Math 441: Test 1 – Solutions

Problem 1: (18 pts) Let $f(x) = \frac{1}{\sqrt{1+x}}$.
(a) Write the first order Taylor polynomial T_1 around the value $x_0 = 3$. Use T_1 to approximate $f(3.1)$.

$$T_1(x) = \frac{1}{2} - \frac{x - x_0}{2(1 + x_0)\sqrt{1 + x_0}} = \frac{1}{2} - \frac{x - 3}{16} .$$

Therefore

$$f(3.1) \approx T_1(3.1) = \frac{1}{2} - \frac{0.1}{16} = \frac{79}{160} .$$

(b) Write the remainder R_1 in Lagrange form and use it to estimate how well you approximated $f(3.1)$.

We have

$$R_1(x) = \frac{3}{8} \frac{(x - x_0)^2}{(1 + \xi)^2\sqrt{1 + \xi}} ,$$

with $\xi \in [x_0, x]$. Here

$$R_1(x) \approx \frac{3}{8} \cdot \frac{0.1^2}{32} = \frac{3}{256} 10^{-2} .$$

Problem 2: (18 pts) Let $x = 2^{12} + 2^{-9} + 2^{-12}$.
(a) Find the machine numbers x_- and x_+ on the hypothetical computer Marc-32 just to the left and right of x respectively.
We write $x = (1 + 2^{-21} + 2^{-24}) \times 2^{12}$; hence the last bit (on position 24) will be discarded, and $x_- = (1 + 2^{-21}) \times 2^{12} = 2^{12} + 2^{-9}$. Therefore $x_+ = (1 + 2^{-21} + 2^{-23}) \times 2^{12} = 2^{12} + 2^{-9} + 2^{-11}$.

(b) Determine $fl(x)$ and the relative error between x and $fl(x)$.
Simple subtraction gives $x - x_- = 2^{-12}$ and $x_+ - x = 2^{-11} - 2^{-12} = 2^{-12}$. Hence x is exactly in the middle between x_- and x_+. Depending on the

convention you can choose either $fl(x) = x_-$ or x_+. In either case the relative error is

$$\frac{|x - fl(x)|}{x} = \frac{2^{-12}}{2^{12} + 2^{-9} + 2^{-12}} = \frac{2^{-24}}{1 + 2^{-21} + 2^{-24}} \ .$$

Problem 3: (18 pts) Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 2 & 5 \end{bmatrix} \ .$$

Use Gaussian elimination to find the LU decomposition of A. Write L as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \ ,$$

and

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \ .$$

In matrix language:

$$E_{32}(-2)E_{21}(1)A = U \Longrightarrow A = \underbrace{E_{21}(-1)E_{32}(2)}_{L} U \ ,$$

therefore

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \ .$$

Problem 4: (14 pts) Let $f(x) = \frac{1}{1+x^2} - \log_2 x$. Find an interval in which the equation $f(x) = 0$ has a solution. How many steps of the bisection method would be required in order to compute the solution with an approximation of 2^{-12}?

$f(1) = \frac{1}{2} > 0$ and $f(2) = \frac{1}{5} - 1 < 0$. Hence f has a zero in $[1, 2]$. If we divide n times the interval $[1, 2]$ we would have located the zero in an interval of length $2^{-n}(2-1) = 2^{-n}$. Hence $n = 12$ locates the solution in an interval of the requested size.

Problem 5: (14 pts) Which of the following sequences converge to 0 in a q-linear or q-quadratic way: a) $\frac{1}{n^4}$, b) $\frac{1}{3^{2^n}}$ (the exponent of the denominator is 2^n)? For q-linearity we need to form x_{n+1}/x_n. For the first sequence the result is $n^4/(n+1)^4 \to 1$, hence $\frac{1}{n^4}$ **does not** converge q-linearly, therefore does not converge q-quadraticaly as well. For the second sequence

$$x_{n+1}/x_n = \frac{3^{2^n}}{3^{2^{n+1}}} = \frac{1}{3^{2^{n+1}-2^n}} = 3^{-2^n} \to 0 \ .$$

As a result, the second sequence converges q-linearly. In order to verify that it also converges q-quadratically we form

$$x_{n+1}/x_n^2 = \frac{3^{2^n}}{3^{2 \cdot 2^{n+1}}} = 1 \ .$$

Therefore it also converges q-quadratically.

Problem 6: (18 pts) Write the Newton iteration for the equation

$$\begin{cases} 4x^2 - y^2 & = & 0 \\ 4xy^2 - x & = & 1 \end{cases} ,$$

and compute the first iterate starting from $(x_0, y_0) = (0, 1)$.

Define $f(x, y) = (4x^2 - y^2, 4xy^2 - x - 1)$. First we need to compute

$$f'(x, y) = \begin{bmatrix} 8x & -2y \\ 4y^2 - 1 & 8xy \end{bmatrix} .$$

The Newton iteration is:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 8x_n & -2y_n \\ 4y_n^2 - 1 & 8x_n y_n \end{bmatrix}^{-1} \begin{bmatrix} 4x_n^2 - y_n^2 \\ 4x_n y_n^2 - x_n - 1 \end{bmatrix} .$$

If we take $(x_0, y_0) = (0, 1)$ then

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} .$$