Math 441, Introduction to Numerical Analysis
Fall 2006

Homeworks 4 and 5 – solutions to selected problems

- page 91, exercise 6
  solution:
  The formula is:

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - (-x_n^{-2})^{-1}(x_n^{-1} - R) = 2x_n - x_n^2 R \ .$$

  The question is for what values of x_0 the process converges. Our proof in class for convergence of Newton's method (using the contraction principle) gives a set of starting points for which x_n converges to R^{-1}. Namely, for $F(x) = 2x - x^2R$, if we find $\delta > 0$ such that on $x \in I_\delta = (x^* - \delta, x^* + \delta), |F'(x)| < 1$, then for all $x_0 \in I_\delta$ the sequence x_n converges to x^* (here $x^* = R^{-1}$). We have

  $$|F'(x)| < 1 \Leftrightarrow 2\,|1 - xR| < 1 \Leftrightarrow x \in \left(R^{-1} - \frac{1}{2R}, R^{-1} + \frac{1}{2R}\right) \ .$$

  This was the general recipe. This is not yet to say that these are **all** values for which x_n converges to the solution. In general it is not possible to characterize all such x_0's. However, for this simple example one can show that for $x_0 \in (0, 2/R)$ the sequence x_n converges to $1/R$. (Hint: $(x_{n+1} - 1/R) = (x_n - 1/R)(1 - x_nR)$).

- page 193, exercises 5, 13
  comment:
  The answers are 'in general no' for both questions. There are plenty of examples. However, for the second question in exercise 5, the answer is positive provided the matrix A is diagonal.

- page 193, exercise 43

  solution:

  Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be the matrix, and $x = (x_1, \dots, x_n)$ be a vector such that $\|x\|_1 = 1$, that is, $\sum_{j=1}^n |x_j| = 1$. Then

$$\begin{aligned} \|Ax\|_\infty &= \max_{i=1}^m \left| \sum_{j=1}^n a_{ij} x_j \right| \le \max_{i=1}^m \sum_{j=1}^n |a_{ij}|\,|x_j| \\ &\le \max_{i=1}^m \max_{j=1}^n |a_{ij}| \underbrace{\sum_{j=1}^n |x_j|}_{1} = \max_{1\le i\le m, 1\le j\le n} |a_{ij}| \ , \end{aligned}$$

  which implies that the subordinate matrix norm satisfies

$$\|A\| \le \max_{1\le i\le m, 1\le j\le n} |a_{ij}| \ .$$

  Now, if we choose i_0, j_0 where the $\max_{1\le i\le m, 1\le j\le n} |a_{ij}|$ is attained, and e_{j_0} is the vector with its j_0^{th} component is 1 and 0 everywhere else, then $\|Ae_{j_0}\|_\infty = |a_{i_0 j_0}| = \max_{1\le i\le m, 1\le j\le n} |a_{ij}|$. Hence

$$\|A\| = \max_{1\le i\le m, 1\le j\le n} |a_{ij}| \ .$$

- page 106, exercise 2

  solution:

  The Mean Value Theorem shows that

$$|F(x) - F(y)| = |F'(\xi)| \ |x - y| \ ,$$

  for some ξ between x and y. **Since F' is continuous on the compact interval** $[a, b]$, there exists $\eta \in [a, b]$ such that

$$|F'(x)| \le |F'(\eta)| = M < 1 \ .$$

  Hence

$$|F(x) - F(y)| = M\,|x - y| \ ,$$

  with $M < 1$. However, if $F([a, b])$ is not included in $[a, b]$ there is no guarrantee that there is a fixed point. Take for example $[a, b] = [0, 1]$, and $F(x) = \frac{x}{2} + 2$. F is a contraction, but $F(x) \ge 2$, therefore it has no fixed point in $[0, 1]$.

- page 106, exercise 4
  solution:
  If $|f'(x)| \leq M < 1$, then by the Mean Value Theorem

$$|f(x) - f(y)| \leq M\,|x - y| \ .$$

Therefore the smallest M with such property is

$$\max_{x \in I} |f'(x)| \ ,$$

where I is the interval of interest, which could be the whole real line.
For a) we see that

$$|F'(x)| = \frac{2x}{(1+x^2)^2} \ ,$$

and a standard calculation shows that

$$\max_{x \in R} |F'(x)| = \left| F'\left(\frac{1}{\sqrt{3}}\right) \right| = \frac{3\sqrt{3}}{8} \stackrel{\text{def}}{=} \lambda \ .$$

For c) we have $F'(x) = \frac{1}{1+x^2}$. If the interval is $[a, b]$ with $0 < a < b$, then, since $F'(x)$ is decreasing on $[a, b]$, we take

$$\lambda = F'(a) = \frac{1}{1+a^2} \ .$$

Note that F is not contractive on $[0, b]$, because it includes a point, namely $x = 0$, where $|F'(x)| = 1$.

- page 106, exercise 13
  solution:
  Define $x_0 = \frac{1}{p}$, and let

$$x_{n+1} = \frac{1}{p + x_n} \ .$$

The continued fraction is $x^* = \lim_{n \to \infty} x_n$, if it exists. If x^* exists, then

$$x^* = \frac{1}{p + x^*} \ ,$$

therefore x^* satisfies the equation

$$x^2 + px - 1 = 0,$$

which has two solutions:

$$x_{1,2} = \frac{-p \pm \sqrt{p^2+4}}{2} \ .$$

We would pick the positive solution since all x_n's are positive. Hence

$$x^* = \frac{-p + \sqrt{p^2+4}}{2} \ .$$

The question of convergence is related to whether

$$f(x) = \frac{1}{p+x}$$

is a contraction. We have

$$|f'(x)| = \frac{1}{(p+x)^2} \leq \frac{1}{p^2} < 1, \text{ for x} > 0.$$

It remains to find a closed interval such that $f(I) \subseteq I$, and $I = [0,1]$ does the job. The contraction mapping theorem applies to show that for **any** $x_0 \in I$, the sequence x_n converges to x^*.