Numerical Demonstration of Finite Element Convergence for Lagrange Elements in COMSOL Multiphysics

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Abstract: The convergence order of finite elements is related to the polynomial order of the basis functions used on each element, with higher order polynomials yielding better convergence orders. However, two issues can prevent this convergence order from being achieved: the lack of regularity of the PDE solution and the poor approximation of curved boundaries by polygonal meshes. We show studies for Lagrange elements of degrees 1 through 5 applied to the classical test problem of the Poisson equation with Dirichlet boundary condition. We consider this problem in two spatial dimensions with smooth and non-smooth data on domains with polygonal and with curved boundaries. The observed convergence orders in the norm of the error between FEM and PDE solution demonstrate that they are limited by the regularity of the solution and are degraded significantly on domains with non-polygonal boundaries. All numerical tests are carried out with COMSOL Multiphysics.

Key words: Poisson equation, a priori error estimate, convergence study, mesh refinement.

1 Introduction

The finite element method (FEM) is widely used as a numerical method for the solution of partial differential equation (PDE) problems, especially for elliptic PDEs such as the Poisson equation with Dirichlet boundary conditions

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = r \qquad \text{on } \partial\Omega, \tag{1.2}$$

where f and r denote given functions on the domain Ω and on its boundary $\partial\Omega$, respectively. Here, the domain $\Omega \subset \mathbb{R}^d$ is assumed to be a bounded, open, simply connected, and convex set in $d = 1, 2, 3$ dimensions with piecewise smooth boundary $\partial\Omega$.

The FEM solution u_h will typically incur an error against the true solution u of the PDE (1.1)–(1.2). This error can be quantified by bounding the norm of the error $u - u_h$ in terms of the mesh spacing h of the finite element mesh. Such estimates have the form $\|u - u_h\| \leq C\, h^q$, where C is a problem-dependent constant independent of h and the constant q indicates the order of convergence of the FEM, as the mesh spacing h decreases. We see from this form of the error estimate that we need $q > 0$ for convergence as $h \to 0$. More realistically, we wish to have for instance $q = 1$ for linear convergence, $q = 2$ for quadratic convergence, or higher values for even faster convergence.

One appropriate norm for FEM errors is the L^2-norm associated with the space $L^2(\Omega)$ of square-integrable functions, that is, the space of all functions $v(\mathbf{x})$ whose square $v^2(\mathbf{x})$ can be integrated over all $\mathbf{x} \in \Omega$ without the integral becoming infinite. The norm is defined concretely as the square root of that integral, namely

$$\|v\|_{L^2(\Omega)} := \left(\int v^2(\mathbf{x})\, d\mathbf{x} \right)^{1/2}. \tag{1.3}$$

Using the L^2-norm to measure the error of the FEM allows the computation of norms of errors also in cases where the solution and its error do not have derivatives. Thus, we use it in the following because it can quantify the expected error behavior for certain highly non-smooth problems.

Lagrange finite elements of degree p, such as available in COMSOL with $p = 1, \ldots, 5$, approximate the PDE solution at several points in each element of the mesh such that the restriction of the FEM solution u_h to each element is a polynomial of degree up to p in each spatial variable and u_h is continuous across all boundaries between neighboring mesh elements throughout Ω. For the case of linear (degree $p = 1$) Lagrange elements, we have the well known a priori bound (e.g., [1, Section II.7])

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^2.$$

We notice that the convergence order is one higher than the polynomial degree used by the Lagrange elements. Analogously, a more general result for using Lagrange elements with degrees $p \geq 1$ is that we can expect an error bound of

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^{p+1}.$$

The first purpose of this note is to demonstrate numerically that for an appropriate example this behavior can be observed for all Lagrange elements available in COMSOL; this is the contents of Section 2.

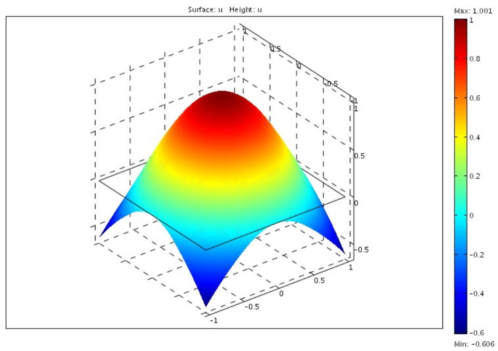
For both results quoted above to hold true in practice, they require a number of assumptions on the problem (1.1)–(1.2) and the finite element method used. One assumption is that the problem has a solution that is sufficiently regular, as expressed by the number of continuous derivatives that it has. In the context of the FEM, it is appropriate to consider so-called weak derivatives. Based on these, we define the Sobolev function spaces $H^k(\Omega)$ of order k of all functions on Ω that have weak derivatives of up to order k that are square-integrable in the sense of the space $L^2(\Omega)$ above. It turns out that the convergence order of the FEM with Lagrange elements with degree p is limited by the regularity order k of the PDE solution $u \in H^k(\Omega)$. Using the concept of weak derivatives, the error bound can then be stated as

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q, \quad q = \min\{k, p + 1\}. \quad (1.4)$$

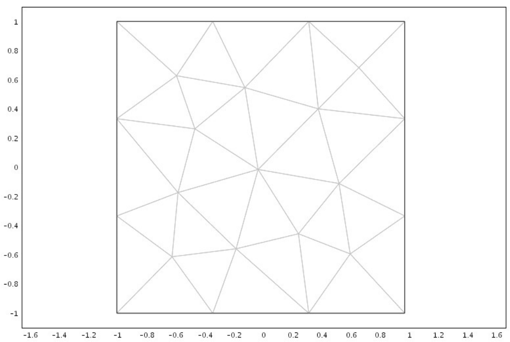
That is, the convergence order of the FEM is regularity order k of the PDE solution or one higher than the polynomial degree p, whichever is smaller. To demonstrate that this limitation of the convergence order applies, we consider a particularly non-smooth problem in Section 3, in which the solution can only be expected to satisfy $u \in H^1(\Omega)$ for $\Omega \subset \mathbb{R}^2$ and thus the convergence order is $q = 1$ for Lagrange elements with any degree $p = 1, \ldots, 5$. This example forces us to conclude that for highly non-smooth problems the computational effort associated with higher-degree FEM is not likely to gain the expected improvement in accuracy and one thus should limit the degree of Lagrange element used to $p = 2$.

In practice, one often encounters PDEs whose domains Ω have curved boundaries $\partial\Omega$. If a mesh with polygonal elements, such as triangles in two dimensions, is used, these boundary curves cannot be represented by the straight edges of the triangular elements. It is clear that the solution accuracy predicted by (1.4) will thus be degraded [3, pages 94–95]. The example in Section 4 answers the questions, for which polynomial degree p of Lagrange elements this error destroys the convergence order of the FEM and if it is still worth the computational effort to use higher-order Lagrange elements. It turns out in two dimensions that the pollution is first visible for Lagrange elements with degree $p = 3$, but that the FEM errors are still smaller for $p = 4$ and $p = 5$, even though the convergence order is reduced from its theoretical value.

All following test problems are designed to have a known true PDE solution u to allow for a direct computation of the error $u - u_h$ and its norm in (1.4). The convergence order q is then estimated from these computational results by the following steps: Starting from some initial mesh, we refine it uniformly repeatedly, which subdivides every triangle into four triangles. If h measures the maximum side length of all triangles, this procedure halves the value of h in each refinement. Let r denotes the number of refinement levels from the initial mesh and $E_r := \|u - u_h\|_{L^2(\Omega)}$ the error norm on that level. Then assuming that $E_r = C\, h^q$, the error for the next coarser mesh with mesh spacing $2h$ is $E_{r-1} = C\,(2h)^q = 2^q\, C\, h^q$. Their ratio is then $R_r = E_{r-1}/E_r = 2^q$ and $Q_r = \log_2(R_r)$ provides us with a computable estimate for q in (1.4) as $h \to 0$. Notice that the technique described here uses the known true PDE solution u; this is in contrast to the technique described in [2] that worked for Lagrange elements with $p = 1$ without knowing the PDE solution u.



(a)



(b)

Figure 1: Smooth problem on polygonal domain: (a) solution, (b) mesh.

Table 1: Convergence study for the smooth test problem on $\Omega = (-1, 1)^2$.

(a) Lagrange elements with $p = 1$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	13	3.049e-01
1	64	41	41	8.387e-02 (1.86)
2	256	145	145	2.177e-02 (1.95)
3	1024	545	545	5.511e-03 (1.98)
4	4096	2113	2113	1.383e-03 (1.99)

(b) Lagrange elements with $p = 2$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	41	1.532e-02
1	64	41	145	1.624e-03 (3.24)
2	256	145	545	1.953e-04 (3.06)
3	1024	545	2113	2.445e-05 (3.00)
4	4096	2113	8321	3.075e-06 (2.99)

(c) Lagrange elements with $p = 3$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	85	2.177e-03
1	64	41	313	1.372e-04 (3.99)
2	256	145	1201	8.586e-06 (4.00)
3	1024	545	4705	5.360e-07 (4.00)
4	4096	2113	18625	3.347e-08 (4.00)

(d) Lagrange elements with $p = 4$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	145	9.729e-05
1	64	41	545	2.407e-06 (5.34)
2	256	145	2113	6.471e-08 (5.22)
3	1024	545	8321	1.866e-09 (5.12)
4	4096	2113	23025	5.591e-11 (5.06)

(e) Lagrange elements with $p = 5$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	221	1.003e-05
1	64	41	841	1.649e-07 (5.93)
2	256	145	3281	2.621e-09 (5.98)
3	1024	545	12961	4.094e-11 (6.00)
4	4096	2113	51521	9.099e-13 (5.49)

2 Smooth Problem

In this section, we test the smooth problem on a polygonal domain, which can be partitioned into the finite element mesh without error. Specifically, we choose the square $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ and supply the right-hand side of (1.1) as

$$f(\mathbf{x}) = \frac{\pi}{2}\left(\frac{1}{\rho}\sin\frac{\pi\rho}{2} + \frac{\pi}{2}\cos\frac{\pi\rho}{2}\right) \tag{2.1}$$

and the Dirichlet boundary condition of (1.2) as

$$r(\mathbf{x}) = \cos\frac{\pi\rho}{2}, \tag{2.2}$$

where $\rho = \sqrt{x^2 + y^2}$ for short. This problem admits the true PDE solution

$$u_{true}(\mathbf{x}) = \cos\frac{\pi\sqrt{x^2 + y^2}}{2}; \tag{2.3}$$

the boundary condition is in fact chosen from the PDE solution. This PDE solution is infinitely often differentiable in the classical sense, and hence the regularity order k does not limit the predicted convergence order $q = p + 1$ for any degree p of the Lagrange finite elements.

Figure 1 shows the solution and the fairly coarse mesh used to compute it. This computation used the default quadratic Lagrange elements. Table 1 shows the results for the convergence study for all Lagrange elements available in COMSOL, ranging over degrees $p = 1, \ldots, 5$. For each refinement level r, we list the number of elements N_e in the mesh, the number of points N_p, the number of degrees of freedom DOF, the true error $E_r = \|u - u_h\|_{L^2(\Omega)}$, and in parentheses the estimate Q_r for the convergence order computed as described in the Introduction. In these tables, we observe that the convergence order estimate Q_r is consistent with the predicted value $q = p + 1$ for all $p = 1, \ldots, 5$. In an elliptic problem, a system of linear equations of dimension equal to the DOF needs to be solved, thus this information characterizes the numerical cost associated with each element and each refinement level.

3 Non-Smooth Problem

For the non-smooth test problem, we choose again $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$. The right-hand side function in (1.1) is set to the Dirac delta 'function'

$$f(\mathbf{x}) = \delta(\mathbf{x}) \tag{3.1}$$

and the Dirichlet data in (1.2) to

$$r(\mathbf{x}) = \frac{-\ln\sqrt{x^2 + y^2}}{2\pi}. \tag{3.2}$$

The Dirac delta 'function' models a point source and is mathematically defined by requiring $\delta(\mathbf{x} - \hat{\mathbf{x}}) = 0$ for all $\mathbf{x} \neq \hat{\mathbf{x}}$ while simultaneously $\int \varphi(\mathbf{x})\, \delta(\mathbf{x} - \hat{\mathbf{x}})\, d\mathbf{x} = \varphi(\hat{\mathbf{x}})$ for any continuous function $\varphi(\mathbf{x})$. Considering for instance the example $\varphi \equiv 1$ and $\hat{\mathbf{x}} = 0$, we have $\int \delta(\mathbf{x})\, d\mathbf{x} = 1$. Thus, the definition implies that δ tends to infinity at the origin 0, thus this 'function' is properly called a distribution in mathematics. The finite element method is nonetheless able to deal with this distribution, because it is based on the weak formulation of the problem. That is, the PDE is integrated with respect to a smooth test function $\varphi(\mathbf{x})$ so that the right-hand side becomes $\int_\Omega \varphi(\mathbf{x})\, \delta(\mathbf{x})\, d\mathbf{x} = \varphi(0)$. If the point 0 is chosen as a mesh point of the FEM mesh, the test function φ evaluated at 0 will equal 1 for the FEM basis function centered at this mesh point and 0 for all others. As explained in the COMSOL manual, a point source modeled by the Dirac delta distribution can be implemented in COMSOL by adding the test function `u_test` at that mesh point to the weak term. For the GUI (graphical user interface) of COMSOL Multiphysics, this is described step-by-step in the Quick Start guide, in the chapter Modeling Physics and Equations, under the section on Specifying Point and Edge Settings. To see how this can be implemented in a script, such as usable for COMSOL Script, an example with results is available in the COMSOL area of the first author's homepage (`www.math.umbc.edu/~gobbert`). A sample FEM solution on a fairly coarse mesh is shown along with its mesh in Figure 2.

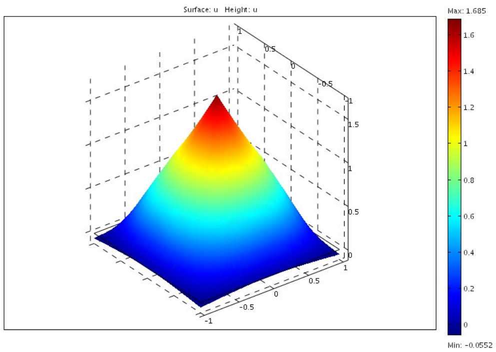
This PDE admits the true PDE solution

$$u(\mathbf{x}) = \frac{-\ln\sqrt{x^2 + y^2}}{2\pi}; \tag{3.3}$$

the boundary condition had been chosen to equal the true solution. Notice that the true solution has a singularity at the origin 0, where it tends to infinity. Thus, the solution is not differentiable everywhere in Ω and thus not in any space of continuous or continuously differentiable functions. However, since $\int v(\mathbf{x})\, \delta(\mathbf{x})\, d\mathbf{x} = v(0)$ for any continuous function $v(0)$ and since the Sobolev space $H^{d/2+\varepsilon}$ is continuously embedded in the space of continuous function $C^0(\Omega)$ in $d = 1, 2, 3$ dimensions for any $\varepsilon > 0$, one can argue that δ is in the dual space of $v \in H^{d/2+\varepsilon}(\Omega)$, that is, $\delta \in H^{-d/2-\varepsilon}(\Omega)$. Using this information in $d = 2$ dimensions, we have $\delta \in H^{-1-\varepsilon}(\Omega)$ for the right-hand side of (1.1). Since the solution u of this second-order elliptic PDE is two orders smoother, we obtain the regularity $u \in H^{1-\varepsilon}(\Omega)$ or $k \approx 1$ in (1.4). In other words, since $k \approx 1$ is smaller than $p + 1$ for any $p \geq 1$, we expect a convergence order of $q \approx 1$ for all Lagrange elements available in COMSOL. This is indeed born out by the values for Q_r in Table 2. However, notice that the absolute errors are still better for the Lagrange element with $p = 2$ than with $p = 1$, therefore, it is worth using these elements; but higher-order Lagrange elements do not give any significantly better results any more.

4 Curved Boundary

Finally, we consider the same data f and r in (2.1) and (2.2), respectively, for the PDE as in Section 2, but we use the unit disk $\Omega = B_1(0) \subset \mathbb{R}^2$ as do-



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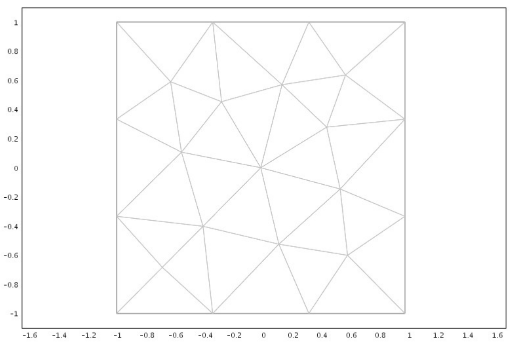


(b)

Figure 2: Non-smooth problem on polygonal domain: (a) solution, (b) mesh.

Table 2: Convergence study for the non-smooth test problem on $\Omega = (-1,1)^2$.

(a) Lagrange elements with $p = 1$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	13	4.589e-02
1	64	41	41	2.468e-02 (0.90)
2	256	145	145	1.256e-02 (0.98)
3	1024	545	545	6.311e-03 (0.99)
4	4096	2113	2113	3.160e-03 (1.00)

(b) Lagrange elements with $p = 2$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	41	9.118e-03
1	64	41	145	4.588e-03 (1.00)
2	256	145	545	2.294e-03 (1.00)
3	1024	545	2113	1.147e-03 (1.00)
4	4096	2113	8321	5.736e-04 (1.00)

(c) Lagrange elements with $p = 3$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	85	6.283e-03
1	64	41	313	3.156e-03 (1.00)
2	256	145	1201	1.578e-03 (1.00)
3	1024	545	4705	7.890e-04 (1.00)
4	4096	2113	18625	3.945e-04 (1.00)

(d) Lagrange elements with $p = 4$

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	145	5.920e-03
1	64	41	545	2.961e-03 (1.00)
2	256	145	2113	1.480e-03 (1.00)
3	1024	545	8321	7.402e-04 (1.00)
4	4096	2113	33025	3.701e-04 (1.00)

(e) Lagrange elements with $p = 5$

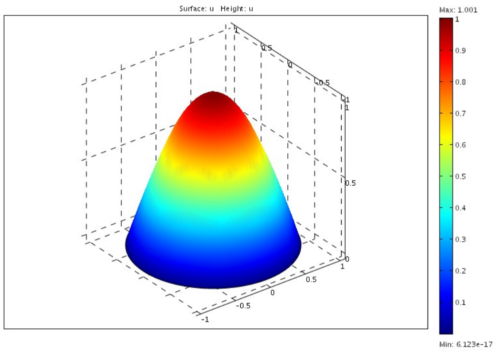
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	221	6.362e-03
1	64	41	841	3.181e-03 (1.00)
2	256	145	3281	1.590e-03 (1.00)
3	1024	545	12961	7.952e-04 (1.00)
4	4096	2113	51521	3.976e-04 (1.00)

main. Also the true PDE solution

$$u_{true}(\mathbf{x}) = \cos \frac{\pi\sqrt{x^2+y^2}}{2} \tag{4.1}$$

is the same as in Section 2, and we notice that the Dirichlet boundary condition is in fact homogeneous $r \equiv 0$ in this case. Figure 3 shows the FEM solution obtained on a fairly coarse mesh and its mesh. The mesh plot in Figure 3 (b) shows clearly that the boundary of the unit disk is approximated by straight edges of the triangles in the mesh.

The pollution effects of the inaccuracies associated with this approximation are visible from the data in Table 3. We see that for $p = 1$ and $p = 2$, the estimate of the convergence orders are still nominal $q = p + 1$. For $p = 3$, where the convergence order should be $q = 4$, we see a slight degradation. Finally for $p = 4$ and $p = 5$, we clearly see that the convergence order does not improve any more over the case of $p = 3$. We notice that the absolute errors still improve with mesh refinement, but likely not enough to justify the larger computational cost incurred by the highest-degree elements, as quantified by its large DOF.



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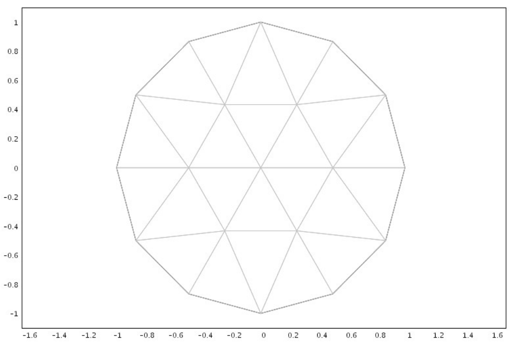

(b)

Figure 3: Smooth problem on domain with curved boundary: (a) solution, (b) mesh.

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Table 3: Convergence study for the smooth test problem on $\Omega = B_1(0) \in \mathbb{R}^2$.

(a) Lagrange elements with $p = 1$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	69	3.345e-02
1	448	249	249	8.567e-03 (1.97)
2	1792	945	945	2.160e-03 (1.99)
3	7168	3681	3681	5.415e-04 (2.00)
4	28672	14529	14529	1.355e-04 (2.00)

(b) Lagrange elements with $p = 2$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	249	5.386e-04
1	448	249	945	7.098e-05 (2.92)
2	1792	945	3681	9.016e-06 (2.98)
3	7168	3681	14529	1.136e-06 (2.99)
4	28672	14529	57729	1.426e-07 (2.99)

(c) Lagrange elements with $p = 3$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	541	4.943e-05
1	448	249	2089	3.333e-06 (3.89)
2	1792	945	8209	2.354e-07 (3.82)
3	7168	3681	32545	1.764e-08 (3.74)
4	28672	14529	129601	1.399e-09 (3.66)

(d) Lagrange elements with $p = 4$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	945	8.115e-06
1	448	249	3681	6.607e-07 (3.62)
2	1792	945	14529	5.788e-08 (3.51)
3	7168	3681	57729	5.151e-09 (3.49)
4	28672	14529	230145	4.582e-10 (3.49)

(e) Lagrange elements with $p = 5$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	1461	2.708e-06
1	448	249	5721	1.735e-07 (3.97)
2	1792	945	22641	1.376e-08 (3.66)
3	7168	3681	90081	1.198e-09 (3.52)
4	28672	14529	359361	1.064e-10 (3.49)