Lectures 16 – contents

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Lecture 16. (Section 6.1)

- **The problem:** Given some information about a function f – values, and/or derivatives at a set of points $x_0, x_1, x_2, \ldots, x_n$ – we would like additional information at, maybe different points, e.g., $f(x), f''(x_2)$.
- **The solution:** Represent the function by a function P for which we have an analytical expression, a process called *interpolation*.
- **Polynomial interpolation:** P is globally polynomial.
- **Theorem:** Given $x_0, x_1, x_2, \ldots, x_n$ pairwise distinct numbers on the real line, and values $y_0, y_1, \ldots, y_n$, there exists a unique polynomial of degree $\leq n$ such that $P(x_i) = y_i, \forall 0 \leq i \leq n$. (Proof is done by the "counting" argument and the fact that it's a linear system that needs to be solved.) P is called the interpolation polynomial.
- Newton form of the interpolation polynomial: we construct a sequence of polynomials $p_0, p_1, \ldots, p_n$, with $P = p_n$, and

  $$p_k(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \cdots c_k(x - x_0)\cdots(x - x_{k-1}) \ .$$

  Computing c_i's is done recursively:

  $$c_k = \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0)\cdots(x_k - x_{k-1})}.$$

- Computing $p_k(x)$ through nested multiplication:

  $$p_k(x) = (\ldots(x - x_{k-1})c_k + c_{k-1})(x - x_{k-2}) + c_{k-3})(x - x_{k-3}) + \cdots + c_0 \ .$$

- **Lagrange formula:**

$$P(x) = \sum_{i=0}^{n} y_k l_k(x),$$

where the *cardinal functions* are

$$l_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} \ .$$

- **Example:** $f(x) = |x|, x_0 = -1, x_1 = 0, x_2 = 1, y_i = f(x_i)$.

- **Theorem on polynomial interpolation error:** If f is $n+1$ times differentiable in $[a, b]$, $x_0, \ldots x_n \in [a, b]$, then for each $x \in [a, b]$ there exists $\xi_x \in (a, b)$ s.t.

$$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i) \ .$$