Math 441, Introduction to Numerical Analysis
Fall 2006

Homework 2 – solutions to selected problems

1. Prove that for all $x \in \mathcal{R}$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, by showing that the remainder $R_n(x) = e^x - T_n(x)$ converges to 0 as $n \to \infty$, where $T_n(x)$ is the n^{th} Taylor polynomial of e^x around 0.
solution:
The Lagrange form of the remainder is:
$$R_n(x) = \frac{f^{[n+1]}(\xi)}{(n+1)!} x^{n+1} ,$$
with $\xi \in [0, x]$ or $[x, 0]$ depending on whether $0 < x$ or $x < 0$. Since $f^{[n+1]}(\xi) = e^{\xi}$, we have $f^{[n+1]}(\xi) \leq e^x$ if $x > 0$, or $f^{[n+1]}(\xi) \leq 1$, if $x < 0$. In either case $|f^{[n+1]}(\xi)| \leq C(x)$, with $C(x)$ being a constant independent of n. Hence
$$|R_n(x)| \leq C(x) \frac{|x|^{n+1}}{(n+1)!} ,$$
and the sequence $a_n = \frac{|x|^{n+1}}{(n+1)!}$ converges to zero because
$$\lim_{n\to\infty} a_{n+1}/a_n < 1$$
(what is the limit?).

2. Which of the following are true (justify your answers)?

   (a) $xy = O(\sqrt{x^2 + y^2})$ as $(x, y) \to 0$.
   solution:
   In the neighborhood $(-1, 1) \times (-1, 1)$ of $(0, 0)$ we have $|x| < \sqrt{x^2 + y^2}$ and $|y| \leq 1$, therefore
   $$|xy| \leq \sqrt{x^2 + y^2} .$$
   The answer is 'yes'.

(b) $\sqrt{x^2+y^2} = O(xy)$ as $(x,y) \to 0$.
solution:
Assume that in a certain neighborhood of V of $(0,0)$ we could find a constant C such that $\sqrt{x^2+y^2} \le C|xy|$. There exists a number $\delta > 0$ such that $(\delta, 0) \in V$. However, $\sqrt{\delta^2+0^2} > C\delta \cdot 0$, which contradicts the assumption. The answer is 'no'.

3. Write a Matlab function that computes $\ln(x)$ with a given approximation from the formula

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots .$$

Verify your answers against the Matlab provided ln function for three numbers of your choice (pick numbers in the interval where the series converges). The header of the function is the following `function value=logtaylor(x, delta)`. Write your function such that the sum contains no more than 10^4 terms.
solution:
The key item is the Lagrange form of the remainder R_n, namely

$$R_n(x) = (-1)^n \frac{(1+\xi)^{-(n+1)}}{n+1} x^{n+1} \ ,$$

which, for $x > 0$ leads to the estimate

$$|R_n(x)| \le \frac{x^{n+1}}{n+1} \ .$$

This means that, in order for $|T_n(x) - \ln(1+x)| \le \delta$ it is sufficient for the **next term in the series** to be $\le \delta$. Please find the code under the 'Codes' tab of the course webpage. Runs:
Type: `abs(log(1.6)-logtaylor(0.6, 0.001))`;
result: `0.00039168`.
Type: `abs(log(1.6)-logtaylor(0.6, 0.0001))`;
result: `3.5903e-05`.
Type: `abs(log(2)-logtaylor(1, 0.001))`;
result: `0.00050025`.
Note that the result is always smaller than the prescribed error (δ), unless more than $N = 10^4$ are needed.

4. Compute the Taylor polynomials T_0, T_1, T_2, T_3 of the function $f(x,y) = \ln(x+y)$ around the point $(x,y) = (1,0)$.

solution:

$$\begin{aligned}
T_0(x,y) &= 0\ ;\\
T_1(x,y) &= (x-1)+y\ ;\\
T_2(x,y) &= (x-1)+y-\frac{1}{2}(x-1)^2-(x-1)y-\frac{1}{2}y^2\ ;\\
T_3(x,y) &= (x-1)+y-\frac{1}{2}(x-1)^2-(x-1)y-\frac{1}{2}y^2\\
&\quad +\frac{1}{3}(x-1)^3+(x-1)^2y+(x-1)y^2+\frac{1}{3}y^3\ ;
\end{aligned}$$

5. page 14, exercise 36.
   solution:
   We have
   $$x^{\frac{1}{5}} = 32^{\frac{1}{5}} + \frac{1}{5}32^{-\frac{4}{5}}(x-32) - \frac{4}{2\cdot 25}\xi^{-\frac{9}{5}}(x-32)^2\ ,$$
   with ξ between x and 32. For $x_0 = 31.999999$ we obtain
   $$x^{\frac{1}{5}} \approx 2 - \frac{1}{16\cdot 5}10^{-6}\ .$$
   The remainder can be estimated by
   $$|R_1(x_0)| \le \frac{2}{25}x_0^{-\frac{9}{5}}10^{-12} \approx \frac{2}{25\cdot 2^9}10^{-12}.$$

6. pages 51-53, exercise 10
   solution:
   $x = 2^3 + 2^{-19} + 2^{-22} = 2^3(1+2^{-22}+2^{-25})$. By discarding all digits after the 23$^{\text{rd}}$ one we obtain
   $$x_- = 2^3(1+2^{-22})\ .$$
   Then
   $$x_+ = 2^3(1+2^{-22}+2^{-23})\ .$$
   Since $x - x_- = 2^{-22}$ and $x_+ - x = 2^{-20}$ we have $\mathrm{fl(x)} = \mathrm{x}_-$. Hence the absolute round-off error here is $|x - \mathrm{fl(x)}| = 2^{-22}$, and the relative round-off error is
   $$\frac{|x-\mathrm{fl(x)}|}{|x|} = \frac{2^{-22}}{2^3(1+2^{-22}+2^{-25})} = \frac{2^{-25}}{1+2^{-22}+2^{-25}} < 2^{-25}\ .$$