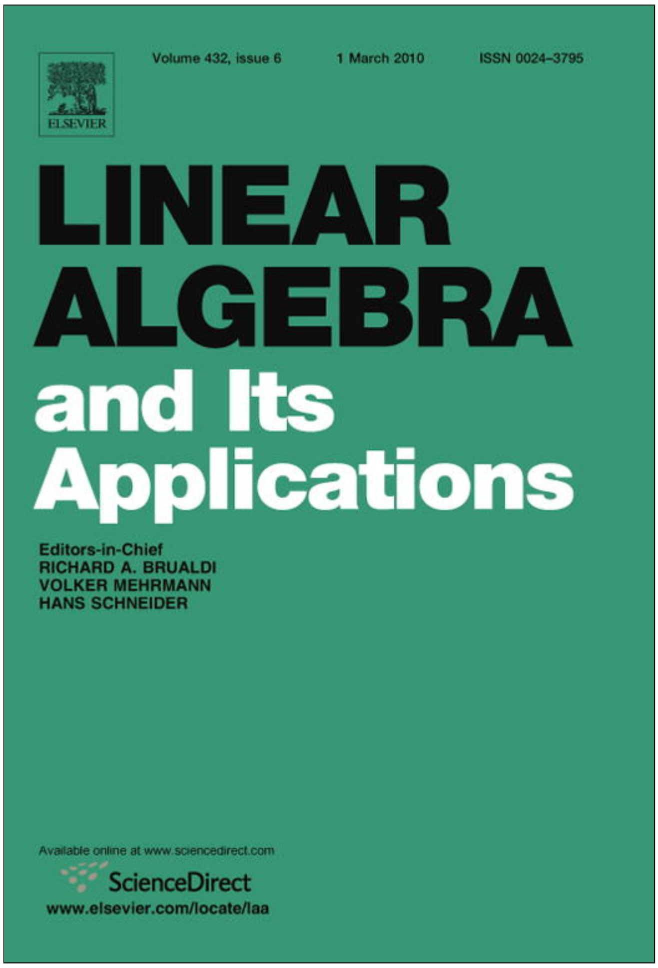
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Schur complements, Schur determinantal and Haynsworth inertia formulas in Euclidean Jordan algebras

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ABSTRACT

In this article, we study the concept of Schur complement in the setting of Euclidean Jordan algebras and describe Schur determinantal and Haynsworth inertia formulas.

1. Introduction

Consider a square complex matrix given in the block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is square. If A is invertible, the Schur complement of A in M is defined by

$$M/A := D - CA^{-1}B.$$

Schur complement plays an important role in various areas including matrix analysis, statistics, numerical analysis, optimization, etc., see e.g. [1,3,13,15,16], and the references therein. Among its numerous properties, perhaps, the most important and useful ones are:

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(1) The Schur determinantal formula: $\det(M) = \det(A)\det(M/A)$.
(2) The Haynsworth formula for inertia of a Hermitian matrix: $\mathrm{In}(M) = \mathrm{In}(A) + \mathrm{In}(M/A)$.
(3) The positive definiteness property for a Hermitian matrix: M is positive definite if and only if A and M/A are positive definite.
(4) The Guttman rank additivity formula for a Hermitian matrix: $\mathrm{rank}(M) = \mathrm{rank}(A) + \mathrm{rank}(M/A)$.

All of the above results can be derived from the following so-called Aitken block-diagonalization formula:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

In fact, taking the determinants both sides of the above formula and using the multiplicative property of the determinant, we get the Schur determinantal formula in (1). When M is Hermitian, by applying the Sylvester's law of inertia [9], we get the Haynsworth formula in (2). Finally, (for M Hermitian) items (3) and (4) follow immediately from item (2).

Can the above results be extended to matrices over quaternions and octonions? In the case of quaternions, the Aitken formula continues to hold. However, the non-existence of a quaternion-valued multiplicative determinant that extends the usual complex determinant [2] prevents us from stating a Schur determinantal formula in this setting. In the case of octonions (which are non-commutative and non-associative), the quantity $D - CA^{-1}B$ is not even defined. In spite of these drawbacks, in this paper, we formulate the concept of Schur complement in Euclidean Jordan algebras and state analogs of above statements (1)–(4).

To explain our results, consider an Euclidean Jordan algebra $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$. For any given idempotent $c \in \mathcal{J}$, we have the Peirce decomposition [5, Chap. IV, Sec. 1]

$$\mathcal{J} = \mathcal{J}(c, 1) \oplus \mathcal{J}\left(c, \frac{1}{2}\right) \oplus \mathcal{J}(c, 0), \tag{1}$$

where

$$\mathcal{J}(c, \gamma) = \{x \in \mathcal{J} : x \circ c = \gamma x\},$$

for $\gamma = 0, \frac{1}{2}, 1$. Given any element $x \in \mathcal{J}$, we write the decomposition

$$x = u + v + w, \tag{2}$$

where $u \in \mathcal{J}(c, 1)$, $v \in \mathcal{J}\left(c, \frac{1}{2}\right)$, and $w \in \mathcal{J}(c, 0)$. When u is invertible in the Euclidean Jordan (sub)algebra $\mathcal{J}(c, 1)$, let u_*^{-1} denote the inverse of u in $\mathcal{J}(c, 1)$. In this case, the *Schur complement* of u in x is defined by

$$x/u := w - P_v(u_*^{-1}), \tag{3}$$

where, for any element $a \in \mathcal{J}$, the quadratic representation P_a is given by

$$P_a(z) = 2a \circ (a \circ z) - (a \circ a) \circ z \quad (z \in \mathcal{J}).$$

The expression $w - P_v(u_*^{-1})$ appears in the works of Loos [10] and Massam and Neher [11], without the name 'Schur complement' and the associated notation. It turns out, see [11] (or Section 2 below), that $x/u \in \mathcal{J}(c, 0)$. Now, any element $x \in \mathcal{J}$ will have a spectral decomposition

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r, \tag{4}$$

where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame in $\mathcal{J}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the (spectral) eigenvalues of x. Then the *determinant*, *inertia*, and *rank* of x are, respectively, defined by

$$\det(x) := \lambda_1 \lambda_2 \cdots \lambda_r, \tag{5}$$

$$\mathrm{In}(x) := (\pi(x), \nu(x), \delta(x)), \quad \text{and} \quad \mathrm{rank}(x) = \pi(x) + \nu(x). \tag{6}$$

Here $\pi(x)$, $\nu(x)$, and $\delta(x)$ are, respectively, the number of positive, negative, and zero eigenvalues of x. Given x and the above decomposition (2), we let $\det^*(u)$ and $\mathrm{In}^*(u)$ denote, respectively, the determinant and inertia of u in the (sub)algebra $\mathcal{J}(c,1)$. Similarly, we let $\det_*(x/u)$ and $\mathrm{In}_*(x/u)$ denote, respectively, the determinant and inertia of x/u in the (sub)algebra $\mathcal{J}(c,0)$. The analogs of properties (1)–(4) (suppressing the '$*$' in the determinant and inertia notations) can now be stated:

(i) $\det(x) = \det(u)\det(x/u)$.
(ii) $\mathrm{In}(x) = \mathrm{In}(u) + \mathrm{In}(x/u)$.
(iii) $x > 0$ in $\mathcal{J}$ if and only if $u > 0$ in $\mathcal{J}(c,1)$ and $x/u > 0$ in $\mathcal{J}(c,0)$.
(iv) $\mathrm{rank}(x) = \mathrm{rank}(u) + \mathrm{rank}(x/u)$.

In the setting of a general Euclidean Jordan algebra, item (i) was proved by Massam and Neher [11] using the so-called Frobenius transformation as a substitute for the Aitken formula; it can also be derived from the work of Loos [10]. In this paper, using homotopy and continuity arguments, and the properties of Frobenius transformations, we prove item (ii); items (iii) and (iv) follow immediately.

2. Euclidean Jordan algebras

Throughout this paper, we let $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra of rank r [5,7,14]. The symmetric cone of $\mathcal{J}$ is the cone of squares $K := \{x^2 : x \in \mathcal{J}\}$. We use the notation $x \geqslant 0$ ($x > 0$) when $x \in K$ (respectively, $x \in interior(K)$). As is well known, any Euclidean Jordan algebra is a product of simple Euclidean Jordan algebras and every simple algebra is isomorphic to the Jordan spin algebra $\mathcal{L}^n$ or to the algebra of all $n \times n$ real/complex/quaternion Hermitian matrices or to the algebra of all 3×3 octonion Hermitian matrices.

Given any $a \in \mathcal{J}$, we let L_a and P_a denote the corresponding Lyapunov transformation and quadratic representation of a on $\mathcal{J}$:

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

These are linear and self-adjoint on $\mathcal{J}$. In addition, $P_a(K) \subseteq K$ for any a. We say that objects a and b in $\mathcal{J}$ *operator commute* if $L_aL_b = L_bL_a$. It is known that a and b operator commute if and only if they have their spectral decompositions with respect to a common Jordan frame.

Corresponding to the decomposition (2), consider the Schur complement x/u given in (3). Then we have

Proposition 1. *The following statements hold*:

(a) $x/u \in \mathcal{J}(c,0)$.
(b) $x/u = x - P_x\left(u_*^{-1}\right)$.

Proof. (a) As $x/u = w - P_v\left(u_*^{-1}\right)$ and $w \in \mathcal{J}(c,0)$, it is enough to show that $P_v\left(u_*^{-1}\right) \in \mathcal{J}(c,0)$. We actually show that for any $z \in \mathcal{J}(c,1)$, $P_v(z) \in \mathcal{J}(c,0)$, that is, $P_v(z) \circ c = 0$. (This is stated on p. 870 in [11] without proof.) Consider

$$P_v(c) = 2v \circ (v \circ c) - v^2 \circ c = 2v \circ \left(\frac{1}{2}v\right) - v^2 \circ c = v^2 \circ (e - c).$$

Then

$$\langle P_v(c), c\rangle = \langle v^2 \circ (e-c), c\rangle = \langle v^2, (e-c)\circ c\rangle = 0.$$

As $c \in K$ and $P_v(K) \subseteq K$, $\langle P_v(c), c\rangle = 0$ implies $P_v(c) \circ c = 0$, see Proposition 6 in [7]. This means that $P_v(c) \in \mathcal{J}(c,0)$. As $\mathcal{J}(c,1)$ is orthogonal to $\mathcal{J}(c,0)$, we have, for any primitive idempotent $f \in \mathcal{J}(c,1)$,

$$\langle f, P_v(c)\rangle = 0.$$

Since P_v is self-adjoint, $\langle P_v(f), c\rangle = 0$. Once again, as $f \in K$ and $P_v(K) \subseteq K$, we have $P_v(f) \circ c = 0$. By the spectral decomposition theorem, any $z \in J(c,1)$ is a finite linear combination of primitive idempotents in $\mathcal{J}(c,1)$. Hence, by the linearity of P_v, we get

$$P_v(z) \circ c = 0$$

for any $z \in \mathcal{J}(c, 1)$. This completes the proof of (a).

(b) We first compute $P_x\left(u_*^{-1}\right)$. Using the facts $\mathcal{J}(c,1) \circ \mathcal{J}(c,0) = \{0\},\ u \circ u_*^{-1} = c,\ w \circ u_*^{-1} = 0,\ v \circ c = \frac{1}{2}v$, and $w \circ c = 0$, we get

$$x \circ u_*^{-1} = c + v \circ u_*^{-1}$$

and

$$2x \circ \left(x \circ u_*^{-1}\right) = 2u + v + 2u \circ \left(v \circ u_*^{-1}\right) + 2v \circ \left(v \circ u_*^{-1}\right) + 2w \circ \left(v \circ u_*^{-1}\right).$$

Also, from $\mathcal{J}(c,1) \circ \mathcal{J}(c,0) = \{0\}$ and $u^2 \circ u_*^{-1} = u$, we have

$$x^2 = (u+v+w)^2 = u^2 + v^2 + w^2 + 2u \circ v + 2v \circ w$$

and

$$x^2 \circ u_*^{-1} = u + v^2 \circ u_*^{-1} + 2(u \circ v) \circ u_*^{-1} + 2(v \circ w) \circ u_*^{-1}.$$

Now, by orthogonality, any primitive idempotent in $\mathcal{J}(c,1)$ operator commutes with any primitive idempotent in $\mathcal{J}(c,0)$ (see e.g., Proposition 6 in [7]); hence, via the spectral decomposition, any element in $\mathcal{J}(c,1)$ operator commutes with any element in $\mathcal{J}(c,0)$. This means that

$$w \circ \left(u_*^{-1} \circ v\right) = L_w L_{u_*^{-1}}(v) = L_{u_*^{-1}} L_w(v) = u_*^{-1} \circ (w \circ v).$$

In addition, u and u_*^{-1} operator commute and so

$$u \circ \left(u_*^{-1} \circ v\right) = u_*^{-1} \circ (u \circ v).$$

A simple algebraic manipulation gives

$$\begin{aligned} x - P_x\left(u_*^{-1}\right) &= x - \left\{2x \circ \left(x \circ u_*^{-1}\right) - x^2 \circ u_*^{-1}\right\} \\ &= w - \left\{2v \circ \left(v \circ u_*^{-1}\right) - v^2 \circ u_*^{-1}\right\} = w - P_v\left(u_*^{-1}\right). \quad \square \end{aligned}$$

Remarks. In the case of algebras of all $n \times n$ real/complex/quaternion Hermitian matrices, the quadratic representation is given by $P_X(Y) = XYX$, and hence the Schur complement reduces to the standard one when the idempotent c is chosen appropriately. The following example illustrates the concept of Schur complement in the algebra $Herm(\mathcal{O}^{3\times 3})$ of 3×3 Hermitian matrices over octonions.

Example. Consider a matrix $M \in Herm(\mathcal{O}^{3\times 3})$, given by

$$M = \begin{bmatrix} p & a & b \\ \bar{a} & q & c \\ \bar{b} & \bar{c} & r \end{bmatrix},$$

where $p,\ q,\ r$ are real numbers, and $a,\ b$, and c are octonions. In this setting, the determinant of M, which is the product of the spectral eigenvalues of M, is given, via the Freudenthal formula (see [4,12]), by

$$\det(M) = pqr + 2\,Re\,\bar{b}(ac) - r|a|^2 - q|b|^2 - p|c|^2.$$

Now let c be the idempotent in $Herm(\mathcal{O}^{3\times 3})$ which has 1 in the $(1,1)$ slot and zeros elsewhere. Then the Peirce decomposition $M = U + V + W$ is given by

$$M = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & q & c \\ 0 & \bar{c} & r \end{bmatrix}.$$

Now, for any two matrices X and Y in $Herm(\mathcal{O}^{3\times 3})$,

$$P_X(Y) = \frac{1}{2}[X(XY) + X(YX) + (XY)X + (YX)X - X^2Y - YX^2].$$

When p is invertible, a straightforward computation shows that

$$M/U = W - P_V\left(U_*^{-1}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q - p^{-1}|a|^2 & c - (\bar{a}b)p^{-1} \\ 0 & \bar{c} - (\bar{b}a)p^{-1} & r - |b|^2p^{-1} \end{bmatrix}.$$

This leads to

$$\det_*(M/U) = (q - p^{-1}|a|^2)(r - |b|^2p^{-1}) - (\bar{c} - (\bar{b}a)p^{-1})(c - (\bar{a}b)p^{-1}),$$

and to $\det(M) = \det^*(U)\det_*(M/U)$.

3. The Frobenius transformation and the Schur determinantal formula

For any two elements $x, y \in \mathcal{J}$, let

$$x \,\Box\, y := L_{x\circ y} + \{L_xL_y - L_yL_x\}.$$

Then given any idempotent c in $\mathcal{J}$ and $z \in \mathcal{J}\left(c, \frac{1}{2}\right)$, the *Frobenius transformation* is given by (see [5, p. 106]),

$$\tau_z := \exp(2z \,\Box\, c) = I + (2z \,\Box\, c) + \frac{1}{2}(2z \,\Box\, c)^2.$$

In the following result, proved by Massam and Neher [11, Proposition 1], item (ii)(a) generalizes the Aitken block diagonal formula and item (ii)(b) is the Schur determinantal formula in Euclidean Jordan algebras.

Theorem 2. *Let c be an idempotent in $\mathcal{J}$.*

(i) *For any $z \in \mathcal{J}\left(c, \frac{1}{2}\right)$ and $x \in \mathcal{J}$,* $\det(\tau_z(x)) = \det(x)$.

(ii) *Consider the Peirce decomposition (2) of any $x \in \mathcal{J}$, with u invertible, and let $z = -2v \circ u_*^{-1}$. Then,*

  (a) $\tau_z(x) = u + x/u$ *and*
  (b) $\det(x) = \det^*(u)\det_*(x/u)$.

4. Invariance of inertia under Frobenius transformations

Theorem 3. *Let c be any idempotent in $\mathcal{J}$. Then for any $z \in \mathcal{J}\left(c, \frac{1}{2}\right)$ and $x \in \mathcal{J}$, we have*

$$\mathrm{In}(\tau_z(x)) = \mathrm{In}(x).$$

Proof. Fix $z \in \mathcal{J}\left(c, \frac{1}{2}\right)$ and $x \in \mathcal{J}$; let $y = \tau_z(x)$. First suppose that x is invertible. Define the homotopy

$$y_t = \tau_{tz}(x) \quad (0 \leqslant t \leqslant 1),$$

which connects $y_1 = y$ and $y_0 = x$. By Theorem 2(i), $\det(y_t) = \det(x)$ for all t and so y_t is also invertible. By the invariance of inertia, see Theorem 10 in [8], $\mathrm{In}(y_t)$ is a constant; in particular,

$$\mathrm{In}(y) = \mathrm{In}(x).$$

Now suppose that x is not invertible; let $x = \sum_1^k x_ie_i + \sum_{k+1}^r 0e_i$ be the spectral decomposition of x with $k < r$ and $x_i \neq 0$ for $1 \leqslant i \leqslant k$. Let $x(\varepsilon) := x + \varepsilon\left(\sum_{k+1}^r e_i\right)$ and $y(\varepsilon) := \tau_z(x(\varepsilon))$. As $x(\varepsilon)$ is invertible, from the previous case,

$$\text{In}(y(\varepsilon)) = \text{In}(x(\varepsilon)).$$

Now for small positive ε, $y(\varepsilon)$ is close to y and hence, by the lower semicontinuity of ν, see Theorem 10 in [8],

$$\nu(y) \leqslant \nu(y(\varepsilon)) = \nu(x(\varepsilon)) = \nu(x),$$

where the last equality comes from the observation that the negative eigenvalues of x are not disturbed when $\varepsilon \left(\sum_{k+1}^{r} e_i\right)$ is added to x. Thus, when x is not invertible and $y = \tau_z(x)$, $\nu(y) \leqslant \nu(x)$. Now, applying this inequality to $x = \tau_{-z}(y)$ we get $\nu(x) \leqslant \nu(y)$. Thus $\nu(y) = \nu(x)$. By working with $(-y) = \tau_z(-x)$, we get $\pi(y) = \pi(x)$ and so $\text{In}(y) = \text{In}(x)$. This completes the proof. □

Remarks. The above result can also be proved in a slightly different way. It has been observed that for any $z \in \mathcal{J}\left(c, \frac{1}{2}\right)$, τ_z belongs to the connected component of the automorphism group $Aut(K)$, see p. 874 in [11]. Also, when V is simple, any transformation in $Aut(K)$ preserves inertia, see Corollary 12 in [8]. The general case (of nonsimple V) can be handled by writing V as a product of simple algebras [5] and applying the previous argument to each component of z and then adding the inertias.

5. The Haynsworth inertia formula in Euclidean Jordan algebras

Given an idempotent $c \in \mathcal{J}$, let $a \in \mathcal{J}(c, 1)$ and $b \in \mathcal{J}(c, 0)$. Then we define $\text{In}^*(a)$:=Inertia of a in the algebra $\mathcal{J}(c, 1)$ and $\text{In}_*(b)$:=Inertia of b in the algebra $\mathcal{J}(c, 0)$. We note that

$$\text{In}(a + b) = \text{In}^*(a) + \text{In}_*(b). \tag{7}$$

We are now ready to state the Haynsworth inertia formula in Euclidean Jordan algebras.

Theorem 4. *Given the decomposition (2) with u invertible, we have*

$$\text{In}(x) = \text{In}^*(u) + \text{In}_*(x/u).$$

Proof. Let $z := -2v \circ u_*^{-1}$. Then from Theorem 2(ii),

$$\tau_z(x) = u + x/u.$$

Now by Theorem 3, we have $\text{In}(x) = \text{In}(u + x/u)$. Since $\text{In}(u + x/u) = \text{In}^*(u) + \text{In}_*(x/u)$, we have the stated result. □

The following result is immediate from the above theorem.

Corollary 5. *Given the decomposition (2) with u invertible, we have*

(i) $x > 0$ $(x \geqslant 0)$ *in* $\mathcal{J}$ *if and only if* $u > 0$ *in* $\mathcal{J}(c, 1)$ *and* $x/u > 0$ *(respectively,* $x/u \geqslant 0$*) in* $\mathcal{J}(c, 0)$.
(ii) $\text{rank}(x) = \text{rank}^*(u) + \text{rank}_*(x/u)$.

6. Concluding remarks

This article is a shortened and revised version of Gowda and Sznajder [6] where the results were proved by case-by-case analysis of the five simple algebras.

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References

[1] D. Carlson, What are Schur complements anyway? Linear Algebra Appl. 74 (1986) 257–275.
[2] N. Cohen, S. De Leo, The quaternionic determinant, Electron. J. Linear Algebra 7 (2000) 100–111.
[3] R.W. Cottle, On manifestations of the Schur complement, Rend. Sem. Mat. Fis. Milano 45 (1975) 31–40.
[4] T. Dray, C.A. Manogue, The octonionic eigenvalue problem, Adv. Appl. Clifford Algebras 8 (1998) 341–364.
[5] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Clarendon Press, Oxford, 1994.
[6] M.S. Gowda, R. Sznajder, Schur complements, Schur determinantal and Haynsworth inertia formulas in Euclidean Jordan algebras, Research Report, Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD, March 31, 2009.
[7] M.S. Gowda, R. Sznajder, J. Tao, Some *P*-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra Appl. 393 (2004) 203–232.
[8] M.S. Gowda, J. Tao, M.M. Moldovan, Some inertia theorems in Euclidean Jordan algebras, Linear Algebra Appl. 430 (2009) 1992–2011.
[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[10] O. Loos, Finiteness condition in Jordan pairs, Math. Z. 206 (1991) 577–587.
[11] H. Massam, E. Neher, On transformations and determinants of Wishart variables on symmetric cones, J. Theoret. Probab. 10 (1997) 867–902.
[12] M.M. Moldovan, M.S. Gowda, Strict diagonal dominance and Geršgorin type theorems in Euclidean Jordan algebras, Linear Algebra Appl. 431 (2009) 148–161.
[13] D.V. Ouellette, Schur complements and statistics, Linear Algebra Appl. 36 (1981) 187–295.
[14] S.H. Schmieta, F. Alizadeh, Extension of primal–dual interior point algorithms to symmetric cones, Math. Progam. Ser. A 96 (2003) 409–438.
[15] G.P.H. Styan, Schur complements and linear statistical models, in: T. Pukkila, S. Puntanen (Eds.), Proc. First Internat. Tampere Seminar on Linear Statistical Models and their Applications, Tampere, Finland, August–September 1983, Dept. of Mathematical Sciences, University of Tampere, 1985, pp. 37–75.
[16] F. Zhang, The Schur Complement and its Applications, Springer, New York, 2005.