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# Some new results on semi-FTvN systems

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ABSTRACT

A semi-FTvN system is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the sharpened Cauchy-Schwarz inequality $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq \|x\|\,\|y\|$ for all $x, y \in \mathcal{V}$. Such a system arises, for example, from a complete hyperbolic polynomial and includes the so-called Fan-Theobald-von Neumann system as a special case. Going beyond Hermitian matrices and Euclidean Jordan algebras, it provides a general framework for studying concepts such as (weak) center, majorization, and doubly stochastic transformation. While our previous work [9] introduced semi-FTvN systems and dealt mainly with commutation principles in optimization, in the present paper, we describe some new results specifically dealing with majorization, the Haar measure on the automorphism group, and the comparison of eigenvalue maps.



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## 1. Introduction

The main objective of the present work is to describe some new results on semi-FTvN systems. Simply put, a semi-FTvN system is defined by the 'sharpened Cauchy-Schwarz inequality'

$$\langle x,y\rangle \leq \langle \lambda(x),\lambda(y)\rangle \leq \|x\|\,\|y\|\,;$$

see below for a formal definition. Such a system arises, for example, from a complete hyperbolic polynomial and includes an FTvN system as a special case. While FTvN systems include Euclidean Jordan algebras, normal decomposition systems, systems induced from complete and isometric hyperbolic polynomials, and have been studied in [7,8,11] for their algebraic and optimization significance, the recent work [9] introduces and explores semi-FTvN systems.

First, some motivation. Recall that a real homogeneous polynomial $p$ of degree $n$, defined over a nonzero finite-dimensional real vector space $\mathcal{V}$, is said to be *hyperbolic* [5] relative to a nonzero vector $e \in \mathcal{V}$ if $p(e) \neq 0$ and, for each $x \in \mathcal{V}$, the roots of the univariate polynomial $t \mapsto p(te - x)$ are real. For such a polynomial, one defines the eigenvalue map $\lambda : \mathcal{V} \to \mathcal{R}^n$ which assigns to each $x \in \mathcal{V}$ the vector of roots of this univariate polynomial arranged in decreasing order. We then say that the triple $(\mathcal{V}, p, e)$ is a hyperbolic system. Such systems appear in numerous fields and have applications in optimization, combinatorics, algebraic geometry, etc.; see [2,5,10,15] for example. Among hyperbolic systems, Euclidean Jordan algebras [4] form a well-studied class, especially for their optimization significance and results dealing with (algebra/cone) automorphism groups, (operator) commutativity, majorization, etc. How can these concepts and results be described for general hyperbolic systems? This question, along with other motivating factors, led us to study the broader framework of semi-FTvN systems, described below.

We start with the following (slightly modified) results of Bauschke et al. [2, Theorem 4.2 and Proposition 4.4],

**Theorem 1.1.** *Consider a hyperbolic system $(\mathcal{V}, p, e)$. Let $n$ be the degree of $p$ and $\lambda : \mathcal{V} \to \mathcal{R}^n$ be the associated eigenvalue map. Then,*

$$\langle x,y\rangle := \frac{1}{4}\Big[\ \|\lambda(x+y)\|^2_{\mathcal{R}^n} - \|\lambda(x-y)\|^2_{\mathcal{R}^n}\ \Big] \quad (x,y \in \mathcal{V}) \tag{1}$$

*defines a semi-inner product on $\mathcal{V}$ (with the associated semi-norm $\|\cdot\|$) satisfying the* sharpened Cauchy-Schwarz inequality

$$\langle x,y\rangle \leq \langle \lambda(x),\lambda(y)\rangle_{\mathcal{R}^n} \leq \|x\|\,\|y\| \quad (\forall\ x,y \in \mathcal{V}). \tag{2}$$

*Moreover, $\langle\cdot,\cdot\rangle$ becomes an inner product (and $\|\cdot\|$ becomes a norm) when $p$ is complete, that is, when $\lambda(x) = 0$ implies $x = 0$.*

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Motivated by condition (2), we introduce a semi-FTvN system.

**Definition 1.2.** A semi-FTvN system is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the condition

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq \|x\|\,\|y\| \quad (\forall\, x, y \in \mathcal{V}). \tag{3}$$

(Note that inner products and norms are taken in possibly different spaces.)

Thus, every complete hyperbolic polynomial induces a semi-FTvN system.

It is easy to see that (3) is equivalent to the (norm-preserving and inner product expanding) conditions

$$\|\lambda(x)\| = \|x\| \quad \text{and} \quad \langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \quad (\forall\, x, y \in \mathcal{V}). \tag{4}$$

Interestingly, these two conditions appear in the formulation of an FTvN system (short for Fan-Theobald-von Neumann system); see [8, Definition 3.1]. In a compressed form, *an FTvN system is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces with $\lambda : \mathcal{V} \to \mathcal{W}$ satisfying the condition*

$$\max\Big\{\langle c, z\rangle :\ z \in [u]\Big\} = \langle \lambda(c), \lambda(u)\rangle \quad (\forall\, c, u \in \mathcal{V}), \tag{5}$$

*where* $[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}$ *denotes the so-called* $\lambda$-orbit *of* $u \in \mathcal{V}$. (The names of Fan, Theobald, and von Neumann appear here because of their results on eigen/singular values of Hermitian/general matrices, see [8].)

There are many examples of FTvN systems. Two simple ones (over a real inner product space $\mathcal{V}$) are: $(i)$ $(\mathcal{V}, \mathcal{V}, \lambda)$, where $\lambda : \mathcal{V} \to \mathcal{V}$ is a linear isometry, and $(ii)$ $(\mathcal{V}, \mathcal{R}, \lambda)$, where $\lambda : \mathcal{V} \to \mathcal{R}$, $\lambda(x) = \|x\|$. Non-trivial examples include $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$ (decreasing rearrangement of the vector $x$), Euclidean Jordan algebras, and normal decomposition systems (Eaton triples). It is known that a complete isometric hyperbolic polynomial [2] induces an FTvN system. Recent articles [7–9,11] describe numerous examples and results dealing with FTvN systems.

Semi-FTvN systems were introduced in [9] by focusing on the commutation principles in optimization. In that paper, the concepts of automorphism group, commutativity, strong commutativity, center, weak center, and unit element were studied. The present paper introduces several new concepts and discusses related examples and results. A summary/outline of our contributions is given below (see Section 2 for definitions).

- In Section 2, we recall some basic concepts and results concerning semi-FTvN systems. In particular, given a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, we consider the corresponding *automorphism group* $\mathcal{G}$, consisting of all invertible (bounded) linear transformations $A$ on $\mathcal{V}$ satisfying the condition $\lambda(Ax) = \lambda(x)$ for all $x \in \mathcal{V}$, and define the $\mathcal{G}$-orbit of any $x \in \mathcal{V}$ by $\langle x\rangle := \{Ax : A \in \mathcal{G}\}$. With $\mathrm{Lie}(\mathcal{G})$ denoting the

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'Lie algebra' of $\mathcal{G}$, we define *commutativity* of $x, y \in \mathcal{V}$ by the condition $\langle Lx, y\rangle = 0$ for all $L \in \mathrm{Lie}(\mathcal{G})$ and *strong commutativity* via the condition $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$.

- In Section 3, we introduce the concepts of *majorization* and *doubly stochastic transformation* in the setting of a semi-FTvN system: We say that $x$ is *majorized* by $y$ if $x$ is in the convex hull of (the $\lambda$-orbit) $[y]$; a linear transformation $D$ is said to be *doubly stochastic* if for every $x \in \mathcal{V}$, $Dx$ is majorized by $x$. Replacing $[y]$ by the $\mathcal{G}$-orbit $\langle y\rangle$, we define *strong majorization* and *strongly doubly stochastic transformation*. We also describe some related properties.
- In Section 4, in the setting of a finite-dimensional semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, we consider the matrix Lie group $\mathcal{G}$. By working with the Haar measure $h$ on $\mathcal{G}$, we show that the map $D : x \mapsto x^* := \int_{A\in\mathcal{G}} Ax\, dh$ is *strongly doubly stochastic and that for any $x \in \mathcal{V}$, $x^*$ is in the weak center* (meaning that $x^*$ commutes with every element of $\mathcal{V}$). With this, we show that the weak center has a nonempty intersection with the convex hull of any $\lambda$-orbit.
- Calling the map $\lambda$ in a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ an eigenvalue map, in Section 5, we introduce a partial order among such maps and discuss equivalence.

## 2. Preliminaries

Throughout this paper, $\mathcal{R}^n$ denotes the usual real $n$-dimensional Euclidean space whose elements are viewed as column vectors.

Semi-FTvN and FTvN systems were formulated in Section 1. In what follows, we recall some definitions and results on semi-FTvN and FTvN systems from earlier works [7–9,11].

**Definition 2.1.** Given a real inner product space $\mathcal{V}$, we say that $\lambda$ is an eigenvalue map on $\mathcal{V}$ if there exists a real inner product space $\mathcal{W}$ such that $(\mathcal{V}, \mathcal{W}, \lambda)$ forms a semi-FTvN system.

**Definition 2.2.** [9] In the setting of a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$,

*(i)* elements $x, y \in \mathcal{V}$ *strongly commute* if $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$;
*(ii)* the *center* $\mathcal{C}$ is the set of all elements in $\mathcal{V}$ that strongly commute with every element of $\mathcal{V}$;
*(iii)* a nonzero element $e \in \mathcal{V}$ is said to be a *unit* when $\mathcal{C} = \mathcal{R}e$.

We remark that the center is always a closed subspace of $\mathcal{V}$ [9].

**Definition 2.3.** [9] Given a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, an invertible linear transformation $A : \mathcal{V} \to \mathcal{V}$ is said to be an *automorphism* if $\lambda(Ax) = \lambda(x)$ for all $x \in \mathcal{V}$.

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Note: Because $\lambda$ is norm-preserving, every automorphism is a bounded linear transformation. Moreover, when $\mathcal{V}$ is finite-dimensional, the invertibility requirement on $A$ in the above definition is superfluous.

We let $\mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda)$ be the set of all such automorphisms. For simplicity, we write

$$\mathcal{G} = \mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda).$$

As $\lambda$ is norm-preserving, every automorphism − in addition to being invertible − is orthogonal (equivalently, inner product-preserving). Hence, because of continuity of $\lambda$ [9], $\mathcal{G}$ is a closed subgroup of the orthogonal group $\mathcal{O}(\mathcal{V})$ (consisting of all invertible norm-preserving linear transformations). Corresponding to $\mathcal{G}$, we define its *'Lie algebra'* as

$$\mathrm{Lie}(\mathcal{G}) := \{L \in \mathcal{L}(\mathcal{V}) : \exp(tL) \in \mathcal{G} \text{ for all } t \in \mathcal{R}\},$$

where $\mathcal{L}(\mathcal{V})$ denotes the set of all bounded linear transformations on $\mathcal{V}$. (Here $\exp(tL)$ denotes the sum of the power series $\sum_{k=0}^{\infty} \frac{t^k L^k}{k!}$. In the above definition, we consider only those $L$ for which this power series converges in $\mathcal{L}(\mathcal{V})$. We note that when $\mathcal{V}$ is finite-dimensional, $\mathcal{G}$ is a matrix Lie group and so $\mathrm{Lie}(\mathcal{G})$ is the corresponding Lie algebra.)

Next, we define the concepts of *commutativity* and *weak center*.

**Definition 2.4.** [9] In the setting of a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$,

- $(i)$ elements $x, y \in \mathcal{V}$ *commute* if $\langle Lx, y\rangle = 0$ for all $L \in \mathrm{Lie}(\mathcal{G})$;
- $(ii)$ the *weak center* $\mathcal{C}_w$ is the set of all elements in $\mathcal{V}$ that commute with every element of $\mathcal{V}$; thus, $\mathcal{C}_w = \{x \in \mathcal{V} : Lx = 0 \text{ for all } L \in \mathrm{Lie}(\mathcal{G})\}$.

**Remarks.** It is known that strong commutativity implies commutativity, see [9, Theorem 3.6]. Also, in the setting of a Euclidean Jordan algebra, commutativity reduces to operator commutativity, see [9, Theorem 4.1].

Recall that for any $x$ in a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$, the $\lambda$-orbit $[x]$ and the $\mathcal{G}$-orbit $\langle x\rangle$ are defined, respectively, by

$$[x] := \{y \in \mathcal{V} : \lambda(y) = \lambda(x)\} \quad \text{and} \quad \langle x\rangle := \{Ax : A \in \mathcal{G}\}.$$

Note: When dealing with more than one eigenvalue map $\lambda$, we denote the above orbits by $[x]_\lambda$ and $\langle x\rangle_\lambda$ respectively. Observe that $\langle x\rangle \subseteq [x]$ for all $x \in \mathcal{V}$. Moreover, when $\mathcal{V}$ is finite-dimensional, these orbits are compact in $\mathcal{V}$.

The following result will be used in the sequel.

**Proposition 2.5.** [9, Proposition 3.10] *For an element $c \in \mathcal{V}$ in a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$, consider the following statements:*

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*(a)* $c \in \mathcal{C}$.
*(b)* $[c] = \{c\}$.
*(c)* $Ac = c$ *for all* $A \in \mathcal{G}$.
*(d)* $c \in \mathcal{C}_w$.

*Then* $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. *Moreover, when* $(\mathcal{V}, \mathcal{W}, \lambda)$ *is an FTvN system, we have* $(a) \Leftrightarrow (b)$.

**Definition 2.6.** A semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ is called *orbit-transitive* if $\langle x \rangle = [x]$ for all $x \in \mathcal{V}$.

It has been observed that all simple Euclidean Jordan algebras and the system $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$ with $\lambda(x) = x^{\downarrow}$ are orbit-transitive, see [14, Theorem 4] and [8, Theorem 7.4].

**Definition 2.7.** Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system and $E$ be a set in $\mathcal{V}$.

*(i)* $E$ is called a *spectral set* if it is of the form $E = \lambda^{-1}(Q)$ for some set $Q \subseteq \mathcal{W}$.
*(ii)* $E$ is called a *weakly spectral set* if $A(E) \subseteq E$ for all $A \in \mathcal{G}$.

It is easy to see that a set $E \subseteq \mathcal{V}$ is spectral (weakly spectral) if and only if for all $x \in E$, $[x] \subseteq E$ (respectively, $\langle x \rangle \subseteq E$). We also note that every spectral set is weakly spectral.

The following refers to an FTvN system.

**Definition 2.8.** [8] For an FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, its *reduced system* $(\mathcal{W}, \mathcal{W}, \mu)$ is an FTvN system such that $\mu \circ \lambda = \lambda$ and $\operatorname{ran} \mu \subseteq \operatorname{ran} \lambda$.

For example, if the FTvN system $(\mathcal{V}, \mathcal{R}^n, \lambda)$ comes from a Euclidean Jordan algebra of rank $n$, then $(\mathcal{R}^n, \mathcal{R}^n, \mu)$, with $\mu(x) = x^{\downarrow}$, is a reduced system of $(\mathcal{V}, \mathcal{R}^n, \lambda)$. For further properties of reduced systems, see [8]. We recall one result here.

**Proposition 2.9.** [8, Proposition 10.2] *Let* $(\mathcal{V}, \mathcal{W}, \lambda)$ *be an FTvN system with its reduced system* $(\mathcal{W}, \mathcal{W}, \mu)$. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *denote their respective centers. Then,*

$$\lambda(\mathcal{V}) \cap -\lambda(\mathcal{V}) = \lambda(\mathcal{C}) = \mathcal{C}'. \tag{6}$$

*In particular, if* $e$ *is a unit in* $(\mathcal{V}, \mathcal{W}, \lambda)$, *then* $\lambda(e)$ *is a unit in* $(\mathcal{W}, \mathcal{W}, \mu)$.

We note that in an FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, the set $\lambda(\mathcal{V})$ is a convex cone, see [8, Proposition 5.2]; thus, $\lambda(\mathcal{V}) \cap -\lambda(\mathcal{V})$ is the lineality space (i.e., the largest linear subspace) of $\lambda(\mathcal{V})$.

Henceforth, we say that $(\mathcal{V}, \mathcal{W}, \lambda)$ is *a finite-dimensional semi-FTvN system* if $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system, where $\mathcal{V}$ is finite-dimensional.

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## 3. Majorization and doubly stochastic transformations

In $\mathcal{R}^n$, the (classical) majorization [12] is defined as follows: Given two column vectors $u, v \in \mathcal{R}^n$, let $u^\downarrow$ and $v^\downarrow$ denote their respective decreasing rearrangements. Then $u$ is majorized by $v$ in $\mathcal{R}^n$ if for all $k \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{k} u_i^\downarrow \leq \sum_{i=1}^{k} v_i^\downarrow \quad \text{and} \quad \sum_{i=1}^{n} u_i^\downarrow = \sum_{i=1}^{n} v_i^\downarrow.$$

We now formulate the concepts of majorization and doubly stochastic transformation in the broader setting of semi-FTvN systems. In what follows, for a set $S$, we write $\operatorname{conv} S$ for the convex hull of $S$ (i.e., the smallest convex set containing $S$).

**Definition 3.1.** Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system with its automorphism group $\mathcal{G}$. Let $x, y \in \mathcal{V}$ and $D : \mathcal{V} \to \mathcal{V}$ be a linear transformation. We say that

- $(i)$ $x$ is *majorized* by $y$ if $x \in \operatorname{conv}[y]$. In this case, we write $x \prec y$.
- $(ii)$ $x$ is *strongly majorized* (or $\mathcal{G}$-majorized) by $y$ if $x \in \operatorname{conv}\langle y\rangle$. In this case, we write $x \prec\!\prec y$.
- $(iii)$ $D$ is said to be *doubly stochastic* (*strongly doubly stochastic*) if $D(x) \prec x$ (respectively, $D(x) \prec\!\prec x$) for all $x \in \mathcal{V}$.

In the setting of an FTvN system, the concept of majorization (Item $(i)$ above) was introduced and studied in [8, Sections 8-10]. In the cited paper, it was observed that this formulation reduces to the classical definition in $\mathcal{R}^n$ (thanks to the results of Hardy, Littlewood, and Pólya, and Birkhoff [3]) and in the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices; see [1, Theorem 7.1]. The concept of $\mathcal{G}$-majorization (called group majorization in the literature) is well-known; see e.g., [13].

In some special settings (for example, Euclidean Jordan algebras and complete hyperbolic polynomials), the given semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ comes with an associated semi-FTvN system $(\mathcal{W}, \mathcal{U}, \mu)$. In this case, one can introduce a different majorization concept on $(\mathcal{V}, \mathcal{W}, \lambda)$ via

$$x \underset{\lambda}{\prec} y \text{ in } (\mathcal{V}, \mathcal{W}, \lambda) \iff \lambda(x) \prec \lambda(y) \text{ in } (\mathcal{W}, \mathcal{U}, \mu).$$

An interesting problem then, is to relate this new majorization to the ones given earlier. For some results about this, see Theorems 3.4 and 3.5 below.

We note that every strongly doubly stochastic transformation is doubly stochastic. The converse holds when $\mathcal{V}$ is orbit-transitive.

One application of majorization comes in the form of proving (convex) inequalities. To elaborate, consider a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$. We say that a function $F : \mathcal{V} \to [-\infty, \infty]$ is a *spectral function* if it is of the form $F = f \circ \lambda$ for some $f : \mathcal{W} \to [-\infty, \infty]$. (Equivalently, $F$ is constant on any $\lambda$-orbit.) We have the following.

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**Proposition 3.2.** *On a semi-FTvN system, consider a convex spectral function $F$. Then, for $x, y \in \mathcal{V}$,*

$$x \prec\prec y \implies x \prec y \implies F(x) \leq F(y).$$

**Proof.** Since $\langle y \rangle \subseteq [y]$, the first implication follows. Suppose $x \prec y$ so that $x = \sum_{i=1}^{k} t_i y_i$, where $t_i$'s are nonnegative with sum 1 and $y_i \in [y]$ for all $i$. We may assume that $F(y) < \infty$, else the inequality $F(x) \leq F(y)$ holds trivially. Since $\lambda(y_i) = \lambda(y)$ for all $i$, $F(y_i) = F(y) < \infty$ for all $i$. Then, by convexity,

$$F(x) \leq \sum_{i=1}^{k} t_i F(y_i) = \sum_{i=1}^{k} t_i F(y) = F(y).$$

This completes the proof. $\square$

To see a specific example, consider the convex function $F(x) = \|x\|$ on $\mathcal{V}$. As $\lambda$ is norm-preserving, $F$ is constant on any $\lambda$-orbit. Hence, $F$ is convex and spectral. Therefore,

$$x \prec y \implies \|x\| \leq \|y\|. \tag{7}$$

Consequently, *every (strongly) doubly stochastic transformation is bounded.*

The following is a modified form of [8, Proposition 8.3], which is originally stated for FTvN systems.

**Proposition 3.3.** *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system and $x, y \in \mathcal{V}$. Then,*

- *(a) $x \prec y$ implies $\langle c, x \rangle \leq \langle \lambda(c), \lambda(y) \rangle$ for every $c \in \mathcal{V}$; the reverse implication holds when $(\mathcal{V}, \mathcal{W}, \lambda)$ is a finite-dimensional FTvN system.*
- *(b) $x \prec y$ and $y \prec x$ if and only if $[x] = [y]$.*
- *(c) $x \prec\prec y$ and $y \prec\prec x$ if and only if $\langle x \rangle = \langle y \rangle$.*

**Proof.** The proofs of $(a)$ and $(b)$ are similar to those for FTvN systems, see [8, Proposition 8.3].

$(c)$ The 'if' part readily follows from the definition. For the 'only if' part, assume $x \prec\prec y$ and $y \prec\prec x$ so that $x = \sum_{i=1}^{k} t_i A_i y$ and $y = \sum_{j=1}^{l} s_j B_j x$, where $t_i, s_j > 0$, $\sum_{i=1}^{k} t_i = \sum_{j=1}^{l} s_j = 1$, and $A_i, B_j \in \mathcal{G}$ for all $i, j$. Since $\|A_i y\| = \|\lambda(A_i y)\| = \|\lambda(y)\| = \|y\|$, we have

$$\|x\| = \left\| \sum_{i=1}^{k} t_i A_i y \right\| \leq \sum_{i=1}^{k} t_i \|A_i y\| = \sum_{i=1}^{k} t_i \|y\| = \|y\|.$$

Similarly, from the condition $y = \sum_{j=1}^{l} s_j B_j x$, we get $\|y\| \leq \|x\|$. Hence, the above inequality becomes an equality. Then, by the strict/uniform convexity of the (inner product) norm, we must have $x = A_i y$ for all $i$; hence $x \in \langle y \rangle$ and (as $\mathcal{G}$ is a group) $\langle x \rangle \subseteq \langle y \rangle$. Similarly, $\langle y \rangle \subseteq \langle x \rangle$; hence, $\langle x \rangle = \langle y \rangle$. $\square$

**Theorem 3.4** *(Majorization in a Euclidean Jordan algebra).* [8, Corollary 10.4] *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank $n$ carrying the trace inner product. Consider the FTvN system $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\lambda$ is the eigenvalue map. Then, for $x, y \in \mathcal{V}$,*

$$x \prec y \text{ in } \mathcal{V} \iff \lambda(x) \prec \lambda(y) \text{ in } \mathcal{R}^n.$$

Our next result deals with majorization in a semi-FTvN system induced by a complete hyperbolic polynomial.

**Theorem 3.5** *(Majorization in a hyperbolic system).* [8, Proposition 10.5] *Let $(\mathcal{V}, \mathcal{R}^n, \lambda)$ be the semi-FTvN system induced by a complete hyperbolic polynomial $p$ of degree $n$. Then, for $x, y \in \mathcal{V}$,*

$$x \prec y \text{ in } \mathcal{V} \implies \lambda(x) \prec \lambda(y) \text{ in } \mathcal{R}^n.$$

*The reverse implication holds if $(\mathcal{V}, \mathcal{R}^n, \lambda)$ is an FTvN system (i.e., when $p$ is both complete and isometric [2]).*

In the result below, we describe some properties of a doubly stochastic transformation. When $D : \mathcal{V} \to \mathcal{V}$ is a linear transformation, we define the *adjoint* of $D$ as a linear transformation $D^* : \mathcal{V} \to \mathcal{V}$ satisfying the condition $\langle D^* x, y \rangle = \langle x, Dy \rangle$ for all $x, y \in \mathcal{V}$. Such a $D^*$ exists, for example, when $\mathcal{V}$ is a Hilbert space, in particular, when $\mathcal{V}$ is finite-dimensional.

**Proposition 3.6.** *Suppose $D$ is a doubly stochastic transformation on a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ with its adjoint $D^*$ (whenever defined). Then, the following statements hold:*

- $(a)$ *For every convex spectral set $K$ in $\mathcal{V}$, $D(K) \subseteq K$.*
- $(b)$ *For any $u$ in the center $\mathcal{C}$, $Du = u$ and $D^* u = u$. In particular, if $e$ is a unit element, then $De = e$ and $D^* e = e$.*
- $(c)$ *If $(\mathcal{V}, \mathcal{W}, \lambda)$ is a finite-dimensional FTvN system, then $D^*$ exists and is doubly stochastic.*

**Proof.** $(a)$ Let $K$ be a convex spectral set. Then for any $x \in K$, we have $[x] \subseteq K$ and $\operatorname{conv}[x] \subseteq K$. Thus, $Dx \in \operatorname{conv}[x] \subseteq K$. It follows that $D(K) \subseteq K$.

$(b)$ Let $u \in \mathcal{C}$. Then, by Proposition 2.5 $(b)$, $[u] = \{u\}$. Hence, $Du \in \operatorname{conv}\{u\} = \{u\}$. Thus, $Du = u$. In particular, if $e$ is a unit, then $De = e$.

Suppose, for contradiction, $D^* u \neq u$. Then (working in the finite-dimensional subspace generated by $u$, $Du$, and $D^* u$), we can find a $d \in \mathcal{V}$ such that $\langle d, D^* u \rangle > \langle d, u \rangle$. Then,

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$$\langle d, u \rangle < \langle d, D^*u \rangle = \langle Dd, u \rangle \leq \langle \lambda(d), \lambda(u) \rangle = \langle d, u \rangle,$$

where the second inequality is due to the fact that $Dd \prec d$ and Proposition 3.3 $(a)$; the last equality is due to the assumption that $u \in \mathcal{C}$. As this cannot occur, we must have $D^*u = u$; in particular, $D^*e = e$.

$(c)$ See the proof of item $(c)$ in [8, Proposition 9.2]. □

**Remarks.** Consider a semi-FTvN system induced by a complete hyperbolic polynomial $p$ of degree $n$ (relative to $e$). Let $D$ be doubly stochastic on $(\mathcal{V}, \mathcal{R}^n, \lambda)$. As the hyperbolicity cone $\Lambda_+ := \lambda^{-1}(\mathcal{R}^n_+)$ is a spectral set and convex (shown by Gårding [5]), we see that $D(\Lambda_+) \subseteq \Lambda_+$. This inclusion may be viewed as the 'nonnegativity' property of $D$. Moreover, as $e$ is a unit in the semi-FTvN system [9, Section 5], we see that $De = e$ and $D^*e = e$. These latter properties can be viewed as 'row stochastic' and 'column stochastic' properties of $D$. *It is unclear if these three properties characterize doubly stochastic transformations,* thus motivating the following problem.

**Problem 1.** Consider a complete hyperbolic system $(\mathcal{V}, p, e)$ and its induced semi-FTvN system $(\mathcal{V}, \mathcal{R}^n, \lambda)$. Suppose a linear transformation $D : \mathcal{V} \to \mathcal{V}$ satisfies the following conditions: $D(\Lambda_+) \subseteq \Lambda_+$, $De = e$, and $D^*e = e$. Can we say that $Dx \prec x$ for all $x \in \mathcal{V}$?

In the following, we write $\mathrm{GL}(\mathcal{V})$ for the set of all (bounded) invertible linear transformations on $\mathcal{V}$ and Ext(S) for the set of all extreme points of a convex set $S$. Recall that $\mathcal{G} = \mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$.

**Theorem 3.7.** *For a finite-dimensional semi-FTvN system* $(\mathcal{V}, \mathcal{W}, \lambda)$*, let* $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ *denote the set of all doubly stochastic transformations on* $\mathcal{V}$*. Then the following statements hold:*

$(i)$ $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ *is compact and convex in* $\mathcal{L}(\mathcal{V})$*.*
$(ii)$ $\mathcal{G} = \left\{A \in \mathrm{GL}(\mathcal{V}) : A, A^{-1} \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)\right\}$.
$(iii)$ $\mathcal{G} \subseteq \mathrm{Ext}\left(\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)\right)$*. Consequently,* $\mathrm{conv}\, \mathcal{G} \subseteq \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$*.*
$(iv)$ *Suppose* $(\mathcal{V}, \mathcal{W}, \lambda)$ *is orbit-transitive. If* $x \prec y$*, then there exists* $D \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ *such that* $x = Dy$*.*

**Proof.** $(i)$ The convexity of $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ is easily seen. For any $D \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$, from (7), we have $\|Dx\| \leq \|x\|$ for all $x \in \mathcal{V}$; consequently, the norm of $D$ in $\mathcal{L}(\mathcal{V})$ is bounded by 1. Thus $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ is bounded in $\mathcal{L}(\mathcal{V})$. To show that $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ is closed, consider a sequence $D_k$ in $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ converging to $D \in \mathcal{L}(\mathcal{V})$. Fix $x \in \mathcal{V}$ and consider the $\lambda$-orbit $[x]$. As $\mathcal{V}$ is finite dimensional, $[x]$ is compact; by a result of Minkowski, see [16, Theorem 3.20], $\mathrm{conv}\,[x]$ is compact. As $D_k x \in \mathrm{conv}\,[x]$ for all $k$, we see that $Dx \in \mathrm{conv}\,[x]$. This proves that $D$ is doubly stochastic; hence, $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ is closed. Since $\mathcal{L}(\mathcal{V})$ is finite dimensional, $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ is compact. Thus, we have $(i)$.

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$(ii)$ Suppose $A \in \mathcal{G}$. Since $\lambda(Ax) = \lambda(x)$ for all $x \in \mathcal{V}$, we have $Ax \in [x]$. It follows that $Ax \prec x$ for all $x$. Thus, $A$ is doubly stochastic. Applying the same reasoning to $A^{-1}$, we see that $A^{-1}$ is also doubly stochastic.

Now suppose $A$ is invertible with both $A$ and $A^{-1}$ doubly stochastic. Then, $Ax \prec x$ and $A^{-1}y \prec y$ for all $x, y \in \mathcal{V}$. Putting $y = Ax$, we see that $Ax \prec x$ and $x \prec Ax$ for all $x$. By Proposition 3.3 $(b)$, we have $[Ax] = [x]$ for all $x$. This means that $\lambda(Ax) = \lambda(x)$ for all $x$; thus, $A \in \mathcal{G}$.

$(iii)$ Suppose $A \in \mathcal{G}$. From $(ii)$, $A \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$. To show that $A$ is an extreme point of $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$, assume $A = \sum_{i=1}^{k} t_i D_i$, where $D_i \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ and $t_i > 0$ for all $i$ with $\sum_{i=1}^{k} t_i = 1$. Then, for any $x \neq 0$ in $\mathcal{V}$,

$$\|x\| = \|Ax\| = \left\| \sum_{i=1}^{k} t_i D_i x \right\| \leq \sum_{i=1}^{k} t_i \|D_i x\| \leq \sum_{i=1}^{k} t_i \|x\| = \|x\|.$$

By the strict convexity of the (inner product) norm, we see that $Ax = D_i x$ for all $i$. As $x$ is arbitrary, $A = D_i$ for all $i$. Thus, $A$ is an extreme point of $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$. The inclusion $\mathrm{conv}\, \mathcal{G} \subseteq \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ comes from the convexity of $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$.

$(iv)$ Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is orbit-transitive and $x \prec y$. Then $x = \sum_{i=1}^{k} t_i y_i$, where $t_i$s are positive with sum 1 and $y_i \in [y]$ for all $i$. As $(\mathcal{V}, \mathcal{W}, \lambda)$ is orbit-transitive, there exists $A_i \in \mathcal{G}$ such that $y_i = A_i y$ for each $i$. Defining $D := \sum_{i=1}^{k} t_i A_i$, we see that $x = Dy$ with $D \in \mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ (from $(iii)$ above). $\square$

**Remarks.** In $\mathcal{R}^n$, Birkhoff's theorem says that the extreme points of the set of all doubly stochastic matrices are, precisely, automorphisms on $\mathcal{R}^n$ (that is, permutation matrices). A similar statement holds in the Jordan spin algebra $\mathcal{L}^n$, see [6, Theorem 10]. However, in general, $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ may contain extreme points other than automorphisms, see [6, Example 11] for the simple Euclidean Jordan algebra $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices.

**Remarks.** In a finite-dimensional semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, let $\mathrm{SDS}(\mathcal{V}, \mathcal{W}, \lambda)$ denote the set of all strongly doubly stochastic transformations. As $\mathcal{V}$ is finite-dimensional, $\mathcal{G}$ is compact in $\mathcal{L}(\mathcal{V})$; consequently, $\langle x \rangle$ is compact in $\mathcal{V}$ for every $x \in \mathcal{V}$. Now, with minor modifications, one can replace $\mathrm{DS}(\mathcal{V}, \mathcal{W}, \lambda)$ in items $(i)$ – $(iii)$ of the above theorem by $\mathrm{SDS}(\mathcal{V}, \mathcal{W}, \lambda)$; Item $(iv)$ should be replaced by:

- If $x \prec\prec y$, then there exists $D \in \mathrm{SDS}(\mathcal{V}, \mathcal{W}, \lambda)$ such that $x = Dy$.

## 4. The Haar measure on $\mathcal{G}$

In $\mathcal{R}^n$, let $\mathbf{1}$ denote the (column) vector of ones. Then, for any $u \in \mathcal{R}^n$, it follows from elementary considerations that

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$$u^* := \frac{\langle u, \mathbf{1}\rangle}{n}\mathbf{1} \prec u.$$

This implies two things: First, the 'average' of $u$ is always majorized by $u$, and second, $u^* \in \mathcal{R}\mathbf{1} \cap \operatorname{conv}[u]$. As $\mathbf{1}$ is a unit in the FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^\downarrow$, the second statement can be interpreted as $\mathcal{C} \cap \operatorname{conv}[u] \neq \emptyset$ for every $u$. Do we have a result of this type in a general setting? In a different form, a question of this type was raised by Zhao [17], see below. Specifically, we ask:

*In a finite-dimensional semi-FTvN system, does the (weak) center have a nonempty intersection with the convex hull of any $\lambda$-orbit?*

We answer this affirmatively (at least for the weak center) by relying on the Haar measure on the automorphism group $\mathcal{G}$, see Theorem 4.5 below.

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system, where $\mathcal{V}$ is finite-dimensional. Then, $\mathcal{G} := \operatorname{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$ is a compact subgroup of the orthogonal group $\mathcal{O}(\mathcal{V})$. On this group $\mathcal{G}$, we consider the Haar measure $h$. Recall that this is a unique probability measure that is both right and left translation invariant [16, Theorem 5.14]. So, using $A$ as a variable of integration,

$$\int\limits_{A\in\mathcal{G}} f(BA)\,dh = \int\limits_{A\in\mathcal{G}} f(A)\,dh = \int\limits_{A\in\mathcal{G}} f(AB)\,dh,$$

for all $B \in \mathcal{G}$ and all real-valued continuous functions $f$ on $\mathcal{G}$.

Now, for any $x \in \mathcal{V}$, the (bounded) linear functional $y \mapsto \int_{A\in\mathcal{G}} \langle y, Ax\rangle\, dh$ defines, via the Riesz representation theorem, a unique element $x^* \in \mathcal{V}$ such that

$$\langle y, x^*\rangle = \int\limits_{A\in\mathcal{G}} \langle y, Ax\rangle\; dh \quad (\forall\, y \in \mathcal{V}). \tag{8}$$

We then write

$$x^* = \int\limits_{A\in\mathcal{G}} \phi_x(A)\,dh, \quad \text{where } \phi_x(A) := Ax, \tag{9}$$

and call it the *average* of $x$. We note that, as a consequence of (8),

$$Bx^* = B\Big(\int\limits_{A\in\mathcal{G}} \phi_x(A)\,dh\Big) = \int\limits_{A\in\mathcal{G}} B\phi_x(A)\,dh \quad (\forall\, B \in \mathcal{G}).$$

**Example 4.1.** Consider the FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \mu)$, where $\mathcal{R}^n$ carries the usual inner product and $\mu(u) = u^\downarrow$. Then,

$$\mathcal{G} = \operatorname{Aut}(\mathcal{R}^n, \mathcal{R}^n, \mu) = \Sigma_n,$$

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where $\Sigma_n$ is the set of all $n \times n$ permutation matrices. It is shown in [9, Example 3.15] that $\mathcal{C} = \mathcal{R}\mathbf{1}$ and $\mathcal{C}_w = \mathcal{R}^n$. Now, as the cardinality of $\Sigma_n$ is $n!$, the Haar measure $h$ on this group is the (discrete) probability measure which assigns the value $\frac{1}{n!}$ to each singleton set in $\Sigma_n$. Now, we take (the column vector) $u = (u_1, u_2, \ldots, u_n)^\mathsf{T}$ in $\mathcal{R}^n$ and consider the sum of all possible permutations of $u$. An easy calculation shows

$$\frac{1}{n!} \sum_{P \in \Sigma_n} Pu = \left( \frac{1}{n} \sum_{i=1}^{n} u_i \right) \mathbf{1} = \sigma_u \mathbf{1}, \tag{10}$$

where $\sigma_u := \dfrac{1}{n} \sum_i u_i = \dfrac{\langle u, \mathbf{1} \rangle}{n}$. Hence, from (10), for any $u \in \mathcal{R}^n$,

$$u^* = \int\limits_{P \in \Sigma_n} \phi_u(P)\, dh = \int\limits_{P \in \Sigma_n} (Pu)\, dh = \frac{1}{n!} \sum_{P \in \Sigma_n} Pu = \left( \frac{1}{n} \sum_{i=1}^{n} u_i \right) \mathbf{1} = \frac{\langle u, \mathbf{1} \rangle}{n}\, \mathbf{1}. \tag{11}$$

**Example 4.2.** Consider the semi-FTvN system $(\mathcal{R}, \mathcal{R}, \lambda)$, where $\lambda(x) = |x|$. Then $\mathcal{G} = \{I, -I\}$, where $I$ denotes the identity transformation, $\mathcal{C} = \{0\}$, and $\mathcal{C}_w = \mathcal{R}1$, see [9, Example 3.13]. The Haar measure on $\mathcal{G}$ is the discrete probability measure that assigns $\frac{1}{2}$ to $\{I\}$ and $\{-I\}$. In this case,

$$\int\limits_{A \in \mathcal{G}} f(A)\, dh = \frac{f(I) + f(-I)}{2}.$$

Then, for any $x$ and $y$,

$$\langle y, x^* \rangle = \int\limits_{A \in \mathcal{G}} \langle y, Ax \rangle\; dh = \frac{1}{2} \Big( \langle y, x \rangle + \langle y, -x \rangle \Big) = 0.$$

Hence, from (9), $x^* = 0$.

**Theorem 4.3.** *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a finite-dimensional semi-FTvN system. For $x \in \mathcal{V}$, we define $x^*$ by (9). Then the following statements hold:*

*(i) For every $x \in \mathcal{V}$, $x^* \in \mathrm{conv}\langle x \rangle \subseteq \mathrm{conv}\,[x]$.*
*(ii) For every convex (weakly) spectral set $E \subseteq \mathcal{V}$ and $x \in E$, we have $x^* \in E$.*
*(iii) The map $D : x \mapsto x^*$ is linear and strongly doubly stochastic.*

**Proof.** $(i)$ Fix $x \in \mathcal{V}$ and assume the contrary, i.e., $x^* \notin \mathrm{conv}\langle x \rangle$. Since $\mathcal{V}$ is finite-dimensional, the $\mathcal{G}$-orbit $\langle x \rangle$ is compact and so $\mathrm{conv}\,\langle x \rangle$ is compact and convex [16, Theorem 3.25]. By a separation theorem [16, Theorem 3.4], there exist $\alpha \in \mathcal{R}$ and $c \in \mathcal{V}$ such that

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$$\langle c, x^* \rangle < \alpha \leq \langle c, z \rangle, \quad (\forall\, z \in \text{conv}\, \langle x \rangle).$$

Now, for any $A \in \mathcal{G}$, we have $Ax \in \langle x \rangle$; hence, $\alpha \leq \langle c, Ax \rangle = \langle c, \phi_x(A) \rangle$. Then,

$$\alpha > \langle c, x^* \rangle = \left\langle c, \int\limits_{A \in \mathcal{G}} \phi_x(A)\, dh \right\rangle = \int\limits_{A \in \mathcal{G}} \langle c, \phi_x(A) \rangle \; dh \geq \alpha,$$

which is a contradiction. Hence, we must have $x^* \in \text{conv}\, \langle x \rangle$.

Items $(ii)$ and $(iii)$ follow from the definitions. $\square$

**Remarks.** It follows from the above theorem and Example 4.1 that in the setting of the FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \mu)$, where $\mu(u) = u^{\downarrow}$,

$$u^* = \frac{\langle u, \mathbf{1} \rangle}{n} \mathbf{1} \prec u \quad (\forall\, u \in \mathcal{R}^n). \tag{12}$$

As noted at the beginning of this section, this can also be seen from the classical definition of majorization in $\mathcal{R}^n$.

**Theorem 4.4.** *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a finite-dimensional semi-FTvN system and $x \in \mathcal{V}$. Define $x^*$ by (9). Then, the following statements hold:*

$(a)$ *$Bx^* = x^* = (Bx)^*$ for all $B \in \mathcal{G}$.*
$(b)$ *$x^* \in \mathcal{C}_w$.*
$(c)$ *$x^* \in \mathcal{C}$ when $(\mathcal{V}, \mathcal{W}, \lambda)$ is a finite-dimensional orbit-transitive FTvN system.*
*If, in addition, $(\mathcal{V}, \mathcal{W}, \lambda)$ has a unit element $e$, then $x^* = \dfrac{\langle x, e \rangle}{\langle e, e \rangle}\, e$.*

**Proof.** $(a)$ Let $B \in \mathcal{G}$. As $B$ is linear and $h$ is left-translation invariant, we have

$$\begin{aligned} Bx^* = B\left( \int\limits_{A \in \mathcal{G}} \phi_x(A)\, dh \right) &= \int\limits_{A \in \mathcal{G}} B(\phi_x(A))\, dh \\ = \int\limits_{A \in \mathcal{G}} \phi_x(BA)\, dh &= \int\limits_{A \in \mathcal{G}} \phi_x(A)\, dh = x^*. \end{aligned}$$

Now, as $h$ is also right-translation invariant, we have

$$\begin{aligned} (Bx)^* = \int\limits_{A \in \mathcal{G}} \phi_{Bx}(A)\, dh &= \int\limits_{A \in \mathcal{G}} \phi_x(AB)\, dh \\ &= \int\limits_{A \in \mathcal{G}} \phi_x(A)\, dh = x^*. \end{aligned}$$

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$(b)$ Since $Bx^* = x^*$ for all $B \in \mathcal{G}$, from Proposition 2.5, we have $x^* \in \mathcal{C}_w$.

$(c)$ Now suppose that the given semi-FTvN system is an FTvN system and orbit-transitive. Let $y \in [x^*]$. As the system is orbit-transitive, there exists $B \in \mathcal{G}$ such that $y = Bx^*$. Since $Bx^* = x^*$ from $(a)$, we have $y = x^*$. Hence, $[x^*] = \{x^*\}$. As our system is assumed to be an FTvN system, Proposition 2.5 implies $x^* \in \mathcal{C}$.

Now, suppose our (finite-dimensional orbit-transitive FTvN) system has a unit $e$. Then $x^* \in \mathcal{C} = \mathcal{R}\, e$. Let $x^* = r\, e$ for some $r \in \mathcal{R}$. As $x \mapsto x^*$ is linear (see Theorem 4.3), $r$ must be linear in $x$. Writing $x^* = \sigma_x e$, where $\sigma_x$ is a linear functional in $x$, we note that $\langle x^*, e\rangle = \sigma_x \langle e, e\rangle$. Now, for any $A \in \mathcal{G}$, we have $Ae = e$ (from Proposition 2.5 $(c)$); hence (as $A$ is inner product-preserving) $\langle Ax, e\rangle = \langle Ax, Ae\rangle = \langle x, e\rangle$ for all $x \in \mathcal{V}$. Then,

$$\sigma_x \langle e, e\rangle = \langle x^*, e\rangle = \left\langle \int_{A\in\mathcal{G}} \phi_x(A)\, dh, e \right\rangle = \int_{A\in\mathcal{G}} \langle \phi_x(A), e\rangle\, dh$$

$$= \int_{A\in\mathcal{G}} \langle Ax, e\rangle\, dh = \int_{A\in\mathcal{G}} \langle x, e\rangle\, dh = \langle x, e\rangle\, h(\mathcal{G}) = \langle x, e\rangle\,.$$

Hence, we have $\sigma_x = \dfrac{\langle x, e\rangle}{\langle e, e\rangle}$, showing $x^* = \sigma_x\, e$. This completes the proof. $\square$

**Theorem 4.5.** *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a finite-dimensional semi-FTvN system. Then, the following statements hold for any $x \in \mathcal{V}$:*

$(a)$ $\mathcal{C}_w \cap \operatorname{conv}[x] \neq \emptyset$,
$(b)$ $\mathcal{C} \cap \operatorname{conv}[x] \neq \emptyset$ *when $(\mathcal{V}, \mathcal{W}, \lambda)$ is an orbit-transitive FTvN system.*

**Proof.** $(a)$ From item $(b)$ in Theorem 4.4, we have $x^* \in \mathcal{C}_w$ and from item $(i)$ in Theorem 4.3, we have $x^* \in \operatorname{conv}[x]$. Thus, $\mathcal{C}_w \cap \operatorname{conv}[x] \neq \emptyset$.

$(b)$ Now assume that $(\mathcal{V}, \mathcal{W}, \lambda)$ is a (finite-dimensional) orbit-transitive FTvN system. From item $(c)$ in Theorem 4.4, we have $x^* \in \mathcal{C}$ and from item $(i)$ in Theorem 4.3, we have $x^* \in \operatorname{conv}[x]$. Thus, $\mathcal{C} \cap \operatorname{conv}[x] \neq \emptyset$. $\square$

Our next example shows that the property $\mathcal{C} \cap \operatorname{conv}[x] \neq \emptyset$ in Item $(b)$ may not hold in an orbit-transitive semi-FTvN system.

**Example 4.6.** [9, Example 3.16] In $\mathcal{R}^2$, let $\mathcal{V} := \{r(1, 2) : r \in \mathcal{R}\}$. Consider $\lambda : \mathcal{V} \to \mathcal{R}^2$ with $\lambda(x) = x^{\downarrow}$. Then, $(\mathcal{V}, \mathcal{R}^2, \lambda)$ is a semi-FTvN system, where for any $x \in \mathcal{V}$, $[x] = \{y \in \mathcal{V} : \lambda(x) = \lambda(y)\} = \{x\}$. It follows that (5) fails and so $(\mathcal{V}, \mathcal{R}^2, \lambda)$ is not an FTvN system. Additionally, $\mathcal{G} = \{I\}$, $\mathcal{C} = \{0\}$, and $\mathcal{C}_w = \mathcal{V}$. While our system is orbit-transitive, for any nonzero $x \in \mathcal{V}$, we have $\mathcal{C} \cap \operatorname{conv}[x] = \emptyset$.

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**Corollary 4.7.** *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be an FTvN system with a corresponding reduced system $(\mathcal{W}, \mathcal{W}, \mu)$. Suppose $(\mathcal{W}, \mathcal{W}, \mu)$ is finite-dimensional and orbit-transitive. Let $L := \lambda(\mathcal{V}) \cap -\lambda(\mathcal{V})$. Then,*

$$L \cap \operatorname{conv}[w] \neq \emptyset \quad (\forall\, w \in \mathcal{W}). \tag{13}$$

**Proof.** Let $\mathcal{C}$ and $\mathcal{C}'$ denote, respectively, the centers of $(\mathcal{V}, \mathcal{W}, \lambda)$ and $(\mathcal{W}, \mathcal{W}, \mu)$. Under the given assumptions, from (6), we have $L = \lambda(\mathcal{V}) \cap -\lambda(\mathcal{V}) = \lambda(\mathcal{C}) = \mathcal{C}'$. Thus, (13) follows from item $(b)$ of Theorem 4.5 applied to $(\mathcal{W}, \mathcal{W}, \mu)$. $\square$

**Remarks.** In [17], Zhao deals with an FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$ with a reduced system $(\mathcal{R}^n, \mathcal{R}^n, \gamma)$ and conjectures that (13) holds generally (without our orbit-transitivity condition). This conjecture and Theorem 4.5 motivate the following problems.

**Problem 2.** Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a finite-dimensional FTvN system with center $\mathcal{C}$. Do we always have $\mathcal{C} \cap \operatorname{conv}[x] \neq \emptyset$?

**Problem 3.** Suppose $(\mathcal{W}, \mathcal{W}, \mu)$ is a finite-dimensional FTvN system with $\mu^2 = \mu$. With $\mathcal{C}'$ denoting the center in $\mathcal{W}$, does it hold that $\mathcal{C}' \cap \operatorname{conv}[w] \neq \emptyset$ for all $w \in \mathcal{W}$?

In support of Problem 2, we have the following result.

**Theorem 4.8.** *Suppose $(\mathcal{V}, \mathcal{R}^n, \lambda)$ is an FTvN system induced by a complete isometric hyperbolic polynomial $p$ of degree $n$. Let $e$ be the direction along which $p$ is hyperbolic. Then, for all $x \in \mathcal{V}$,*

$$x^{**} := \frac{\langle x, e\rangle}{n} e \in \mathcal{C} \cap \operatorname{conv}[x]. \tag{14}$$

*In particular, this conclusion holds when $\mathcal{V}$ is a Euclidean Jordan algebra of rank $n$ with unit element $e$ (and carrying the trace inner product).*

**Proof.** Let $x \in \mathcal{V}$. As our FTvN system comes from a complete hyperbolic polynomial, it is known that $\mathcal{C} = \mathcal{R}\, e$ (see [9, Page 28]). Hence, $x^{**} = \frac{\langle x,e\rangle}{n} e \in \mathcal{C}$. We show that $x^{**} \in \operatorname{conv}[x]$, equivalently, $x^{**} \prec x$. Now, from Theorem 3.5, $x^{**} \prec x$ in $\mathcal{V}$ if and only if $\lambda(x^{**}) \prec \lambda(x)$ in $\mathcal{R}^n$. We verify the latter condition. As $\lambda(z + re) = \lambda(z) + r\mathbf{1}$ for every $z \in \mathcal{V}$ and $r \in \mathcal{R}$, we have

$$\lambda(x^{**}) = \frac{\langle x, e\rangle}{n} \mathbf{1} = \frac{\langle \lambda(x), \mathbf{1}\rangle}{n} \mathbf{1} \prec \lambda(x),$$

with the second equality coming from (1) and the (majorization) inequality coming from (12). We thus have $\lambda(x^{**}) \prec \lambda(x)$, or equivalently, $x^{**} \prec x$. This completes the proof. $\square$

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**Remarks.** Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a finite-dimensional orbit-transitive FTvN system with a unit element $e \in \mathcal{V}$. Let $(\mathcal{W}, \mathcal{W}, \mu)$ be a reduced system of $(\mathcal{V}, \mathcal{W}, \lambda)$, which is also orbit-transitive. For an $x \in \mathcal{V}$, we consider $x^* \in \mathcal{V}$, $\lambda(x^*) \in \mathcal{W}$, and $\lambda(x)^* \in \mathcal{W}$. We ask: *Is there a relationship between $\lambda(x^*)$ and $\lambda(x)^*$?* We provide an answer below.

For $x \in \mathcal{V}$, by item $(c)$ in Theorem 4.4, we have

$$x^* = \sigma_x e = \frac{\langle x, e\rangle}{\langle e, e\rangle} e.$$

As $\sigma_{-x} = -\sigma_x$, without loss of generality, let $\sigma_x \geq 0$. In this case, by the positive homogeneity of $\lambda$,

$$\lambda(x^*) = \lambda(\sigma_x\, e) = \sigma_x\, \lambda(e) = \frac{\langle x, e\rangle}{\langle e, e\rangle}\, \lambda(e).$$

(Note that $\lambda(e)$ is a unit element in $\mathcal{W}$, see Proposition 2.9) Moreover, as $e$ commutes with every element in $\mathcal{V}$, we have $\langle \lambda(x), \lambda(e)\rangle = \langle x, e\rangle$. It follows that

$$\lambda(x)^* = \frac{\langle \lambda(x), \lambda(e)\rangle}{\langle \lambda(e), \lambda(e)\rangle}\, \lambda(e) = \frac{\langle x, e\rangle}{\langle e, e\rangle}\, \lambda(e).$$

We conclude that $\lambda(x^*) = \lambda(x)^*$ for all $x \in \mathcal{V}$.

## 5. A partial order and equivalence among eigenvalue maps

Recall that $\lambda$ is an eigenvalue map on (a real inner product space) $\mathcal{V}$ if there is a $\mathcal{W}$ such that $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system. In this section, we consider two semi-FTvN systems having the same domain. Our objective is to compare the corresponding eigenvalue maps.

**Definition 5.1.** Given two eigenvalue maps $\lambda$ and $\mu$ on the same real inner product space $\mathcal{V}$, we define a partial order $\lambda \leq \mu$ as follows:

$$\lambda \leq \mu \quad \text{if and only if} \quad \big\langle \lambda(x), \lambda(y)\big\rangle_{\mathcal{W}_1} \leq \big\langle \mu(x), \mu(y)\big\rangle_{\mathcal{W}_2} \quad (\forall\, x, y \in \mathcal{V}),$$

where $(\mathcal{V}, \mathcal{W}_1, \lambda)$ and $(\mathcal{V}, \mathcal{W}_2, \mu)$ are semi-FTvN systems corresponding to $\lambda$ and $\mu$ respectively. We also say that $\lambda$ and $\mu$ are equivalent and write $\lambda \sim \mu$ when $\lambda \leq \mu$ and $\mu \leq \lambda$.

**Example 5.2.** Let $\mathcal{V}$ be a real inner product space. Define $\lambda_0 : \mathcal{V} \to \mathcal{V}$ by $\lambda_0(x) := x$ for all $x \in \mathcal{V}$ and $\lambda_\infty : \mathcal{V} \to \mathcal{R}$ by $\lambda_\infty(x) := \|x\|$ for all $x \in \mathcal{V}$. Then, for any eigenvalue map $\lambda$ on $\mathcal{V}$,

$$\lambda_0 \leq \lambda \leq \lambda_\infty.$$

This is due to the fact that $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq \|x\|\,\|y\|$.

**Example 5.3.** On $\mathcal{R}^n$, define

$$\lambda(x) := x^{\downarrow} \quad \text{and} \quad \mu(x) = |x|^{\downarrow},$$

where $x^{\downarrow}$ is the decreasing rearrangement of $x$ and $|x|$ is the vector of absolute values of the entries of $x$. It is known that $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$ and $(\mathcal{R}^n, \mathcal{R}^n, \mu)$ are FTvN systems, see [8, Example 4.3]. Since

$$\langle x, y\rangle \leq \left\langle x^{\downarrow}, y^{\downarrow}\right\rangle \leq \left\langle\, |x|^{\downarrow}, |y|^{\downarrow}\,\right\rangle \quad (\forall\, x, y \in \mathcal{R}^n),$$

we conclude that $\lambda \leq \mu$.

**Example 5.4.** Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ and $(\mathcal{W}, \mathcal{U}, \nu)$ are two semi-FTvN systems. Then, by the inner product expanding property of $\nu$, $\nu \circ \lambda$ is an eigenvalue map on $\mathcal{V}$ with $\lambda \leq \nu \circ \lambda$.

**Example 5.5.** On any inner product space $\mathcal{V}$, every linear isometry $S$ is an eigenvalue map of the (semi-)FTvN system $(\mathcal{V}, \mathcal{V}, S)$. Moreover, any two linear isometries are equivalent.

**Example 5.6.** Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system. Then, for any linear isometry $S : \mathcal{W} \to \mathcal{W}$, we have $\lambda \sim S\lambda$.

We show below that if $\lambda \sim \mu$, then $\mu = S\lambda$ for some linear isometry $S$.

**Proposition 5.7.** *Suppose $\lambda : \mathcal{V} \to \mathcal{W}_1$ and $\mu : \mathcal{V} \to \mathcal{W}_2$ are eigenvalue maps with $\lambda \leq \mu$. Let $[x]_\lambda$ and $[x]_\mu$ denote, respectively, the $\lambda$-orbit and $\mu$-orbit of $x$; $\langle x\rangle_\lambda$ and $\langle x\rangle_\mu$ are defined similarly. Then, the following statements hold for $x, y \in \mathcal{V}$:*

- *(i)* $\lambda(x) = \lambda(y)$ *implies* $\mu(x) = \mu(y)$.
- *(ii)* $[x]_\lambda \subseteq [x]_\mu$.
- *(iii)* $\mathrm{Aut}(\mathcal{V}, \mathcal{W}_1, \lambda) \subseteq \mathrm{Aut}(\mathcal{V}, \mathcal{W}_2, \mu)$; *consequently* $\langle x\rangle_\lambda \subseteq \langle x\rangle_\mu$.
- *(iv) If $x$ and $y$ (strongly) commute relative to $\mu$, then $x$ and $y$ (strongly) commute relative to $\lambda$.*
- *(v) If $x$ is (strongly) majorized by $y$ relative to $\lambda$, then $x$ is (strongly) majorized by $y$ relative to $\mu$.*

**Proof.** *(i)* Let $\lambda(x) = \lambda(y)$. Then, as $\lambda$ and $\mu$ are norm-preserving,

$$\|\mu(x)\| = \|x\| = \|\lambda(x)\| = \|\lambda(y)\| = \|y\| = \|\mu(y)\|.$$

Consequently,

$$\|x\|^2 = \|\lambda(x)\|^2 = \langle \lambda(x), \lambda(y)\rangle \leq \langle \mu(x), \mu(y)\rangle \leq \|\mu(x)\|\,\|\mu(y)\| = \|x\|\,\|y\| = \|x\|^2.$$

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It follows that in $\mathcal{W}_2$, $\langle \mu(x), \mu(y) \rangle = \|\mu(x)\| \, \|\mu(y)\|$. Hence, by the equality case in the Cauchy-Schwarz inequality, there exists $r \geq 0$ such that $\mu(x) = r\mu(y)$. As the norms of $\mu(x)$ and $\mu(y)$ are equal, we must have $r = 1$, proving the equality $\mu(x) = \mu(y)$.

$(ii)$ This follows from the definition of orbits and $(i)$.

$(iii)$ Suppose $A \in \mathrm{Aut}(\mathcal{V}, \mathcal{W}_1, \lambda)$. Then $\lambda(Ax) = \lambda(x)$ for all $x \in \mathcal{V}$. Thus, from $(i)$, we have $\mu(Ax) = \mu(x)$ for all $x \in \mathcal{V}$. This means $A \in \mathrm{Aut}(\mathcal{V}, \mathcal{W}_2, \mu)$.

$(iv)$ First suppose that $x$ strongly commutes with $y$ relative to $\mu$. Then

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \leq \langle \mu(x), \mu(y) \rangle = \langle x, y \rangle \, ,$$

showing that $x$ strongly commutes with $y$ relative to $\lambda$. When $x$ commutes with $y$ relative to $\mu$, we have $\langle Lx, y \rangle = 0$ for all $L \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}, \mathcal{W}_2, \mu))$. Then, because of $(iii)$, $\langle Lx, y \rangle = 0$ for all $L \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}, \mathcal{W}_1, \lambda))$. This says that $x$ commutes with $y$ relative to $\lambda$.

Finally, $(v)$ follows from $(ii)$ and $(iii)$. $\square$

**Remarks.** In the above result, we see that $\lambda \leq \mu \implies [x]_\lambda \subseteq [x]_\mu$ for all $x \in \mathcal{V}$. We now address the reverse implication. Suppose $(\mathcal{V}, \mathcal{W}_1, \lambda)$ is an FTvN system and $(\mathcal{V}, \mathcal{W}_2, \mu)$ is a semi-FTvN system with $[x]_\lambda \subseteq [x]_\mu$ for all $x \in \mathcal{V}$. Then, using (5), for all $c, x \in \mathcal{V}$,

$$\langle \lambda(c), \lambda(x) \rangle_{\mathcal{W}_1} = \sup_{u \in [x]_\lambda} \langle c, u \rangle_{\mathcal{V}} \leq \sup_{u \in [x]_\mu} \langle c, u \rangle_{\mathcal{V}} \leq \langle \mu(c), \mu(x) \rangle_{\mathcal{W}_2} \, ,$$

where the last inequality comes from the sharpened Cauchy-Schwarz inequality in $(\mathcal{V}, \mathcal{W}_2, \mu)$. Hence, in this setting, $\lambda \leq \mu$.

Our next result is about the equivalence of eigenvalue maps. We saw in Example 5.6 that a composition of an eigenvalue map $\lambda$ and an isometry $S$ is again an eigenvalue map that is equivalent to the given map $\lambda$. The following result addresses the converse.

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system. We let

$$\mathrm{span}(\lambda) := \mathrm{span}(\lambda(\mathcal{V})).$$

We note that when $(\mathcal{V}, \mathcal{W}, \lambda)$ is an FTvN system, $\lambda(\mathcal{V})$ is a convex cone [7, Corollary 2.8] and so, $\mathrm{span}(\lambda) = \lambda(\mathcal{V}) - \lambda(\mathcal{V})$.

**Proposition 5.8.** *Suppose $(\mathcal{V}, \mathcal{W}_1, \lambda)$ and $(\mathcal{V}, \mathcal{W}_2, \mu)$ are two semi-FTvN systems with $\lambda \sim \mu$. Then, there exists a surjective inner product-preserving linear map $S : \mathrm{span}(\lambda) \to \mathrm{span}(\mu)$ such that $\mu = S\lambda$.*

**Proof.** From our hypothesis, $\lambda \sim \mu$, so for all $x, y \in \mathcal{V}$, $\langle \lambda(x), \lambda(y) \rangle = \langle \mu(x), \mu(y) \rangle$. Without loss of generality, let $\mathcal{W}_1 := \mathrm{span}(\lambda)$ and $\mathcal{W}_2 := \mathrm{span}(\mu)$. We define the map $S : \mathcal{W}_1 \to \mathcal{W}_2$ as follows. Consider any $x \in \mathrm{span}(\lambda)$ and write $x = \sum_{i=1}^k \alpha_i \lambda(x_i)$, where $\alpha_i \in \mathcal{R}$ and $x_i \in \mathcal{V}$ for all $i$. Define

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$$S(x) := \sum_{i=1}^{k} \alpha_i \mu(x_i).$$

To see that $S$ is well defined, assume $x$ has another representation $x = \sum_{j=1}^{l} \beta_j \lambda(y_j)$. We need to show that $\sum_{i=1}^{k} \alpha_i \mu(x_i) = \sum_{j=1}^{l} \beta_j \mu(y_j)$. Now, as $\sum_{i=1}^{k} \alpha_i \lambda(x_i) = \sum_{j=1}^{l} \beta_j \lambda(y_j)$, we have

$$0 = \left\| \sum_{i=1}^{k} \alpha_i \lambda(x_i) - \sum_{j=1}^{l} \beta_j \lambda(y_j) \right\|^2.$$

Writing the right-hand side as an inner product and expanding it, we observe that a typical term in this expansion is of the form $\gamma \langle \lambda(u), \lambda(v) \rangle$ for some $\gamma \in \mathcal{R}$ and $u, v \in \mathcal{V}$. As $\lambda \sim \mu$, we can replace this typical term by $\gamma \langle \mu(u), \mu(v) \rangle$. Then the above equality becomes

$$0 = \left\| \sum_{i=1}^{k} \alpha_i \mu(x_i) - \sum_{j=1}^{l} \beta_j \mu(y_j) \right\|^2.$$

This shows that $\sum_{i=1}^{k} \alpha_i \mu(x_i) = \sum_{j=1}^{l} \beta_j \mu(y_j)$. Thus, $S$ is well-defined. Clearly, $S$ is linear and surjective. Once again, using the equivalence $\lambda \sim \mu$, we verify that $S$ is inner product-preserving; consequently, $S$ is an isometry and injective. Clearly, $\mu(x) = S\lambda(x)$ for all $x \in \mathcal{V}$, showing the equality $\mu = S\lambda$. $\square$

## Declaration of competing interest

There is nothing to declare.

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## Data availability

No data was used for the research described in the article.

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