Lecture 1 – brief summary

Thursday, 08/30/2006

1. The main objectives of numerical analysis are the design, analysis and practical evaluation of numerical methods for solving mathematical problems. Typically, the mathematical problem is to solve an equation of some sort. A priori knowledge about the existence, uniqueness (if), and various characteristics of the solution are essential ingredients in the design and analysis of a numerical method (scheme).

2. **Example:** The *forward Euler* method for solving the initial value problem

$$\frac{d}{dt}x = -a\,x \ , \quad x(0) = x_0 \ , \tag{1}$$

with $a > 0$. The equation has the exact solution $x(t) = x_0 e^{-at}$. Fix a *time step* $h > 0$ and define $t_n = nh$. The goal of a numerical method for solving (1) is to approximate the values of the exact solution at the "times" t_n, that is, $x_n = x(t_n)$. Our program consists in: **A.** developing the method, and **B.** analysing it.
A. We will construct a sequence U_n such that $U_n \approx x_n$. It is natural to define $U_0 = x_0$. The remainder of the terms will be defined inductively. The main step is to derive a difference equation from (1) for U_n to satisfy. Consider the value of the two sides of (1) at t_n. If we consider the approximation

$$\frac{d}{dt}x(t_n) \approx \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} = \frac{x(t_{n+1}) - x(t_n)}{h} \ , \tag{2}$$

then we can write

$$\frac{U_{n+1} - U_n}{h} \approx \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \approx \frac{d}{dt}x(t_n) = -a\,x(t_n) \approx -aU_n \ .$$

By setting an equal sign between the ends we derive the difference equation that now defines U_n:

$$\frac{U_{n+1} - U_n}{h} = -aU_n \ , \tag{3}$$

or

$$U_{n+1} = (1 - ah)U_n \ , \tag{4}$$

Equation (4) can be easily solved:

$$U_n = (1 - ah)^n U_0 \ , \tag{5}$$

B. The question raised here is whether U_n approximates x_n. More precisely, we want to learn whether for fixed $n > 0$

$$\lim_{h \to 0} U_n = x_n \ ,$$

and how fast the above convergence is. The first thing to notice is that if $(1 - ah) \leq -1$ (i.e. $h \geq 2/a$) then the sequence U_n diverges, thus rendering the method *unstable*. The method is *stable* only for $h < 2/a$. In order to resolve the convergence issue we compute the difference

$$E(h) = U_n(h) - x(nh) = \Big((1 - ah)^n - e^{-anh}\Big)\, x_0 \ . \tag{6}$$

Taylor's theorem states (under certain conditions) that

$$E(h) = E(0) + E'(0)h + \frac{E''(\xi)}{2} h^2 \ , \tag{7}$$

for some $\xi \in (0, h)$. An easy calculation shows that $E(0) = 0$, $E'(0) = 0$, and $E''(0) \neq 0$. By continuity we do not expect that $E''(\xi) \neq 0$ for $\xi \approx 0$. Therefore we conclude that

$$E(h) = \Theta(n, h)h^2 \ ,$$

with Θ being a bounded function of n, h. Hence our conclusion is that U_n converges to x_n as $h \to 0$, and the order of convergence (exponent of h) is 2.