow (iv)$: Assume that (ii) holds. Let $0 \neq x \geq 0$. If $x > 0$, (iv) trivially holds as $T \in \pi(K)$. So, suppose rank $(x) = k$, where $1 \leq k < n$. We write the spectral decomposition $x = x_1e_1 + x_2e_2 + \cdots + x_ke_k + 0e_{k+1} + \cdots + 0e_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame and $x_i > 0$ for all $i = 1, 2, \ldots, k$. Let

$$c := e_1 + e_2 + \cdots + e_k.$$

Then, in the (sub)algebra $\mathcal{V}(c,1)$, $x > 0$. We show that $T(x) \notin \mathcal{V}(c,1)$. Assume the contrary, i.e., $T(x) \in \mathcal{V}(c,1)$; then, as $T \in \pi(K)$, $T(x) \in \mathcal{V}_+(c,1)$. *We now claim that* $T(\mathcal{V}(c,1)) \subseteq \mathcal{V}(c,1)$. To see this, take any $y \in \mathcal{V}(c,1)$. Then, for small positive ε, $x \pm \varepsilon y > 0$ in $\mathcal{V}(c,1)$. As $T \in \pi(K)$ and $\mathcal{V}_+(c,1) \subseteq K$, we have $T(x \pm \varepsilon y) \geq 0$. Since $T(x)$ (which is in $\mathcal{V}_+(c,1)$) is now the average of two elements (namely, $T(x+\varepsilon y)$ and $T(x-\varepsilon y)$) in K, these two elements belong to the face $\mathcal{V}_+(c,1)$; consequently, their difference $2\varepsilon T(y)$ (equivalently, $T(y)$) belongs to $\mathcal{V}(c, 1)$. This proves the claim. However, this contradicts (ii) and so we must have $T(x) \notin \mathcal{V}(c, 1)$. Since $x \in \mathcal{V}(c, 1)$, this implies that

$$z := x + T(x) = (I + T)x \notin \mathcal{V}(c, 1).$$

Now, with $y := T(x)$, we apply Lemma 4.2 to see that

$$\text{rank } z > \text{ rank } x = k.$$

At this stage, we have proved that

$$0 \neq x \in K, \text{ rank } x < n \Rightarrow \text{rank } (I + T)x > \text{rank } x.$$

So, starting with an element $0 \neq x \in K$ and applying $I + T$ repeatedly, we reach an element in K with rank n in at most $n - 1$ steps. Thus, (iv) holds.

$(iv) \Rightarrow (v)$: Assume that (v) does not hold. Then, there exist $0 \neq x \geq 0$ and a positive number t such that $exp(tT)x \not> 0$. (We note that $exp(tT) \in \pi(K)$.) Then, for some (primitive) idempotent c, $\langle exp(tT)x, c\rangle = 0$. Expanding $exp(tT)$ and using the positivity of T and its powers, we see that $\langle T^m(x), c\rangle = 0$ for all $m = 0, 1, 2, \ldots$ Hence, $\langle (I + T)^{n-1}x, c\rangle = 0$. This shows that (iv) does not hold. Hence, we have the implication $(iv) \Rightarrow (v)$.

$(v) \Rightarrow (ii)$: Assume the negation of (ii). Then, for some idempotent $c\,(\neq 0, e)$, we have $T\big(\mathcal{V}(c, 1)\big) \subseteq \mathcal{V}(c, 1)$. As $c \in \mathcal{V}(c, 1)$, we have $T^m(c) \in \mathcal{V}(c, 1)$ for all $m = 0, 1, 2, \ldots$ Thus, $exp(T)c \in \mathcal{V}(c, 1)$. In particular, $exp(T)c \not> 0$, contradicting (v) for $x = c$ and $t = 1$.

In summary, we have proved the implications $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$. Thus, all the statements in the theorem are equivalent. This completes the proof. ◻

4.4 The completely mixed property of symmetric/skew-symmetric Lyapunov-like transformations

In the next two results, we deal with the completely mixed property of a symmetric/skew-symmetric Lyapunov-like transformation on a Euclidean Jordan algebra. Recall that in this setting, every symmetric Lyapunov-like transformation is of the form L_a for some $a \in \mathcal{V}$ and a skew-symmetric Lyapunov-like transformation is a derivation, see [26]. When the rank of $\mathcal{V}$ is one, we can write $\mathcal{V} = \mathcal{R}e$. In this case, e is the only strategy vector. Hence, *when the rank of $\mathcal{V}$ is one, every linear transformation on $\mathcal{V}$is completely mixed.*

Theorem 4.3 *Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n, where $n \geq 2$. For $a \in \mathcal{V}$, the following statements are equivalent:*

(i) *L_a is completely mixed.*
(ii) *$a > 0$ or $a < 0$.*
(iii) *$v(L_a) \neq 0$.*

Proof $(i) \Rightarrow (ii)$: Suppose L_a is completely mixed. Consider the spectral decomposition $a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$. If possible, let $a_1 = 0$. Then, by (7),

$$L_a^T(e_1) = L_a(e_1) = 0.$$

As $e_1 \not> 0$, we reach a contradiction to (i). Hence, $a_i \neq 0$ for all i. Now suppose, if possible, there is an index k such that $1 \leq k < n$ such that $a_i > 0$ for $1 \leq i \leq k$ and $a_i < 0$ for $k+1 \leq i \leq n$. Let $c := e_1 + e_2 + \cdots + e_k$. Then, by (7), $L_a(c) = \sum_{i=1}^{k} a_i e_i \geq 0$ and $L_a^T(e-c) = \sum_{i=k+1}^{n} a_i e_i \leq 0$. We see that

$$L_a^T\Big(\frac{e-c}{n-k}\Big) \leq 0 \leq L_a\Big(\frac{c}{k}\Big).$$

As $\frac{c}{k} \not> 0$, we reach a contradiction to (i). Hence, $a > 0$ or $a < 0$.

$(ii) \Rightarrow (iii)$: We first show that $a > 0 \Rightarrow v(L_a) > 0$.

Suppose $a > 0$ and $v(L_a) \leq 0$. Then, there exists $y \geq 0$ with $\langle y, e\rangle = 1$ (in particular, $y \neq 0$) such that $L_a^T(y) \leq v(L_a)\, e \leq 0$. As $a > 0$, we have $\langle L_a^T(y), a\rangle \leq 0$, that is, $\langle y, L_a(a)\rangle \leq 0$. Since $y \geq 0$ and $L_a(a) = a^2 > 0$, we must have $y = 0$, yielding a contradiction. Hence $a > 0 \Rightarrow v(L_a) > 0$.

When $a < 0$, we have $-a > 0$ and so $v(L_{-a}) > 0$. Since $L_{-a} = -L_a$ and L_a is self-adjoint, this gives $-v(L_a) > 0$, that is, $v(L_a) < 0$.

$(iii) \Rightarrow (i)$: As L_a is Lyapunov-like, this follows from Theorem 1.2. □

Suppose the linear transformation L is skew-symmetric, that is, $L + L^T = 0$. In this case, the game (L, e) is called a *symmetric game* [5]. For such an L, the value is zero (see Theorem 1.1, Item (2)). It is easy to see that a skew-symmetric Z-transformation is Lyapunov-like.

Theorem 4.4 *Let D be a skew-symmetric Z-transformation (=derivation) on a Euclidean Jordan algebra $\mathcal{V}$. Then, the following are equivalent:*

(i) The game (D, e) is completely mixed.
(ii) D is space-irreducible.
(iii) $ker(D)$ is one dimensional.

Proof Since D is skew-symmetric, its value is zero. As D is a Z-transformation, by Theorem 3.2, (i) and (ii) are equivalent.

$(i) \Rightarrow (iii)$: This follows from Theorem 1.1, Item (6).

$(iii) \Rightarrow (i)$: As D is a derivation, $D(e) = 0$, see [26], Proposition 3. Furthermore, as D is skew-symmetric, $D^T(e) = 0$. Now suppose $(\overline{x}, \overline{y})$ is any optimal strategy pair for the game (D, e) so that $D^T(\overline{y}) \leq 0 \leq D(\overline{x})$. Since (e, e) is also an optimal strategy pair and $e > 0$, we have $D^T(\overline{y}) = 0 = D(\overline{x})$, see Theorem 1.1, Item (1). So, when (iii) holds, $\overline{x}$ is a positive multiple of e; thus, $\overline{x} > 0$. This proves (i). □

We refer to the last part of Example 4.1 for an illustration of the above theorem.

5 Concluding remarks

In the setting of a self-dual cone in a finite dimensional real inner product space, we showed that a Z-transformation with value zero is completely mixed if and only if it is space-irreducible. We also gave a finer characterization of cone-irreducibility for a positive transformation on a Euclidean Jordan algebra and generalized a recent result of Parthasarathy et al. to the setting of a symmetric cone. As mentioned in the Introduction, one can look for the properties of Z-transformations with value zero in regard to dynamical systems and complementarity theory. These are left for future study.

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References

1. Berman, A., Plemmons, R.J.: Nonnegative matrices in the Mathematical Sciences. SIAM, Philadelphia (1994)
2. Elsner, L.: Quasimonotonie und Ungleichungen in halbgeordneten Rämen. Linear Alg. Appl. **8**, 249–261 (1974)
3. Encinas, A.M., Mondal, S., Sivakumar, K.C.: On an analogue of a property of singular M-matrices, for the Lyapunov and the Stein operators. Linear and Multilinear Algebra **73**, 1–16 (2025)
4. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford University Press, Oxford (1994)
5. Gokulraj, S., Chandrashekaran, A.: On symmetric linear games. Linear Algebra Appl. **562**, 44–54 (2019)
6. Gokulraj, S., Chandrashekaran, A.: Linear complementarity problems and bilinear games. Applications of Mathematics. **65**, 665–675 (2020)
7. Gowda, M.S.: Completely mixed linear games on a self-dual cone. Linear Algebra Appl. **498**, 219–230 (2016)
8. Gowda, M.S.: Completely mixed linear games for Z-transformations over self-dual cones. arXiv:2310.13464v1 (2023)
9. Gowda, M.S., Ravindran, G.: On the game-theoretic value of a linear transformation relative to a self-dual cone. Linear Algebra Appl. **469**, 440–463 (2015)
10. Gowda, M.S., Sznajder, R.: Automorphism invariance of linear transformations on Euclidean Jordan algebras. Math. Oper. Res. **31**, 109–123 (2006)
11. Gowda, M.S., Sznajder, R., Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. **393**, 203–232 (2004)
12. Gowda, M.S., Tao, J.: Z-transformations on proper and symmetric cones. Math. Prog. Series B. **117**, 195–222 (2009)

13. Gowda, M.S., Tao, J., Ravindran, G.: On the P-property of Z and Lyapunov-like transformations on Euclidean Jordan algebras. Linear Algebra Appl. **436**, 2201–2209 (2012)
14. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
15. Idel, M.: A review of matrix scaling and Sinkhorn's normal form for matrices and positive maps. arXiv:1609.06349 (2016)
16. Kaplansky, I.: A contribution to von Neumann's theory of games. Ann. of Math. **46**, 474–479 (1945)
17. Kaplansky, I.: A contribution to von Neumann's theory of games. II. Linear Algebra Appl. **226–228**, 371–372 (1995)
18. Karlin, S.: Mathematical Methods and Theory in Games, Programming, and Economics. Addison Wesley Publishing Company, Reading, MA (1959)
19. Li, W.: Characterizations of singular irreducible M-matrices. Linear and Multilinear Alg. **38**, 241–247 (1995)
20. Orlitzky, M.: Positive Operators, Z-Operators, Lyapunov Rank, and Linear Games on Closed Convex Cones. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.), University of Maryland, Baltimore County (2017)
21. Orlitzky, M.: On Elsner's irreducibility concept. UMBC Optimization Seminar. March 28 (2024)
22. Parthasarathy, T., Ravindran, G., Kumar, S.: On completely mixed games. Jour. Optim. Theory and Appl. **201**, 313–322 (2024)
23. Raghavan, T.E.S.: Completely mixed games and M-matrices. Linear Algebra Appl. **21**, 35–45 (1978)
24. Robert, P.: On the group inverse of a linear transformation. J. Math. Anal. Appl. **22**, 658–669 (1968)
25. Schneider, H., Vidyasagar, M.: Cross-positive matrices. SIAM J. Num. Anal. **7**, 508–519 (1970)
26. Tao, J., Gowda, M.S.: A representation theorem for Lyapunov-like transformations on Euclidean Jordan algebras. Inter. Game Theory Rev. 13, 1340034*1-11 (2013)
27. Vandergraft, G.: Spectral properties of matrices having invariant cones. SIAM J. Appl. Math. **16**, 1208–1222 (1968)
28. Wolf, M.M.: Quantum Channels and Operations, Guided Tour. Lecture notes, https://mediatum.ub.tum.de (2012)

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