Numerical Demonstration of Finite Element Convergence for Lagrange Elements in COMSOL Multiphysics

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Problem Statement

- Problem: Assess the quality of a FEM solution quantitatively for all Lagrange elements with polynomial degrees $1 \leq p \leq 5$ available in COMSOL.
- Approach: Use guidance from the a priori error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q, \qquad \text{as } h \to 0$$

  with a constant C independent of h and the convergence order $q > 0$. Here, h is the maximum side length of the elements in the triangulation.
- Concrete goal: Confirm that solutions on a sequence of meshes, that are progressively uniformly refined, behaves as predicted by the error estimate.

Computational Convergence Study

- Consider the FEM solution u_h on a sequence of meshes with uniform refinement levels $r = 0, 1, 2, \ldots$, and let $E_r := \|u - u_h\|_{L^2(\Omega)}$ denote the norm of the error.
- Then assuming that $E_r = C\, h^q$, the error for the next coarser mesh with mesh spacing $2h$ is $E_{r-1} = C\,(2h)^q = 2^q\, C\, h^q$. Their ratio is then $R_r = E_{r-1}/E_r = 2^q$ and $Q_r = \log_2(R_r)$ provides us with a computable estimate for q as $h \to 0$.

r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	13	3.049e-01
1	64	41	41	8.387e-02 (1.86)
2	256	145	145	2.177e-02 (1.95)
3	1024	545	545	5.511e-03 (1.98)
4	4096	2113	2113	1.383e-03 (1.99)

- What value do we expect for the convergence order q?

FEM Theory for Lagrange Elements

- For linear Lagrange elements (polynomial degree $p = 1$), optimal convergence order is $q = p + 1 = 2$ in
$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q = C\, h^2$$
- For Lagrange FEM with polynomial degree $p = 1, \ldots, 5$, as available in COMSOL, we expect $q = p + 1$ in
$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q = C\, h^{p+1},$$
provided that
  - the solution u is smooth enough: $u \in H^k(\Omega)$ with $k \geq p + 1$,
  - the domain Ω is bounded, convex, and simply connected,
  - and the domain boundary $\partial\Omega$ piecewise smooth, i.e., the domain Ω can be triangulated without error.
- For Lagrange FEM with polynomial degree $p = 1, \ldots, 5$, if the solution is $u \in H^k(\Omega)$, then
$$\|u - u_h\|_{L^2(\Omega)} \leq C\, h^q, \quad q = \min\{k, p + 1\}.$$

Three Example Problems

Poisson problem with Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega, \\ u &= r \quad \text{on } \partial\Omega. \end{aligned}$$

- Example 1: Smooth problem with polygonal domain

  $$u \in H^k(\Omega) \text{ with } k = \infty \text{ and polygonal domain } \Omega \subset \mathbb{R}^2$$

- Example 2: Non-smooth problem with polygonal domain

  $$u \in H^k(\Omega) \text{ with } k = 1 \text{ and polygonal domain } \Omega \subset \mathbb{R}^2$$

- Example 3: Smooth problem with curved boundary

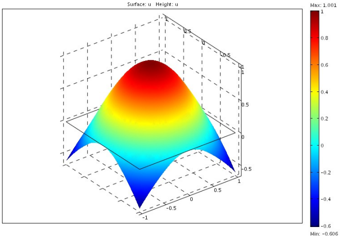
  $$u \in H^k(\Omega) \text{ with } k = \infty \text{ and } \Omega \subset \mathbb{R}^2 \text{ with curved boundary}$$

Example 1: Smooth Problem on Polygonal Domain

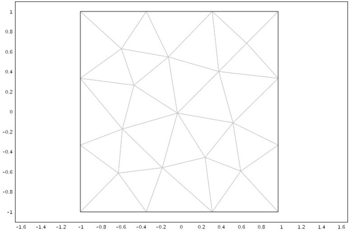
$$u_{true}(\mathbf{x}) = \cos \frac{\pi\sqrt{x^2+y^2}}{2} \text{ on } \Omega = (-1,1)^2 \subset \mathbb{R}^2$$

$$u_{true} \in H^k(\Omega) \text{ infinitely often differentiable} \Longrightarrow k = \infty$$

$$\Longrightarrow \text{convergence order } q = \min\{k, p+1\} = p+1$$



Solution



Mesh

Example 1: Smooth Problem on Polygonal Domain

Lagrange elements with $p = 1$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	13	3.049e-01
1	64	41	41	8.387e-02 (1.86)
2	256	145	145	2.177e-02 (1.95)
3	1024	545	545	5.511e-03 (1.98)
4	4096	2113	2113	1.383e-03 (1.99)

Lagrange elements with $p = 2$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	41	1.532e-02
1	64	41	145	1.624e-03 (3.24)
2	256	145	545	1.953e-04 (3.06)
3	1024	545	2113	2.445e-05 (3.00)
4	4096	2113	8321	3.075e-06 (2.99)

Example 1: Smooth Problem on Polygonal Domain

Lagrange elements with $p = 3$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	85	2.177e-03
1	64	41	313	1.372e-04 (3.99)
2	256	145	1201	8.586e-06 (4.00)
3	1024	545	4705	5.360e-07 (4.00)
4	4096	2113	18625	3.347e-08 (4.00)

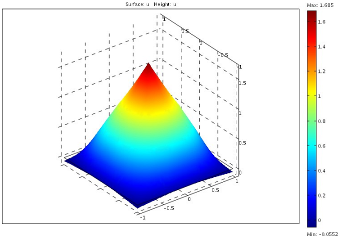
Lagrange elements with $p = 4$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	145	9.729e-05
1	64	41	545	2.407e-06 (5.34)
2	256	145	2113	6.471e-08 (5.22)
3	1024	545	8321	1.866e-09 (5.12)
4	4096	2113	23025	5.591e-11 (5.06)

Example 2: Non-Smooth Problem on Polygonal Domain

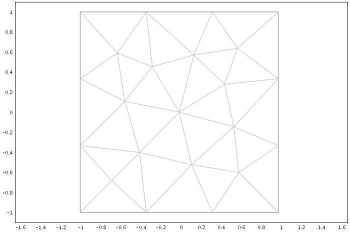
$$u_{true}(\mathbf{x}) = \frac{-\ln\sqrt{x^2+y^2}}{2\pi} \text{ on } \Omega = (-1,1)^2 \subset \mathbb{R}^2$$

$$f = \delta(\mathbf{x}) = \text{ Dirac delta distribution} \Longrightarrow u_{true} \in H^k(\Omega) \text{ with } k = 1$$

$$\Longrightarrow \text{convergence order } q = \min\{k, p+1\} = 1 \text{ for all } p = 1, \ldots, 5$$



Solution



Mesh

Example 2: Non-Smooth Problem on Polygonal Domain

Lagrange elements with $p = 1$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	13	4.589e-02
1	64	41	41	2.468e-02 (0.90)
2	256	145	145	1.256e-02 (0.98)
3	1024	545	545	6.311e-03 (0.99)
4	4096	2113	2113	3.160e-03 (1.00)

Lagrange elements with $p = 2$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	41	9.118e-03
1	64	41	145	4.588e-03 (1.00)
2	256	145	545	2.294e-03 (1.00)
3	1024	545	2113	1.147e-03 (1.00)
4	4096	2113	8321	5.736e-04 (1.00)

Example 2: Non-Smooth Problem on Polygonal Domain

Lagrange elements with $p = 3$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	85	6.283e-03
1	64	41	313	3.156e-03 (1.00)
2	256	145	1201	1.578e-03 (1.00)
3	1024	545	4705	7.890e-04 (1.00)
4	4096	2113	18625	3.945e-04 (1.00)

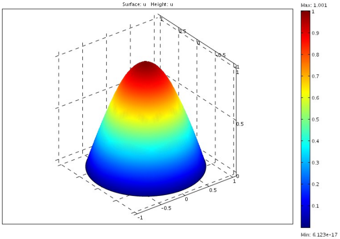
Lagrange elements with $p = 4$				
r	N_e	N_p	DOF	E_r (Q_r)
0	16	13	145	5.920e-03
1	64	41	545	2.961e-03 (1.00)
2	256	145	2113	1.480e-03 (1.00)
3	1024	545	8321	7.402e-04 (1.00)
4	4096	2113	33025	3.701e-04 (1.00)

Example 3: Smooth Problem with Curved Boundary

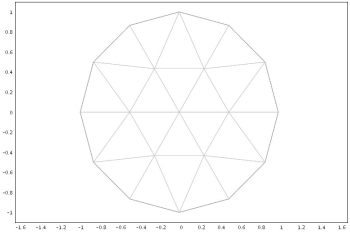
$$u_{true}(\mathbf{x}) = \cos \frac{\pi\sqrt{x^2+y^2}}{2} \text{ on unit ball } \Omega = B_1(0) \subset \mathbb{R}^2$$

$$u_{true} \in H^k(\Omega) \text{ infinitely often differentiable} \Longrightarrow k = \infty$$

$$\text{triangulation of } \Omega \text{ incurs error} \Longrightarrow q = \min\{k, p+1\} < p+1$$



Solution



Mesh

Example 3: Smooth Problem with Curved Boundary

Lagrange elements with $p = 1$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	69	3.345e-02
1	448	249	249	8.567e-03 (1.97)
2	1792	945	945	2.160e-03 (1.99)
3	7168	3681	3681	5.415e-04 (2.00)
4	28672	14529	14529	1.355e-04 (2.00)

Lagrange elements with $p = 2$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	249	5.386e-04
1	448	249	945	7.098e-05 (2.92)
2	1792	945	3681	9.016e-06 (2.98)
3	7168	3681	14529	1.136e-06 (2.99)
4	28672	14529	57729	1.426e-07 (2.99)

Example 3: Smooth Problem with Curved Boundary

Lagrange elements with $p = 3$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	541	4.943e-05
1	448	249	2089	3.333e-06 (3.89)
2	1792	945	8209	2.354e-07 (3.82)
3	7168	3681	32545	1.764e-08 (3.74)
4	28672	14529	129601	1.399e-09 (3.66)

Lagrange elements with $p = 4$

r	N_e	N_p	DOF	E_r (Q_r)
0	112	69	945	8.115e-06
1	448	249	3681	6.607e-07 (3.62)
2	1792	945	14529	5.788e-08 (3.51)
3	7168	3681	57729	5.151e-09 (3.49)
4	28672	14529	230145	4.582e-10 (3.49)

Conclusions and Future Work

Conclusions:

- COMSOL: behaves as predicted by theory for Lagrange elements on triangular meshes in 2-D, also in 1-D and on tetragonal meshes in 3-D; see a technical report available from my webpage
- Education: COMSOL can be used to demonstrate FEM theory
- Applications: tests of this type can guide choice of finite element
- Support: structure of driver scripts available from my webpage `www.math.umbc.edu/~gobbert`

Future Work:

- Extensions: quadrilateral meshes in 2-D and 3-D, other elements than Lagrange elements
- Extension to error based on reference solution; this requires more information about internal data structures of COMSOL!
- Alternative: isoparametric elements to allow for curved boundaries
- Alternative: a posteriori error estimates (to measure error from computable quantities); also needs more info. about COMSOL!