MULTIGRID PRECONDITIONERS FOR THE NEWTON-KRYLOV METHOD IN THE OPTIMAL CONTROL OF THE STATIONARY NAVIER-STOKES EQUATIONS *

ANA MARIA SOANE† AND ANDREI DRĂGĂNESCU ‡

Abstract. In this work we construct multigrid preconditioners to be used in the Newton-Krylov method for a distributed optimal control problem constrained by the stationary Navier-Stokes equations. These preconditioners are shown to be of optimal order with respect to the convergence properties of the discrete methods use to solve the Navier-Stokes equations.

Key words. multigrid methods, PDE-constrained optimization, Navier-Stokes equations, finite elements

AMS subject classifications. 65F08, 65K15, 65N21, 65N55, 90C06

1. Introduction.

We consider the optimal control problem

$$\min_{y,p,u} J(y,p,u) = \frac{\gamma_y}{2}\|y-y_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\gamma_p}{2}\|p-p_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{\mathbf{L}^2(\Omega)}, \tag{1}$$

subject to the stationary Navier-Stokes equations

$$\begin{aligned} -\nu\Delta y + (y\cdot\nabla)y + \nabla p &= u \quad \text{in } \Omega,\\ \operatorname{div} y &= 0 \quad \text{in } \Omega,\\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain. The goal of the control problem is to find a force u that gives rise to a velocity y and/or pressure p to match a known target velocity y_d, respectively pressure p_d. Since this problem is ill-posed, we consider a standard Tikhonov regularization for the force, with the regularization parameter β being a fixed positive number. The constants γ_y, γ_p are nonnegative, not both zero.

Optimal control problems constrained by the Navier-Stokes equations have been studied in many papers, see e.g. [11, 12, 13, 5] and the references therein, where both optimality conditions and numerical methods are addressed, for the unconstrained, control-constrained, or mixed control-state constrained problems. For a comprehensive overview of optimal flow control we refer to [10]. This paper focuses on the efficient solution of the linear systems arising in the solution process of (1)–(2), specifically on the design of multigrid preconditioners for the reduced Hessian in the Newton-CG method. To the best of our knowledge, this has not been addressed in the literature for the Navier-Stokes optimal control problem. For the Stokes optimal control problem, the design of efficient preconditioners for the Karush-Kuhn-Tucker (KKT) system is addressed in [15, 17] and the case of the reduced KKT system is discussed in [7]. This paper is an extension of the work on the Stokes optimal control problem

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†Department of Mathematics, United States Naval Academy, Annapolis, Maryland 21402 (soane@usna.edu)

‡Department of Mathematics and Statistics, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, Maryland 21250 (draga@umbc.edu).

in [7]; essentially, we show that for low to moderate Reynolds numbers the constructed preconditioners display the same optimal behavior as in the case of Stokes flow. While the design of the preconditioner is rooted in [7], the analysis presents several challenges due to the presence of the convection term in the constraints. Therefore, the main contribution in this work lies in the analysis of the multigrid preconditioner for the case when the constraints are formed by the Navier-Stokes equations; the analysis is restricted to the two-dimensional case.

The paper is organized as follows. In Section 2, we introduce the optimal control problem and review results that will be needed in the sequel. In Section 3, we introduce the discrete optimal control problem and prove finite element estimates that will be needed for the multigrid analysis. Section 4 contains the main result of the paper, the analysis of the two-grid preconditioner. In Section 5, we present numerical experiments that illustrate our theoretical results. Conclusions are given in Section 6.

2. Problem formulation.

2.1. Preliminaries. In this section we introduce notations and review some classical results regarding the Navier-Stokes equations. We use standard notation for the Sobolev spaces $H^m(\Omega)$ and for their vector-valued counterparts we use the boldface notation. We denote by $\tilde{\mathbf{H}}^{-m}(\Omega)$ the dual (with respect to $\mathbf{L}^2$-inner product) of $\mathbf{H}^m(\Omega) \cap \mathbf{H}^1_0(\Omega)$ and define $Q = L^2_0(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p\, dx = 0\}$, $X = \mathbf{H}^1_0(\Omega)$, and $V = \{v \in \mathbf{H}^1_0(\Omega) : (\operatorname{div} v, q) = 0,\ \forall q \in Q\}$. Throughout this paper we write $(\cdot,\cdot)$ for the inner product in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$, according to context, if there is no risk of misunderstanding. The $\mathbf{H}^m(\Omega)$ or $H^m(\Omega)$-norm will be denoted by $\|\cdot\|_m$, while $\|\cdot\|$ denotes the $\mathbf{L}^2(\Omega)$ or $L^2(\Omega)$-norm. Furthermore, define the norm in V' by

$$\|u\|_{V'} = \sup_{\phi \in V\setminus\{0\}} (u,\phi)/\|\nabla\phi\|.$$

To define the weak formulation of (2), we introduce the bilinear forms

$$a(y,\phi) = \nu(\nabla y, \nabla\phi) = \nu\sum_{i=1}^{2}\int_\Omega \nabla y_i \cdot \nabla\phi_i\, dx \quad \forall y,\phi \in X, \tag{3}$$

$$b(\phi,p) = -\int_\Omega p \operatorname{div}\phi\, dx \quad \forall \phi \in X, \forall p \in Q, \tag{4}$$

and the trilinear form

$$c(y;\phi,\psi) = ((y\cdot\nabla)\phi,\psi) \quad \forall y,\phi,\psi \in \mathbf{H}^1(\Omega). \tag{5}$$

A weak formulation of the Navier-Stokes equations is given by:

Given $u \in \mathbf{H}^{-1}(\Omega)$, find $(y,p) \in X \times Q$ satisfying

$$\begin{aligned} a(y,\phi) + c(y;y,\phi) + b(\phi,p) &= \langle u,\phi\rangle \quad \forall \phi \in X,\\ b(y,q) &= 0 \qquad\quad\ \forall q \in Q, \end{aligned} \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{H}^1_0(\Omega)$ and $\mathbf{H}^{-1}(\Omega)$. Following [16], the system (6) can be written equivalently as:

Find $y \in V$ that satisfies

$$a(y,\phi) + c(y;y,\phi) = \langle u,\phi\rangle \quad \forall \phi \in V. \tag{7}$$

We recall here a standard result regarding the existence of solution of (6) and uniqueness for small data, see e.g. [8, 16]. For $\mathbf{H}^2$ regularity see [4].

THEOREM 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary. Then for any $\nu > 0$ and $u \in \mathbf{H}^{-1}(\Omega)$ there exists at least one solution $(y, p) \in V \times Q$ of the stationary Navier-Stokes problem (6) that satisfies the estimate*

$$\|\nabla y\| \leq \nu^{-1}\|u\|_{V'}. \tag{8}$$

Moreover, the solution is unique if the data satisfies the smallness condition

$$\mathcal{M}\nu^{-2}\|u\|_{V'} < 1, \text{ with } \mathcal{M} = \sup_{\phi,\psi,\chi \in X\setminus\{0\}} \frac{|c(\phi;\psi,\chi)|}{\|\nabla\phi\|\|\nabla\psi\|\|\nabla\chi\|}. \tag{9}$$

If Ω is convex and polygonal, and $u \in \mathbf{L}^2(\Omega)$, then $y \in \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$ and

$$\|y\|_2 + \|p\|_1 \leq C(1 + \|u\|^3). \tag{10}$$

Throughout this paper we will assume Ω to be convex, so that the $\mathbf{H}^2$-regularity of the Navier-Stokes problem is ensured. We state here some well-known results concerning the trilinear form defined in (5), that will be needed in the sequel [3, 8, 13].

LEMMA 2. *The trilinear form $c(y;\phi,\psi)$ defined in (5) has the following properties:*

$$\begin{aligned}
&c(y;\phi,\psi) = -c(y;\psi,\phi) \quad \forall y \in V, \forall\phi,\psi \in \mathbf{H}^1(\Omega),\\
&c(y;\phi,\phi) = 0 \quad \forall y \in V, \phi \in \mathbf{H}^1(\Omega),\\
&c(y;\phi,\psi) = ((\nabla\phi)^T\psi, y) \quad \forall y,\phi,\psi \in \mathbf{H}^1(\Omega),\\
&|c(y;\phi,\psi)| \leq \|y\|_1\|\phi\|_1\|\psi\|_1 \quad \forall y,\phi,\psi \in V,\\
&|c(y;\phi,\psi)| \leq \mathcal{M}\|\nabla y\|\|\nabla\phi\|\|\nabla\psi\| \quad \forall y,\phi,\psi \in X,\\
&|c(y;\phi,\psi)| \leq C\|u\|_1\|\phi\|_1\|\psi\|_1 \quad \forall y,\phi,\psi \in \mathbf{H}^1(\Omega),\\
&|c(y;\phi,\psi)| \leq C\|y\|\|\phi\|_2\|\psi\|_1 \quad \forall y,\psi \in X, \phi \in \mathbf{H}^2(\Omega),\\
&|c(y;\phi,\psi)| \leq C\|y\|_1\|\phi\|_2\|\psi\| \quad \forall y,\psi \in X, \phi \in \mathbf{H}^2(\Omega),
\end{aligned} \tag{11}$$

with $\mathcal{M}$ given in (9) and C independent of y, ϕ, ψ.

Proof. While the others are standard, we prove here only the last estimate. Using Hölder's inequality and the embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$, we have

$$|c(y;\phi,\psi)| = |((y\cdot\nabla)\phi,\psi)| \leq \|y\|_{L^4(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}\|\psi\|_{L^2(\Omega)} \leq C\|y\|_1\|\phi\|_2\|\psi\|. \qquad \square$$

When discretizing (6) using finite elements, in order to preserve the antisymmetry in the last two arguments of the trilinear form c on the finite element spaces, it is standard to introduce a modified trilinear form [12, 16]

$$\tilde{c}(y;\phi,\psi) = \frac{1}{2}\{c(y;\phi,\psi) - c(y;\psi,\phi)\} \quad \forall y,\phi,\psi \in X, \tag{12}$$

that has the following properties:

$$\begin{aligned}
&c(y;\phi,\psi) = \tilde{c}(y;\phi,\psi) \quad \forall y \in V, \phi,\psi \in X,\\
&\tilde{c}(y;\phi,\psi) = -\tilde{c}(y;\psi,\phi) \quad \forall y,\phi,\psi \in X,\\
&\tilde{c}(y;\psi,\psi) = 0 \quad \forall y,\phi \in X,\\
&|\tilde{c}(y;\phi,\psi)| \leq \mathcal{M}\|\nabla y\|\|\nabla\phi\|\|\nabla\psi\| \quad \forall y,\phi,\psi \in X,
\end{aligned} \tag{13}$$

for the same $\mathcal{M} = \mathcal{M}(\Omega)$ as in (9). Thus, another variational formulation of (6) is:

Given $u \in \mathbf{H}^{-1}(\Omega)$*, find* $(y,p) \in X \times Q$ *satisfying*

$$\begin{aligned} a(y,\phi) + \tilde{c}(y;y,\phi) + b(\phi,p) &= \langle u,\phi\rangle \quad \forall \phi \in X, \\ b(y,q) &= 0 \qquad\quad \forall q \in Q. \end{aligned} \tag{14}$$

We define the set of admissible controls $U = \{u : \mathbf{L}^2(\Omega) : \|u\| < \nu^2/(\mathcal{M}\kappa)\}$, with $\mathcal{M}$ defined in (9) and κ the embedding constant of $\mathbf{L}^2(\Omega)$ into V'. By Theorem 1, the Navier-Stokes equations have a unique solution for each $u \in U$ on the right hand side of (6). We introduce the control-to-state operators $Y : U \to V$, $P : U \to Q$ that assign to each $u \in U \subset \mathbf{L}^2(\Omega)$ the corresponding Navier-Stokes velocity $y = Y(u)$ and pressure $p = P(u)$, and rewrite problem (1) in reduced form as

$$\min_{u \in U} \hat{J}(u) = \frac{\gamma_y}{2}\|Y(u) - y_d\|^2 + \frac{\gamma_p}{2}\|P(u) - p_d\|^2 + \frac{\beta}{2}\|u\|^2. \tag{15}$$

Throughout this paper we will assume that the target velocity field y_d is from $\mathbf{H}^1(\Omega)$ and the target pressure p_d is from Q.

We note that for all pairs $(y(u), u)$ with $u \in U$, we have

$$\nu > \mathcal{M}(y), \quad \text{with } \mathcal{M}(y) := \sup_{v \in X} \frac{|c(v;y,v)|}{\|\nabla v\|^2}, \tag{16}$$

which ensures the ellipticity of the linearized equations about y.

Lemma 3. *Let* $u \in U$ *and* $y = Y(u) \in V$*. Then for every* $g \in V'$ *there exists a unique weak solution* $(w,r) \in X \times Q$ *of the linearized Navier-Stokes system*

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)y + (y\cdot\nabla)w + \nabla r &= g \quad \text{in } \Omega, \\ \operatorname{div} w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{17}$$

and

$$\|\nabla w\| \le \frac{2}{\nu}\|g\|_{V'}. \tag{18}$$

If Ω *is a convex and polygonal, and* $g \in \mathbf{L}^2(\Omega)$*, then* $w \in \mathbf{H}^2(\Omega)$*,* $r \in H^1(\Omega)$*, and*

$$\|w\|_2 \le C(y)\|g\|. \tag{19}$$

Proof. Existence and uniqueness follows from the Lax-Milgram lemma, using (16) to prove the ellipticity of the associated bilinear form. For the proof of (18) see [19], Corollary 3.7. To prove (19), we note that for $g \in \mathbf{L}^2(\Omega)$, we have $(w\cdot\nabla)y$, $(y\cdot\nabla)w$ $\in \mathbf{L}^2(\Omega)$ (see estimates below); thus by rewriting (17) as

$$-\nu\Delta w + \nabla r = g - (w\cdot\nabla)y - (y\cdot\nabla)w,$$

we can use standard regularity results for the Stokes equations to obtain

$$\|\nabla\nabla w\| \le C_1(\Omega)(\|g\| + \|(w\cdot\nabla)y\| + \|(y\cdot\nabla)w\|). \tag{20}$$

We have

$$\|(w\cdot\nabla)y\|^2 = \int_\Omega |(w\cdot\nabla)y|^2 dx \le \int_\Omega |w|^2|\nabla y|^2 dx \le \|w\|^2_{\mathbf{L}^4(\Omega)}\|\nabla y\|^2_{\mathbf{L}^4(\Omega)},$$

which implies

$$\|(w\cdot\nabla)y\| \le C\|w\|_1\|\nabla y\|_1 \le C_1(y)\|g\|, \tag{21}$$

since $\mathbf{H}^1(\Omega)\hookrightarrow \mathbf{L}^4(\Omega)$. Similarly, it can be shown that

$$\|(y\cdot\nabla)w\| \le C\|y\|_1\|\nabla w\|_{\mathbf{L}^4(\Omega)} \le C_2(y)\|\nabla w\|^{1/2}\|\nabla\nabla w\|^{1/2},$$

where we used Ladyzhenskaya's inequality,

$$\|\nabla w\|_{\mathbf{L}^4(\Omega)} \le C\|\nabla w\|^{1/2}\|\nabla\nabla w\|^{1/2}.$$

Finally, using Young's inequality we obtain

$$\begin{aligned}\|(y\cdot\nabla)w\| &\le C_2(y)\Big(\frac{1}{2}C_2(y)C_1(\Omega)\|\nabla w\| + \frac{1}{2C_2(y)C_1(\Omega)}\|\nabla\nabla w\|\Big)\\ &= \frac{1}{2}C_2^2(y)C_1(\Omega)\|\nabla w\| + \frac{1}{2C_1(\Omega)}\|\nabla\nabla w\|.\end{aligned}$$

Substituting in (20) gives

$$\|\nabla\nabla w\| \le C_1(\Omega)\Big(\|g\| + C_1(y)\|g\| + \frac{1}{2}C_2^2(y)C_1(\Omega)\|\nabla w\| + \frac{1}{2C_1(\Omega)}\|\nabla\nabla w\|\Big),$$

from which (19) follows immediately. ∎

We recall here the following results from [5] regarding the differentiability of the solution operators Y, P.

THEOREM 4. *Let $u\in U$ and $y=Y(u)$. The control-to-state operators Y, P are twice Fréchet differentiable at u and their derivatives $w=Y'(u)g$, $r=P'(u)g$ and $\lambda=Y''(u)[g_1,g_2]$, $\mu=P''(u)[g_1,g_2]$ are given by the unique weak solutions of the systems:*

$$\begin{aligned} -\nu\Delta w + (w\cdot\nabla)y + (y\cdot\nabla)w + \nabla r &= g \quad \text{in } \Omega,\\ \operatorname{div} w &= 0 \quad \text{in } \Omega,\\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{22}$$

and

$$\begin{aligned} -\nu\Delta\lambda + (y\cdot\nabla)\lambda + (\lambda\cdot\nabla)y + \nabla\mu = -(Y'(u)g_1\cdot\nabla)Y'(u)g_2 &\\ - (Y'(u)g_2\cdot\nabla)Y'(u)g_1 &\quad \text{in } \Omega,\\ \operatorname{div}\lambda = 0 &\quad \text{in } \Omega,\\ \lambda = 0 &\quad \text{on } \partial\Omega, \end{aligned} \tag{23}$$

respectively.

LEMMA 5. *Let $u\in U$, $y=Y(u)$, and $Y'(u)^*$ be the adjoint of $Y'(u)$. Then $z=Y'(u)^*g$ is the first component of the unique weak solution (z,ρ) of the system*

$$\begin{aligned} -\nu\Delta z - (y\cdot\nabla)z + (\nabla y)^T z + \nabla\rho &= g \quad \text{in } \Omega,\\ \operatorname{div} z &= 0 \quad \text{in } \Omega,\\ z &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{24}$$

If $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain then $z\in\mathbf{H}^2(\Omega)$, $\rho\in H^1(\Omega)$ and

$$\|z\|_2 \le C(y)\|g\|. \tag{25}$$

Proof. See [19, Theorem 3.10] and Lemma 3. ∎

2.2. Optimality conditions. We derive next the first-order necessary optimality conditions associated with the optimal control problem (15). For $g \in \mathbf{L}^2(\Omega)$,

$$\hat{J}'(u)g = \gamma_y(Y(u) - y_d, Y'(u)g) + \gamma_p(P(u) - p_d, P'(u)g) + \beta(u, g),$$

therefore

$$\nabla \hat{J}(u) = \gamma_y Y'(u)^*(Y(u) - y_d) + \gamma_p P'(u)^*(P(u) - p_d) + \beta u. \tag{26}$$

Thus, the optimal control u is the solution of the non-linear equation

$$\gamma_y Y'(u)^*(Y(u) - y_d) + \gamma_p P'(u)^*(P(u) - p_d) + \beta u = 0. \tag{27}$$

The reduced Hessian is computed using the second variation of $\hat{J}$: if $g_1, g_2 \in \mathbf{L}^2(\Omega)$

$$\begin{aligned}\hat{J}''(u)[g_1, g_2] &= \gamma_y(Y'(u)g_2, Y'(u)g_1) + \gamma_y(Y(u) - y_d, Y''(u)[g_2, g_1]) \\ &\quad + \gamma_p(P'(u)g_2, P'(u)g_1) + \gamma_p(P(u) - p_d, P''(u)[g_2, g_1]) + \beta(g_1, g_2).\end{aligned} \tag{28}$$

We use different approaches in proving the main multigrid results, depending on whether the pressure term is present in the cost functional (1) or not, therefore we will derive the reduced Hessian for the two cases separately.

2.2.1. Velocity control only. We consider first the case of velocity control only, i.e., $\gamma_y = 1, \gamma_p = 0$. In this case the second variation of $\hat{J}$ becomes

$$\hat{J}''(u)[g_1, g_2] = (Y'(u)g_2, Y'(u)g_1) + (Y(u) - y_d, Y''(u)[g_2, g_1]) + \beta(g_1, g_2). \tag{29}$$

We denote by L and M the solution operators of (22), such that $Lg = Y'(u)g$, $Mg = P'(u)g$. Although L, M depend on $y = y(u)$ in (22), we use the notation L, M instead of $L(u)$, $M(u)$, for simplicity, when there is no risk of misunderstanding. Cf. Theorem 4, $\lambda = Y''(u)[g_1, g_2]$ is the solution of

$$\begin{aligned} a(\lambda, \phi) + c(y; \lambda, \phi) + c(\lambda; y, \phi) + b(\phi, \mu)& \\ = -c(Lg_1; Lg_2; \phi) - c(Lg_2; Lg_1, \phi) &\quad \forall \phi \in X, \\ b(\lambda, q) = 0 &\quad \forall q \in Q. \end{aligned} \tag{30}$$

Similarly, we let $z = L^*(Y(u) - y_d)$. Note that is the solution of

$$\begin{aligned} a(z, \phi) + c(y; \phi, z) + c(\phi; y, z) + b(\phi, \rho) = (y - y_d, \phi) &\quad \forall \phi \in X, \\ b(z, q) = 0 &\quad \forall q \in Q. \end{aligned} \tag{31}$$

By taking $\phi = z$ in (30) and $\phi = \lambda$ in (31) we obtain

$$-c(Lg_1; Lg_2; z) - c(Lg_2; Lg_1, z) = (Y(u) - y_d, \lambda). \tag{32}$$

Using this in (29) we get

$$\begin{aligned}\hat{J}''(u)[g_1, g_2] &= (Lg_1, Lg_2) - c(Lg_1; Lg_2, z) - c(Lg_2; Lg_1, z) + \beta(g_1, g_2) \\ &= (Lg_1, Lg_2) + ((Lg_1 \cdot \nabla)z, Lg_2) - ((\nabla Lg_1)^T z, Lg_2) + \beta(g_1, g_2).\end{aligned}$$

The Hessian operator associated with $\hat{J}$, defined by $(H_\beta(u)v, g) = \hat{J}''(u)[v, g]$, is

$$H_\beta(u)v = \beta v + L^*Lv + L^*((Lv \cdot \nabla) - (\nabla Lv)^T)L^*(y - y_d). \tag{33}$$

To simplify the presentation we introduce the notation

$$A(u)v = L^*Lv, \quad \mathcal{C}(u)v = L^*((Lv \cdot \nabla) - (\nabla Lv)^T)L^*(Y(u) - y_d),$$

that we will use throughout the paper. Note that

$$(\mathcal{C}(u)v, v) = -2c(Lv; Lv, L^*(Y(u) - y_d)). \tag{34}$$

2.2.2. Mixed/pressure control. Here we consider the general case of mixed velocity/pressure control or pressure control only, i.e, $\gamma_p \neq 0$.

Let $(\tilde{z}, \tilde{\rho})$ be the solution of the problem

$$\begin{aligned} a(\tilde{z}, \phi) + c(y; \phi, \tilde{z}) + c(\phi; y, \tilde{z}) + b(\phi, \tilde{\rho}) &= \gamma_y (y - y_d, \phi) \quad \forall \phi \in X, \\ b(\tilde{z}, q) &= \gamma_p (p - p_d, q) \quad \forall q \in Q, \end{aligned} \tag{35}$$

which is the weak form of the problem

$$\begin{aligned} -\nu \Delta \tilde{z} - (y \cdot \nabla)\tilde{z} + (\nabla y)^T \tilde{z} + \nabla \tilde{\rho} &= \gamma_y (y - y_d) \quad \text{in } \Omega, \\ \operatorname{div} \tilde{z} &= \gamma_p (p_d - p) \quad \text{in } \Omega, \\ \tilde{z} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{36}$$

By taking $\phi = \lambda$ in (35), $\phi = \tilde{z}$ in (30), and using $b(\lambda, \tilde{\rho}) = 0$, $b(\tilde{z}, \mu) = \gamma_p (p - p_d, \mu)$ we obtain

$$\gamma_y (y - y_d, \lambda) + \gamma_p (p - p_d, \mu) = -c(Lg_1; Lg_2, \tilde{z}) - c(Lg_2; Lg_1, \tilde{z}).$$

Thus, the second variation of the reduced cost functional (28) becomes

$$\begin{aligned} \hat{J}''(u)[g_1, g_2] = \gamma_y (Y'(u)g_2, Y'(u)g_1) + \gamma_p (P'(u)g_2, P'(u)g_1) + \beta(g_1, g_2) \\ - c(Lg_1; Lg_2, \tilde{z}) - c(Lg_2; Lg_1, \tilde{z}) + \beta(g_1, g_2) \end{aligned} \tag{37}$$

and the reduced Hessian is given by

$$H_\beta(u)v = \beta v + \gamma_y L^* L v + \gamma_p M^* M v + L^*((Lv \cdot \nabla)\tilde{z} - (\nabla L v)^T \tilde{z}). \tag{38}$$

We introduce the notation

$$\tilde{\mathcal{C}}(u)v = L^*((Lv \cdot \nabla)\tilde{z} - (\nabla L v)^T \tilde{z}) \tag{39}$$

and note that

$$(\tilde{\mathcal{C}}(u)v, v) = -2c(Lv; Lv, \tilde{z}). \tag{40}$$

Note that if we take $\gamma_y = 1$, $\gamma_p = 0$ in (35), then (35) is the adjoint linearized Navier-Stokes system and in this case (38) reduces to (29).

3. Discretization and approximation results. The strategy we adopt is to first discretize the Navier-Stokes system, then optimize the cost functional J in (1) subject to the resulting discrete constraints.

3.1. Finite element approximation. In this section we collect several approximation results pertaining to the finite element approximation of the Navier-Stokes equations and the linearized/adjoint linearized Navier-Stokes equations.

We consider a shape regular quasi-uniform quadrilateral mesh $\mathscr{T}_h$ of $\bar{\Omega}$, and we assume that the mesh $\mathscr{T}_h$ results from a coarser regular mesh $\mathscr{T}_{2h}$ from one uniform refinement. We use the Taylor-Hood $\mathbf{Q}_2 - \mathbf{Q}_1$ finite elements to discretize the state equation. The velocity field y is approximated in the space $X_h^0 = X_h \cap \mathbf{H}_0^1(\Omega)$, where

$$X_h = \{v_h \in C(\bar{\Omega})^2 : v_h|_T \in \mathbf{Q}_2(T)^2 \text{ for } T \in \mathscr{T}_h\}$$

and the pressure p is approximated in the space

$$Q_h = \{q_h \in C(\Omega) \cap L_0^2(\Omega) : q_h|_T \in \mathbf{Q}_1(T) \text{ for } T \in \mathscr{T}_h\},$$

where $\mathbf{Q}_k(T)$ is the space of polynomials of degree less than or equal to k in each variable [2]. The control variable u is approximated by continuous piecewise biquadratic polynomial vector functions from X_h. We also introduce the space

$$V_h = \{v_h \in X_h^0 : (\operatorname{div} v_h, q_h) = 0 \ \forall q_h \in Q_h\} \tag{41}$$

and note that $V_h \nsubseteq V$.

REMARK 1. *The choice to work with quadrilateral $\mathbf{Q}_2 - \mathbf{Q}_1$ Taylor-Hood elements was made for convenience and clarity of exposition; our analysis can be extended to triangular $\mathbf{P}_2 - \mathbf{P}_1$ elements as well as other stable mixed finite elements.*

For a given control $u_h \in X_h \cap U$, the solution (y_h, p_h) of the discrete state equation is given by

$$\begin{aligned} a(y_h, \phi_h) + \tilde{c}(y_h; y_h, \phi_h) + b(\phi_h, p_h) &= (u_h, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(y_h, q_h) &= 0 \qquad\qquad \forall q_h \in Q_h. \end{aligned} \tag{42}$$

Let Y_h and P_h be the solution mappings of the discretized state equation, defined analogously to their continuous counterparts. The discretized, reduced optimal control problem reads

$$\min_{u_h} \hat{J}_h(u_h) = \frac{\gamma_y}{2}\|Y_h(u_h) - y_d^h\|^2 + \frac{\gamma_p}{2}\|P_h(u_h) - p_d^h\|^2 + \frac{\beta}{2}\|u_h\|^2, \tag{43}$$

where y_d^h, p_d^h are the L^2-projections of the data onto X_h, respectively Q_h.

We denote by L_h, M_h the solution operators of the discretized linearized Navier-Stokes equations (about y_h), i.e., $L_h g = w_h$, $M_h g = r_h$, where

$$\begin{aligned} a(w_h, \phi_h) + \tilde{c}(y_h; w_h, \phi_h) + \tilde{c}(w_h; y_h, \phi_h) + b(\phi_h, r_h) \\ = (g, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(w_h, q_h) = 0 \quad \forall q_h \in Q_h, \end{aligned} \tag{44}$$

We remark that, as in the continuous case, $z_h = L_h^* g$ satisfies

$$\begin{aligned} a(z_h, \phi_h) + \tilde{c}(y_h; \phi_h, z_h) + \tilde{c}(\phi_h; y_h, z_h) + b(\phi_h, \rho_h) \\ = (g, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(z_h, q_h) = 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{45}$$

3.2. A priori estimates.

LEMMA 6. *Let π_h be the L^2-orthogonal projection onto X_h. The following approximation properties hold:*

$$\|(I - \pi_h)v\|_{\tilde{\mathbf{H}}^{-k}(\Omega)} \le Ch^k \|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \quad k = 1, 2, \tag{46}$$

$$\|(I - \pi_h)u\|_{\tilde{\mathbf{H}}^{-1}(\Omega)} \le Ch^2 \|u\|_1 \quad \forall u \in \mathbf{H}^1(\Omega), \tag{47}$$

with C independent of h.

Proof. The estimate (46) is a standard result (e.g., see [6]). For (47), let $I_h : \mathbf{H}^1(\Omega) \to X_h$ be the interpolant introduced by Scott and Zhang in [18]. We have

$$\begin{aligned} \|u - \pi_h u\|_{\tilde{\mathbf{H}}^{-1}(\Omega)} &= \sup_{v \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{(u - \pi_h u, v)}{\|v\|_1} = \sup_{v \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{(u - \pi_h u, v - I_h v)}{\|v\|_1} \\ &\le \sup_{v \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{\|u - \pi_h u\|\|v - I_h v\|}{\|v\|_1} \le Ch\|u - \pi_h u\|, \end{aligned}$$

where we have used $\|v - I_h v\| \le Ch\|v\|_1$ (see [18, (4.6)]). Moreover,

$$\|u - \pi_h u\| \le \|u - I_h u\| \le ch\|u\|_1,$$

which combined with the previous estimate leads to (47). ∎

THEOREM 7. *Let $u \in U$ and $y = Y(u) \in V \cap \mathbf{H}^2(\Omega)$ (so that $\nu > \mathcal{M}(y)$), and L, M be the velocity/pressure operators of the linearized Navier-Stokes equations about y, and L_h, M_h their discrete counterparts. There exists constants C, $C_1 = C_1(y)$, $C_2 = C_2(y)$, and $C_3 = C_3(y)$ such that the following hold:*

(a) *smoothing:*

$$\|Lv\| \le C_1\|v\|_{\tilde{\mathbf{H}}^{-2}(\Omega)} \quad \forall v \in \mathbf{L}^2(\Omega), \tag{48}$$

$$\|Mv\| \le C_2\|v\|_{\tilde{\mathbf{H}}^{-1}(\Omega)} \quad \forall v \in \mathbf{L}^2(\Omega). \tag{49}$$

(b) *approximation:*

$$\|Y(u) - Y_h(u)\| \le Ch^2\|u\| \quad \forall u \in U, \tag{50}$$

$$\|Lv - L_h v\|_1 \le C_1 h\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{51}$$

$$\|Lv - L_h v\| \le C_1 h^2\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{52}$$

$$\|Mv - M_h v\| \le C_2 h\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{53}$$

$$\|L^* v - L_h^* v\|_1 \le C_3 h\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{54}$$

$$\|L^* v - L_h^* v\| \le C_3 h^2\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{55}$$

(c) *stability:*

$$\|Y_h(u)\| \le C\|u\| \quad \forall u \in U, \tag{56}$$

$$\|L_h v\| \le C_1\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{57}$$

$$\|M_h v\| \le C_2\|v\| \quad \forall v \in \mathbf{L}^2(\Omega), \tag{58}$$

$$\|L_h^* v\| \le C_3\|v\| \quad \forall v \in \mathbf{L}^2(\Omega). \tag{59}$$

Proof. The statement at (a) is similar to the case of the Stokes problem [7]. For (50) in (b) see [9], page 32, and for (51)–(55) see [11]. The stability in (c) follows from (8), (a), and (b). ∎

REMARK 2. *Theorem 7 and Lemma 6 imply that there is a constant $C > 0$ independent of h such that*

$$\|L(I - \pi_h)v\| \le Ch^2\|v\| \quad \forall v \in \mathbf{L}^2(\Omega) \tag{60}$$

and

$$\|M(I - \pi_h)v\| \le Ch\|v\| \quad \forall v \in \mathbf{L}^2(\Omega). \tag{61}$$

For a polygonal domain $\Omega \subset \mathbb{R}^2$, the weighted Sobolev space $W_0^{1,0}(\Omega)$ is defined to be the class of functions for which the following norm is finite:

$$\|w\|^2_{W_0^{1,0}(\Omega)} = \int_\Omega |\nabla w|^2 dx + \int_\Omega \delta(x)^{-2}|w|^2 dx,$$

where $\delta(x) = \min\{\text{dist}(x, P) : P \text{ a vertex of } \Omega\}$.

THEOREM 8. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, $u \in U$, $y = Y(u) \in V$ and $f \in \mathbf{L}^2(\Omega)$, $g \in W_0^{1,0}(\Omega)$, $\int_\Omega g dx = 0$. Furthermore, let $\tilde{z} = \tilde{L}(f,g)$, $\tilde{\rho} = \tilde{M}(f,g)$ be the weak solution of*

$$\begin{aligned} -\nu\Delta\tilde{z} - (y\cdot\nabla)\tilde{z} + (\nabla y)^T\tilde{z} + \nabla\tilde{\rho} &= f \quad \text{in } \Omega, \\ \operatorname{div}\tilde{z} &= g \quad \text{in } \Omega, \\ \tilde{z} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{62}$$

Then $\tilde{z} \in \mathbf{H}^2(\Omega)$, $\tilde{\rho} \in H^1(\Omega)$ and there exists a constant $C = C(\Omega, y) > 0$ such that

$$\|\tilde{z}\|_{\mathbf{H}^2(\Omega)} + \|\nabla\tilde{\rho}\| \le C(\|f\|_{\mathbf{L}^2(\Omega)} + \|g\|_{W_0^{1,0}(\Omega)}). \tag{63}$$

Moreover, if $\tilde{z}_h$ is the velocity of the corresponding discrete problem, then

$$\|\tilde{z} - \tilde{z}_h\|_1 \le Ch(\|f\|_{\mathbf{L}_2(\Omega)} + \|g\|_{W_0^{1,0}(\Omega)}), \quad \|\tilde{z}_h\|_1 \le C(\|f\|_{\mathbf{L}_2(\Omega)} + \|g\|_{W_0^{1,0}(\Omega)}). \tag{64}$$

Proof. The existence of a unique solution $(\tilde{z}, \tilde{\rho}) \in X \times Q$ of (62) and the estimate

$$\|\tilde{z}\|_{\mathbf{H}^1(\Omega)} + \|\tilde{\rho}\| \le C(\|f\|_{-1} + \|g\|), \tag{65}$$

follow from standard results for saddle point problems [1]. In [14], it is shown that under the hypotheses of the theorem, the solution of the generalized Stokes system

$$\begin{aligned} -\nu\Delta z + \nabla\rho &= f \quad \text{in } \Omega, \\ \operatorname{div} z &= g \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

satisfies $z \in \mathbf{H}^2(\Omega)$, $\rho \in H^1(\Omega)$ and

$$\|z\|_{\mathbf{H}^2(\Omega)} + \|\nabla\rho\| \le C(\|f\| + \|g\|_{W^0_{0,1}}).$$

Using this result together with (65), it is straightforward to show (63) using the same approach as in Lemma 3. For finite element spaces X_h, Q_h that satisfy the inf-sup condition, we have

$$\|\tilde{z} - \tilde{z}_h\|_{\mathbf{H}^1(\Omega)} + \|\tilde{\rho} - \rho_h\| \le C(\inf_{\phi_h \in X_h} \|\tilde{z} - \phi_h\|_{\mathbf{H}^1(\Omega)} + \inf_{q_h \in Q_h} \|\tilde{\rho} - q_h\|),$$

which combined with interpolation estimates yields (64). ∎

4. Two-grid preconditioner. We begin with the description of the discrete Hessian in Section 4.1, followed by the construction and analysis of the two-grid preconditioner in Section 4.2. The velocity control and mixed/pressure control are treated separately, as the form of the Hessian differs significantly in the two cases.

4.1. The discrete Hessian. As in the continuous case, the discrete Hessian operator at $u \in U \cap X_h$ is defined by the equality

$$(H^h_\beta(u)v, g) = \hat{J}''_h(u)[v, g], \quad \forall v, g \in X_h. \tag{66}$$

4.1.1. Velocity control. As in the continuous case, when $\gamma_p = 0$ we have

$$\nabla \hat{J}_h(u) = Y_h'(u)^*(Y_h(u) - y_d^h) + \beta u, \quad u \in U \cap X_h, \tag{67}$$

with the second variation of the discrete cost functional being given by

$$\hat{J}_h''(u)[g_1, g_2] = (Y_h'(u)g_2, Y_h'(u)g_1) + (Y_h(u) - y_d^h, Y_h''(u)[g_2, g_1]) + \beta(g_1, g_2). \tag{68}$$

The second variation $\lambda_h = Y_h''(u)[g_1, g_2] \in X_h^0$ is the solution of

$$\begin{aligned} a(\lambda_h, \phi_h) &+ \tilde{c}(y_h; \lambda_h, \phi_h) + \tilde{c}(\lambda_h; y_h, \phi_h) + b(\phi_h, \mu_h) \\ &= -\tilde{c}(Y_h'(u)g_1; Y_h'(u)g_2, \phi_h) - \tilde{c}(Y_h'(u)g_2; Y_h'(u)g_1, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(\lambda_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{69}$$

The discrete adjoint $z_h = Y_h'(u)^*(y_h - y_d^h) = L_h^*(Y_h(u) - y_d^h)$ is the solution of

$$\begin{aligned} a(z_h, \phi_h) &+ \tilde{c}(y_h; \phi_h, z_h) + \tilde{c}(\phi_h; y_h, z_h) + b(\phi_h, \rho_h) \\ &= (y_h - y_d^h, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(z_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{70}$$

Using the same approach as in the continuous case, we obtain

$$-\tilde{c}(L_h g_1; L_h g_2, z_h) - \tilde{c}(L_h g_2; L_h g_1, z_h) = (y_h - y_d^h, \lambda_h)$$

and

$$\hat{J}_h''(u)[g_1, g_2] = (L_h g_1, L_h g_2) - \tilde{c}(L_h g_1; L_h g_2, z_h) - \tilde{c}(L_h g_2; L_h g_1, z_h) + \beta(g_1, g_2).$$

Hence, the discrete Hessian is given by

$$H_\beta^h(u)v = \beta v + L_h^* L_h v + \mathcal{C}_h(u)v = \beta v + A_h(u)v + \mathcal{C}_h(u)v, \tag{71}$$

where

$$(\mathcal{C}_h(u)v, v) = -2\tilde{c}(L_h v; L_h v, z_h). \tag{72}$$

4.1.2. Mixed/pressure control. Similarly with the derivation in Section 2.2.2, in the case of mixed/pressure control, the discrete Hessian takes the form

$$H_\beta^h(u)v = \beta v + \gamma_y L_h^* L_h + \gamma_p M_h^* M_h + \tilde{\mathcal{C}}_h(u)v, \tag{73}$$

where

$$(\tilde{\mathcal{C}}_h(u)v, v) = -2\tilde{c}(L_h v; L_h v, \tilde{z}_h) \tag{74}$$

and $\tilde{z}_h$ is the solution of the discrete problem (35).

4.2. Two-grid preconditioner for discrete Hessian. In this section, we construct and analyze a two-grid preconditioner for the discrete Hessian $H_\beta^h(u)$ defined in (71) and (73). The construction is a natural extension of the technique used for the optimal control of the Stokes equations in [7], and is the same for both velocity- and mixed/pressure control. Let $X_h = X_{2h} \oplus W_{2h}$ be the L^2-orthogonal decomposition, where we consider on X_h the Hilbert-space structure inhertied from $\mathbf{L}^2(\Omega)$. Let π_{2h} be the L^2-projector onto X_{2h}. For $u \in U \cap X_h$ we define the two-grid preconditioner

$$T_\beta^h(u) = H_\beta^{2h}(\pi_{2h}u)\pi_{2h} + \beta(I - \pi_{2h}). \tag{75}$$

It is worth noting that

$$(T_\beta^h(u))^{-1} = (H_\beta^{2h}(\pi_{2h}u))^{-1}\pi_{2h} + \beta^{-1}(I - \pi_{2h}). \tag{76}$$

We should remark that the difference between the preconditioner in (75) and the one in [7] is given by the dependence of the Hessian on the control u, which forces us to choose a coarse-level control $u_c \in X_{2h}$ at which the coarse Hessian $H_\beta^{2h}(u_c)$ in (75) is computed. The natural choice is $u_c = \pi_{2h}u$.

4.2.1. Analysis for the case of velocity control. Cf. (71) and (75),

$$T_\beta^h(u) = (\beta I + A_{2h}(\pi_{2h}u) + \mathcal{C}_{2h}(\pi_{2h}u))\pi_{2h} + \beta(I - \pi_{2h}). \tag{77}$$

LEMMA 9. *Let $u \in U \cap X_h$ and $y = Y(u)$, $p = P(u)$, $\bar{p} = P(\pi_{2h}u)$, $\bar{y} = Y(\pi_{2h}u)$. Also, let $v \in X_h$ and $w = L(u)v$, $q = M(u)v$, $\bar{w} = L(\pi_{2h}u)v$, $\bar{q} = M(\pi_{2h}u)v$. Then there exists a constant $K = K(u, \nu, \Omega) > 0$ such that*

$$|y - \bar{y}|_1 \le K\|u - \pi_{2h}u\|_{\tilde{\mathbf{H}}^{-1}(\Omega)}, \tag{78}$$

$$\|p - \bar{p}\| \le Kh^2\|u\|_1, \tag{79}$$

and a constant C independent of h such that

$$\|w - \bar{w}\|_1 \le Ch^2\|u\|_1\|v\|, \tag{80}$$

$$\|q - \bar{q}\| \le Ch^2\|u\|_1\|v\|. \tag{81}$$

Proof. Since y and $\bar{y}$ are the solutions of the Navier-Stokes equations with forcing u, $\pi_{2h}u$, respectively, we have

$$a(y, \phi) + c(y; y, \phi) = (u, \phi) \quad \forall \phi \in V,$$
$$a(\bar{y}, \phi) + c(\bar{y}; \bar{y}, \phi) = (\pi_{2h}u, \phi) \quad \forall \phi \in V.$$

By taking $\phi = y - \bar{y}$ and subtracting the equations we obtain

$$a(y - \bar{y}, y - \bar{y}) + c(y - \bar{y}; y, y - \bar{y}) + c(\bar{y}; y - \bar{y}, y - \bar{y}) = (u - \pi_{2h}u, y - \bar{y}).$$

Given that $c(\bar{y}; y - \bar{y}, y - \bar{y}) = 0$, we obtain

$$\begin{aligned} \nu|y - \bar{y}|_1^2 &= (u - \pi_{2h}u, y - \bar{y}) - c(y - \bar{y}; y, y - \bar{y}) \\ &\le \|u - \pi_{2h}u\|_{\tilde{\mathbf{H}}^{-1}}\|y - \bar{y}\|_1 + \mathcal{M}(y)|y - \bar{y}|_1^2. \end{aligned}$$

Since $\mathcal{M}(y)) < \nu$ and $y, \bar{y} \in X = \mathbf{H}_0^1(\Omega)$, we get

$$(\nu - \mathcal{M}(y))|y - \bar{y}|_1^2 \le \|u - \pi_{2h}u\|_{\tilde{\mathbf{H}}^{-1}}\|y - \bar{y}\|_1 \le C\|u - \pi_{2h}u\|_{\tilde{\mathbf{H}}^{-1}}|y - \bar{y}|_1,$$

which implies (78). From the weak formulations of the Navier-Stokes equations in X, with forcing u, $\pi_{2h}u$ respectively, we have

$$b(\phi, p - \bar{p}) = (u - \pi_{2h}u, \phi) - a(y - \bar{y}, \phi) + c(\bar{y}; \bar{y}, \phi) - c(y; y, \phi).$$

Thus for $\phi \in X$

$$\begin{aligned}|b(\phi, p-\bar{p})| &\leq \|u-\pi_{2h}u\|_{-1}\|\phi\|_1 + \nu\|y-\bar{y}\|_1\|\phi\|_1 + |c(\bar{y}; y-\bar{y}, \phi) + c(\bar{y}-y; y, \phi)| \\ &\leq \|u-\pi_{2h}u\|_{-1}\|\phi\|_1 + \nu\|y-\bar{y}\|_1\|\phi\|_1 + \|y-\bar{y}\|_1\|\phi\|_1(\|\bar{y}\|_1 + \|y\|_1).\end{aligned}$$

Then, from the inf-sup condition

$$\beta^*\|q-\bar{q}\| \leq \sup_{0\neq\phi\in X} \frac{|b(q-\bar{q},\phi)|}{\|\nabla\phi\|}, \tag{82}$$

combined with (78), (47), we obtain

$$\|p-\bar{p}\| \leq C(\nu, u, \beta^*)h^2\|u\|_1.$$

Recall that (w, q) (resp. $(\bar{w}, \bar{q})$) satisfy the linearized Navier-Stokes equations (22) about y (resp. $\bar{y}$) with with forcing v, whose weak form in V read:

$$a(w, \phi) + c(w; y, \phi) + c(y; w, \phi) = (v, \phi) \quad \forall \phi \in V, \tag{83}$$

$$a(\bar{w}, \phi) + c(\bar{w}; \bar{y}, \phi) + c(\bar{y}; \bar{w}, \phi) = (v, \phi) \quad \forall \phi \in V. \tag{84}$$

By taking $\phi = w - \bar{w}$ in the equations above and subtracting we obtain

$$\begin{aligned}-a(w-\bar{w}, w-\bar{w}) =& c(w; y; w-\bar{w}) + c(y; w, w-\bar{w}) \\ &- c(\bar{w}; \bar{y}, w-\bar{w}) - c(\bar{y}; \bar{w}, w-\bar{w}).\end{aligned} \tag{85}$$

We have

$$\begin{aligned}&c(w; y, w-\bar{w}) - c(\bar{w}; \bar{y}, w-\bar{w}) = c(w; y-\bar{y}, w-\bar{w}) + c(w-\bar{w}; \bar{y}, w-\bar{w}) \\ &c(y; w, w-\bar{w}) - c(\bar{y}; \bar{w}; w-\bar{w}) = c(y-\bar{y}; w, w-\bar{w}),\end{aligned}$$

where we used $c(\bar{y}; w-\bar{w}, w-\bar{w}) = 0$ (see Lemma 2). Using these in (85), we obtain

$$\nu|w-\bar{w}|_1^2 = |c(y-\bar{y}; w; w-\bar{w}) + c(w; y-\bar{y}, w-\bar{w}) + c(w-\bar{w}; \bar{y}; w-\bar{w})|.$$

From the continuity of the trilinear form c and (16) we get

$$\nu|w-\bar{w}|_1^2 \leq \mathcal{M}\left(|y-\bar{y}|_1|w|_1|w-\bar{w}|_1 + |w|_1|y-\bar{y}|_1|w-\bar{w}|_1\right) + \mathcal{M}(\bar{y})|w-\bar{w}|_1^2$$

which leads to

$$(\nu - \mathcal{M}(\bar{y}))|w-\bar{w}|_1^2 \leq 2\mathcal{M}|w|_1|y-\bar{y}|_1|w-\bar{w}|_1.$$

Since $\|\pi_{2h}u\| \leq \|u\|$, $\pi_{2h}u \in U$, and so $\nu - \mathcal{M}(\bar{y}) > 0$; hence we obtain

$$|w-\bar{w}|_1 \leq C|y-\bar{y}|_1|w|_1 \overset{(18),(47),(78)}{\leq} Ch^2\|u\|_1\|v\|. \tag{86}$$

with C depending on ν, y, κ, $\mathcal{M}$, but not on h. To prove (81), we consider the weak formulations of (83) and (84) in X

$$\begin{aligned}&a(w, \phi) + c(w; y, \phi) + c(y; w, \phi) + b(q, \phi) = (v, \phi) \quad \forall \phi \in X, \\ &a(\bar{w}, \phi) + c(\bar{w}; \bar{y}, \phi) + c(\bar{y}; \bar{w}, \phi) + b(\bar{q}, \phi) = (v, \phi) \quad \forall \phi \in X,\end{aligned}$$

from which we obtain

$$\begin{aligned} b(q-\bar{q},\phi) &= -a(w-\bar{w},\phi) - c(w;y,\phi) - c(y;w,\phi) + c(\bar{w};\bar{y},\phi) + c(\bar{y};\bar{w},\phi) \\ &= -a(w-\bar{w},\phi) - c(w;y-\bar{y},\phi) - c(w-\bar{w};\bar{y},\phi) - c(y;w-\bar{w},\phi) \\ &\qquad - c(y-\bar{y};\bar{w},\phi), \quad \forall \phi \in X. \end{aligned}$$

Thus, $\forall \phi \in X$

$$|b(q-\bar{q},\phi)| \le C|\phi|_1 \left(|w-\bar{w}|_1 + |w-\bar{w}|_1(|y|_1+|\bar{y}|_1) + |y-\bar{y}|_1(|w|_1+|\bar{w}|_1)\right).$$

Using the inf-sup condition (82) we obtain

$$\begin{aligned} \|q-\bar{q}\| &\le C(|w-\bar{w}|_1 + |y-\bar{y}|_1(|w|_1+|\bar{w}|_1) + |w-\bar{w}|_1(|y|_1+|\bar{y}|_1)) \\ &\overset{(86)}{\le} C|y-\bar{y}|_1(|\bar{w}|_1 + |w|_1(1+|y|_1+|\bar{y}|_1)) \\ &\le C|y-\bar{y}|_1\|v\| \overset{(47),(78)}{\le} Ch^2\|u\|_1\|v\|. \end{aligned}$$

□

LEMMA 10. *Let $u \in U \cap X_h$ and $y = Y(u)$, $\bar{y} = Y(\pi_{2h}u)$. Also, let $v \in X_h$ and $z = L^*(y-y_d)$, $\bar{z} = L^*(\bar{y}-y_d)$. Then there exists a constant $C = C(u, y_d)$ independent of h such that*

$$\|z-\bar{z}\| \le Ch^2\|u\|_1. \tag{87}$$

Proof. Recall that z and $\bar{z}$ are solutions of

$$\begin{aligned} a(z,\phi) + c(y;\phi,z) + c(\phi;y,z) &= (y-y_d,\phi) \quad \forall \phi \in V \\ a(\bar{z},\phi) + c(\bar{y};\phi,\bar{z}) + c(\phi;\bar{y},\bar{z}) &= (\bar{y}-y_d,\phi) \quad \forall \phi \in V. \end{aligned}$$

By taking $\phi = z - \bar{z}$ in the previous equations and subtracting them we obtain

$$\begin{aligned} &\nu|z-\bar{z}|_1^2 \\ &\le |(y-\bar{y},z-\bar{z})| + |c(y-\bar{y};z-\bar{z},\bar{z})| + |c(z-\bar{z};y-\bar{y},z)| + |c(z-\bar{z};\bar{y},z-\bar{z})| \\ &\le C_1\|y-\bar{y}\|_{\mathbf{H}^{-1}}|z-\bar{z}|_1 + \|y-\bar{y}\|_1\|z-\bar{z}\|_1(\|\bar{z}\|_1+\|z\|_1) + \mathcal{M}(\bar{y})|z-\bar{z}|_1^2, \end{aligned}$$

which gives

$$(\nu - \mathcal{M}(\bar{y}))|z-\bar{z}|_1^2 \le |z-\bar{z}|_1(C_1\|y-\bar{y}\| + \|y-\bar{y}\|_1(\|z\|_1+\|\bar{z}\|_1)).$$

Hence,

$$|z-\bar{z}|_1 \le \|y-\bar{y}\|_1(C_1 + C_2\|y-y_d\| + C_3\|\bar{y}-y_d\|) \overset{(78),(47)}{\le} C(u,y_d)h^2\|u\|_1.$$

from which (87) follows immediately. □

LEMMA 11. *Let $u \in U \cap X_h$, $y = Y(u)$, $y_h = Y_h(u)$. Also, let $z = L^*(y-y_d)$ and $z_h = L_h^*(y_h - y_d^h)$. Then there exists $C = C(u, y_d)$ independent of h so that*

$$\|y_h - y_d^h\| \le C(\|u\| + \|y_d\|_1), \tag{88}$$

$$\|z - z_h\|_k \le Ch^{2-k}\|u\|, \quad k = 0, 1. \tag{89}$$

Proof. We have

$$\|y_h - y_d^h\| \le \|y_h\| + \|y_d\| + \|y_d - y_d^h\| \overset{(56)}{\le} C\|u\| + \|y_d\| + \|y_d - y_d^h\|$$
$$\le C\|u\| + \|y_d\| + \|y_d - I_h y_d\| \le C(\|u\| + \|y_d\| + h\|y_d\|_1).$$

For $h < 1$ this leads to (88). To prove (89), recall that z and z_h satisfy (31) and (70), respectively. Let $(\bar{z}_h, \bar{\rho}_h)$ be the solution of

$$\begin{aligned} a(\bar{z}_h, \phi_h) + \tilde{c}(y_h; \phi_h, \bar{z}_h) + \tilde{c}(\phi_h; y_h, \bar{z}_h) + b(\phi_h, \bar{\rho}_h) \\ = (y - y_d, \phi_h) \quad \forall \phi_h \in X_h^0, \\ b(\bar{z}_h, q_h) = 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{90}$$

From (54)-(55), we have

$$\|z - \bar{z}_h\|_k \le Ch^{2-k}\|y - y_d\|, \quad k = 0, 1. \tag{91}$$

By taking $\phi_h = z_h - \bar{z}_h$ in (70) and (90) and subtracting the equations we obtain

$$\nu|z_h - \bar{z}_h|_1^2 + \tilde{c}(y_h; z_h - \bar{z}_h, z_h - \bar{z}_h) + \tilde{c}(z_h - \bar{z}_h; y_h, z_h - \bar{z}_h) + b(z_h - \bar{z}_h, \rho_h - \bar{\rho}_h)$$
$$= (y - y_h, z_h - \bar{z}_h) - (y_d - y_d^h, z_h - \bar{z}_h),$$

which, by using (12) and $(y_d - y_d^h, z_h - \bar{z}_h) = 0$, simplifies to

$$\nu|z_h - \bar{z}_h|_1^2 + \tilde{c}(z_h - \bar{z}_h; y_h, z_h - \bar{z}_h) = (y - y_h, z_h - \bar{z}_h).$$

Thus,

$$\nu|z_h - \bar{z}_h|_1^2 \le \|y - y_h\|\|z_h - \bar{z}_h\| + \mathcal{M}(y_h)|z_h - \bar{z}_h|_1^2.$$

Since $\nu - \mathcal{M}(y_h) > 0$, we obtain

$$\|z_h - \bar{z}_h\|_1 \le C\|y - y_h\| \overset{(50)}{\le} Ch^2\|u\|,$$

which combined with (91) proves the lemma. ∎

THEOREM 12. *Given $u \in U \cap X_h$, there exists a constant $C = C(\Omega, u, y_d)$ such that*

$$\|(H_\beta^h(u) - T_\beta^h(u))v\| \le Ch^2\|v\| \quad \forall v \in X_h. \tag{92}$$

Proof. Cf. (71) and (77),

$$T_\beta^h(u) - H_\beta^h(u) = A_{2h}(\pi_{2h}u)\pi_{2h} - A_h(u) + \mathcal{C}_{2h}(\pi_{2h}u)\pi_{2h} - \mathcal{C}_h(u).$$

We first estimate

$$\begin{aligned} A_{2h}(\pi_{2h}u)\pi_{2h} - A_h(u) = [A_{2h}(\pi_{2h}u) - A(\pi_{2h}u)]\pi_{2h} + A(\pi_{2h}u)(\pi_{2h} - I) \\ + A(\pi_{2h}u) - A(u) + A(u) - A_h(u). \end{aligned} \tag{93}$$

For any $v \in X_h$ we have

$$|(A(u) - A_h(u))v, v)| = |(L^*Lv - L_h^*L_hv, v)| = |\|Lv\|^2 - \|L_hv\|^2|$$
$$\le \|(L - L_h)v\|(\|Lv\| + \|L_hv\|) \le Ch^2\|v\|^2,$$

which implies

$$\|(A(u) - A_h(u))v\| \le Ch^2\|v\|,$$

since $A(u) - A_h(u)$ is symmetric on X_h. Similarly, it can be shown that

$$\|(A_{2h}(\pi_{2h}u) - A(\pi_{2h}u))\pi_{2h}v\| \le Ch^2\|v\|.$$

For the second term in (93) we have

$$\|A(\pi_{2h}u)(\pi_{2h} - I)v\| = \|L^*(\pi_{2h}u)L(\pi_{2h}u)(\pi_{2h} - I)v\| \overset{(60)}{\le} Ch^2\|v\|.$$

Finally, we have

$$\begin{aligned}
&|(A(\pi_{2h}u)v - A(u)v, v)| = |(L^*(\pi_{2h}u)L(\pi_{2h}u)v - L^*(u)L(u)v, v)| \\
&\quad = |\|L(\pi_{2h}u)v\|^2 - \|L(u)v\|^2| \le \|(L(\pi_{2h}u)v - L(u)v\|(\|L(\pi_{2h}u)v\| + \|L(u)v\|) \\
&\quad \overset{(80)}{\le} Ch^2\|u\|_1\|v\|,
\end{aligned}$$

which implies

$$\|(A(\pi_{2h}u) - A(u))v\| \le Ch^2\|v\|.$$

Combining this with the previous estimates we obtain

$$\|(A_{2h}(\pi_{2h}u) - A_h(u))v\| \le Ch^2\|v\|. \tag{94}$$

Next, we estimate

$$\begin{aligned}
\mathcal{C}_{2h}(\pi_{2h}u)\pi_{2h} - \mathcal{C}_h(u) = (&\mathcal{C}_{2h}(\pi_{2h}u) - \mathcal{C}(\pi_{2h}u))\pi_{2h} + \mathcal{C}(\pi_{2h}u)(\pi_{2h} - I) \\
&+ \mathcal{C}(\pi_{2h}u) - \mathcal{C}(u) + \mathcal{C}(u) - \mathcal{C}_h(u).
\end{aligned} \tag{95}$$

We begin by estimating the term $\|\mathcal{C}(u)v - \mathcal{C}_h(u)v\|$. Let $y = Y(u)$, $y_h = Y_h(u)$, $z = L^*(y - y_d)$, $z_h = L_h^*(y_h - y_d^h)$. Cf. (34) and (72),

$$(\mathcal{C}(u)v, v) = -2c(Lv; Lv, z) \text{ and } (\mathcal{C}_h(u)v, v) = -2\tilde{c}(L_hv; L_hv, z_h).$$

We have $c(Lv; Lv, z) = \tilde{c}(Lv; Lv, z)$ since $Lv \in V$. Therefore,

$$\begin{aligned}
|(\mathcal{C}(u)v - \mathcal{C}_h(u)v, v)| &= |2\tilde{c}(Lv; Lv, z) - 2\tilde{c}(L_hv; L_hv, z_h)| \\
&\le |c(Lv; Lv, z) - c(L_hv; L_hv, z_h)| \\
&\quad + |c(Lv; z, Lv) - c(L_hv; z_h, L_hv)|.
\end{aligned}$$

The first term in the inequality above can be bounded by

$$\begin{aligned}
&|c(L_hv; L_hv, z_h) - c(Lv; Lv, z)| \\
&\qquad \le |c((L_h - L)v; L_hv, z_h)| + |c(Lv; L_hv, z_h - z)| + |c(Lv; z, (L_h - L)v|,
\end{aligned}$$

where we used $c(Lv; (L_h - L)v, z) = -c(Lv; z, (L_h - L)v)$, since $Lv \in V$.
We have

$$\begin{aligned}
&|c((L_h - L)v; L_hv, z_h)| \\
&\le |c((L_h - L)v; (L_h - L)v, z_h)| + |c((L_h - L)v; Lv, z_h)| \\
&\le \|(L_h - L)v\|_1\|(L_h - L)v\|_1\|z_h\|_1 + \|(L_h - L)v\|\|Lv\|_2\|z_h\|_1 \\
&\overset{(51),(52)}{\le} Ch^2\|v\|^2\|z_h\|_1 \overset{(59)}{\le} Ch^2\|v\|^2\|y_h - y_d^h\| \overset{(88)}{\le} C(u, y_d)h^2\|v\|^2,
\end{aligned}$$

and

$$\begin{aligned}|c(Lv; L_h v, z_h - z)| &\le |c(Lv; (L_h - L)v, z_h - z)| + |c(Lv; Lv, z_h - z)| \\ &\le \|Lv\|_1 \|(L_h - L)v\|_1 \|z_h - z\|_1 + C\|Lv\|_1 \|Lv\|_2 \|z_h - z\| \\ &\overset{(51),(19),(89)}{\le} Ch^2 \|v\|^2 \|y_h - y_d^h\| \overset{(88)}{\le} C(u, y_d) h^2 \|v\|^2.\end{aligned}$$

Combining these estimates with

$$|c(Lv; z, (L_h - L)v)| \le C\|Lv\|_1 \|z\|_2 \|(L_h - L)v\| \overset{(25),(52)}{\le} C(u, y_d) h^2 \|v\|^2,$$

we obtain

$$|c(L_h v; L_h v, z_h) - c(Lv; Lv, z)| \le C(u, y_d) h^2 \|v\|^2.$$

Similarly,

$$\begin{aligned}&|c(L_h v; z_h, L_h v) - c(Lv; z, Lv)| \\ &\quad \le |c((L_h - L)v; z_h, L_h v)| + |c(Lv; z_h, (L_h - L)v)| + |c(Lv; Lv, z - z_h)| \\ &\quad \le |c((L_h - L)v; z_h - z, L_h v)| + |c((L_h - L)v; z, L_h v)| \\ &\qquad + |c(Lv; z_h - z, (L_h - L)v)| + |c(Lv; z, (L_h - L)v)| + |c(Lv; Lv, z - z_h)| \\ &\quad \le \|(L_h - L)v\|_1 \|z_h - z\|_1 \|L_h v\|_1 + C\|(L_h - L)v\| \|z\|_2 \|L_h v\|_1 \\ &\qquad + \|Lv\|_1 \|z_h - z\|_1 \|(L_h - L)v\|_1 + C\|Lv\|_1 \|z\|_2 \|(L_h - L)v\| \\ &\qquad + C\|Lv\|_1 \|Lv\|_2 \|z - z_h\| \overset{(25),(51),(52),(89)}{\le} C(u, y_d) h^2 \|v\|^2.\end{aligned}$$

Using the same approach, it can be shown that

$$\|(\mathcal{C}_{2h}(\pi_{2h} u) - \mathcal{C}(\pi_{2h} u) \pi_{2h} v\| \le Ch^2 \|v\|.$$

Let $\bar{z} = L^*(Y(\pi_{2h} u) - y_d)$. The third term in (95) can be bounded as

$$\begin{aligned}|(\mathcal{C}(\pi_{2h} u)v - \mathcal{C}(u)v, v)| &= 2|c(Lv; Lv, \bar{z}) - c(Lv; Lv, z)| = 2|c(Lv; Lv, \bar{z} - z)| \\ &\le C\|Lv\|_1 \|Lv\|_2 \|\bar{z} - z\| \overset{(19)}{\le} C\|v\|^2 \|\bar{z} - z\| \\ &\overset{(87)}{\le} C(u, y_d) h^2 \|u\|_1 \|v\|^2.\end{aligned}$$

Finally let $w = (\pi_{2h} - I)v$. With $L = L(\pi_{2h} u)$, we have

$$\begin{aligned}&|(\mathcal{C}(\pi_{2h} u)(\pi_{2h} - I)v, v)| \\ &= |((Lw \cdot \nabla)\bar{z} - (\nabla Lw)^T \bar{z}, Lv)| \le |((Lw \cdot \nabla)\bar{z}), Lv)| + |(\nabla Lw)^T \bar{z}, Lv)| \\ &= |c(Lw; \bar{z}, Lv)| + |c(Lv; Lw, \bar{z})| \overset{(11)}{=} |c(Lw; \bar{z}, Lv)| + |c(Lv; \bar{z}, Lw)| \\ &\overset{(11)}{\le} C\|Lw\| \|\bar{z}\|_2 \|Lv\|_1 \overset{(60)}{\le} Ch^2 \|v\| \|\bar{z}\|_2 \|v\| \overset{(25)}{\le} C(u, y_d) h^2 \|v\|^2.\end{aligned}$$

□

To assess the quality of the preconditioner we use the spectral distance introduced in [6], defined for two symmetric positive definite operators $T_1, T_2 \in \mathcal{L}(V_h)$ as

$$d_h(T_1, T_2) = \sup_{w \in V_h \setminus \{0\}} \left| \ln \frac{(T_1 w, w)}{(T_2 w, w)} \right|. \tag{96}$$

COROLLARY 1. *Let $u \in U \cap X_h$. If $\mathcal{C}_h(u)$ is symmetric positive definite then*

$$d(H^h_\beta(u), T^h_\beta(u)) \le \frac{C}{\beta} h^2, \tag{97}$$

for $h < h_0(\beta, \Omega, L)$.

Proof.

$$\left| \frac{(T^h_\beta(u)v, v)}{(H^h_\beta(u)v, v)} - 1 \right| \le \frac{C}{\beta} \frac{h^2 \|v\|^2}{\|v\|^2 + \beta^{-1}(\|L_h v\|^2 + (\mathcal{C}_h(u)v, v))} \le \frac{C}{\beta} h^2.$$

Assuming $C\beta^{-1}h_0^2 = \alpha < 1$, and $0 < h \le h_0$. Hence $T^h_\beta(u)$ is positive definite and

$$\begin{aligned} \sup_{v \in X_h \setminus \{0\}} \left| \ln \frac{(T^h_\beta(u)v, v)}{(H^h_\beta(u)v, v)} \right| &\le \frac{|\ln(1-\alpha)|}{\alpha} \sup_{v \in X_h \setminus \{0\}} \left| \frac{(T^h_\beta(u)v, v)}{(H^h_\beta(u)v, v)} \right| \\ &\le \frac{|\ln(1-\alpha)|}{\alpha} \frac{C}{\beta} h^2 \le \frac{C}{\beta} h^2, \end{aligned}$$

where we also used that for $\alpha \in (0, 1)$, $x \in [1-\alpha, 1+\alpha]$ we have

$$\frac{\ln(1+\alpha)}{\alpha} |1 - x| \le |\ln x| \le \frac{|\ln(1-\alpha)|}{\alpha} |1 - x|.$$ ◻

4.2.2. Analysis for the case of mixed/pressure control. Recall from (73) that in the case of mixed/pressure control, the discrete Hessian takes the form

$$H^h_\beta(u)v = \beta v + \gamma_y A_h v + \gamma_p B_h v + \tilde{\mathcal{C}}_h(u)v,$$

where $B_h = M_h^* M_h$ and $A_h = L_h^* L_h$ as in (71). Following the definition in (75), the two-grid preconditioner takes the form

$$T^h_\beta(u) = (\beta I + \gamma_y A_{2h}(\pi_{2h}u) + \gamma_p B_{2h}(\pi_{2h}u) + \tilde{\mathcal{C}}_{2h}(\pi_{2h}u))\pi_{2h} + \beta(I - \pi_{2h}). \tag{98}$$

LEMMA 13. *Let $u \in U \cap X_h$ and $y = Y(u)$, $p = P(u)$, $\bar{y} = Y(\pi_{2h}u)$, $\bar{p} = P(\pi_{2h}u)$. Also, let $v \in X_h$ and $\tilde{z} = \tilde{L}(\gamma_y(y - y_d), \gamma_p(p_d - p))$, $\hat{z} = \tilde{L}(\gamma_y(\bar{y} - y_d), \gamma_p(p_d - \bar{p}))$, with $\tilde{L}$ defined in Theorem 8. Then there exists a constant $C = C(u, y_d, p_d, \gamma_y, \gamma_p)$ independent of h such that*

$$\|\tilde{z} - \hat{z}\|_1 \le Ch\|u\|_1^{1/2}. \tag{99}$$

Proof. Recall that $(\tilde{z}, \tilde{\rho})$ is the solution of (35), and $(\hat{z}, \hat{\rho})$ satisfies

$$\begin{aligned} a(\hat{z}, \phi) + c(\bar{y}; \phi, \hat{z}) + c(\phi; \bar{y}, \hat{z}) + b(\phi, \hat{\rho}) &= \gamma_y(\bar{y} - y_d, \phi) \quad \forall \phi \in X, \\ b(\hat{z}, q) &= \gamma_p(\bar{p} - p_d, q) \quad \forall q \in Q, \end{aligned} \tag{100}$$

By subtracting (100) from (35) we obtain

$$\begin{aligned} a(\tilde{z} - \hat{z}, \phi) + c(y; \phi, \tilde{z} - \hat{z}) + c(\phi; y, \tilde{z} - \hat{z}) + b(\phi, \tilde{\rho} - \bar{\rho}) &= \\ \gamma_y(y - \bar{y}) + c(\bar{y} - y; \phi, \hat{z}) &+ c(\phi; \bar{y} - y, \hat{z}) \\ b(\tilde{z} - \bar{z}, q) = \gamma_p(\bar{p} - p), \quad \forall \phi \in X, q \in Q&, \end{aligned}$$

which represents the weak form of

$$
\begin{aligned}
-\nu\Delta(\tilde z-\hat z)-(y\cdot\nabla)(\tilde z-\hat z)+(\nabla y)^T(\tilde z-\hat z)+\nabla(\tilde\rho-\hat\rho)&=\gamma_y(y-\bar y)+((y-\bar y)\cdot\nabla)\hat z\\
&\quad-(\nabla(y-\bar y))^T\hat z \quad \text{in } \Omega,\\
\operatorname{div}(\tilde\rho-\hat\rho)&=\gamma_p(\bar p-p)\quad \text{in } \Omega\\
\tilde z-\hat z&=0\quad \text{on } \partial\Omega.
\end{aligned}
$$

Using (65), we get

$$\|\tilde z-\hat z\|_1\le C(\gamma_y\|y-\bar y\|+\gamma_p\|p-\bar p\|+\|((y-\bar y)\cdot\nabla)\hat z\|+\|(\nabla(y-\bar y))^T\hat z\|). \tag{101}$$

From (21), we have

$$\|((y-\bar y)\cdot\nabla)\hat z\|\le C\|y-\bar y\|_1\|\nabla\hat z\| \overset{(78),(47),(65)}{\le} Ch^2(\gamma_y\|\bar y-y_d\|+\gamma_p\|\bar p-p_d\|).$$

Of the four terms in the right hand side of (101), only the last is of order one in h:

$$
\begin{aligned}
\|(\nabla(y-\bar y))^T\hat z\|&\le\|\nabla(y-\bar y)\|_{\mathbf{L}^4(\Omega)}\|\hat z\|_{\mathbf{L}^4(\Omega)}\le C\|\nabla(y-\bar y)\|^{1/2}\|\nabla\nabla(y-\bar y)\|^{1/2}\|\hat z\|_1\\
&\overset{(10),(78),(47),(65)}{\le} C(u)h\|u\|_1^{1/2}(\|\bar y-y_d\|+\|\bar p-p_d\|).
\end{aligned}
$$

Using these estimates together with (78), (79), (47) in (101) we obtain

$$\|\tilde z-\hat z\|_1\le C(u)h\|u\|_1^{1/2}(\gamma_y\|\bar y-y_d\|+\gamma_p\|\bar p-p_d\|). \qquad \square$$

THEOREM 14. *Let $u\in U\cap X_h$ be so that $p=P(u)\in W_0^{1.0}(\Omega)$. If $p_d\in W_0^{1,0}(\Omega)\cap Q$, then there exists a constant $C=C(\Omega,u,y_d,p_d,\gamma_y,\gamma_p)$ such that*

$$\|(H_\beta^h(u)-T_\beta^h(u))v\|\le Ch\|v\|\quad \forall v\in X_h.$$

Proof. For any $u\in U\cap X_h$, we have

$$
\begin{aligned}
T_\beta^h(u)-H_\beta^h(u)&=\gamma_y(A_{2h}(\pi_{2h}u)\pi_{2h}-A_h(u))+\gamma_p(B_{2h}(\pi_{2h}u)\pi_{2h}-B_h(u))\\
&\quad+\tilde{\mathcal{C}}_{2h}(\pi_{2h}u)\pi_{2h}-\tilde{\mathcal{C}}_h(u).
\end{aligned}
$$

We use the same approach as in the case of velocity control only. We have already shown in (94) that the first term is $O(h^2\|v\|)$. The second term is estimated similarly:

$$
\begin{aligned}
B_{2h}(\pi_{2h}u)\pi_{2h}-B_h(u)&=[B_{2h}(\pi_{2h}u)-B(\pi_{2h}u)]\pi_{2h}+B(\pi_{2h}u)(\pi_{2h}-I)\\
&\quad+B(\pi_{2h}u)-B(u)+B(u)-B_h(u).
\end{aligned} \tag{102}
$$

For any $v\in X_h$ we have

$$
\begin{aligned}
|((B(u)-B_h(u))v,v)|&=|(M^*Mv-M_h^*M_hv,v)|=|\|Mv\|^2-\|M_hv\|^2|\\
&\le\|Mv-M_hv\|(\|Mv\|+\|M_hv\|)\le Ch\|v\|^2,
\end{aligned}
$$

where we used (53) and (58). Similarly, it can be shown

$$\|(B_{2h}(\pi_{2h}u)-B(\pi_{2h}u))v\|\le Ch\|v\|^2,\quad \forall v\in X_h.$$

The second term in (102) can be bounded as

$$\|B(\pi_{2h}u)(\pi_{2h}-I)v\| = \|M^*(\pi_{2h}u)M(\pi_{2h}u)(\pi_{2h}-I)v\| \overset{(61)}{\leq} Ch\|v\|. \tag{103}$$

Finally, we have

$$\begin{aligned}|(B(\pi_{2h}u)v - B(u)v, v)| &= |(M^*(\pi_{2h}u)M(\pi_{2h}u)v - M^*(u)M(u)v, v)| \\ &= |\|M(\pi_{2h}u)v\|^2 - \|M(u)v\|^2| \\ &\leq \|(M(\pi_{2h}u)v - M(u)v\|(\|M(\pi_{2h}u)v\| + \|M(u)v\|) \overset{(81)}{\leq} Ch^2\|u\|_1\|v\|,\end{aligned}$$

which gives

$$\|(B(\pi_{2h}u) - B(u))v\| \leq Ch^2\|u\|_1\|v\|. \tag{104}$$

Next, we estimate

$$\begin{aligned}\tilde{\mathcal{C}}_{2h}(\pi_{2h}u)\pi_{2h} - \tilde{\mathcal{C}}_h(u) = [\tilde{\mathcal{C}}_{2h}(\pi_{2h}u) - \tilde{\mathcal{C}}(\pi_{2h}u)]\pi_{2h} + \tilde{\mathcal{C}}(\pi_{2h}u)(\pi_{2h}-I) \\ + \tilde{\mathcal{C}}(\pi_{2h}u) - \tilde{\mathcal{C}}(u) + \tilde{\mathcal{C}}(u) - \tilde{\mathcal{C}}_h(u).\end{aligned} \tag{105}$$

We first estimate the term $\|\tilde{\mathcal{C}}(u)v - \tilde{\mathcal{C}}_h(u)v\|$, and recall that

$$(\tilde{\mathcal{C}}(u)v, v) = -2c(Lv; Lv, \tilde{z}) \text{ and } (\tilde{\mathcal{C}}_h(u)v, v) = -2\tilde{c}(L_hv; L_hv, \tilde{z}_h),$$

with $\tilde{z} = \tilde{L}(\gamma_y(y-y_d), \gamma_p(p_d-p))$, $\tilde{z}_h = \tilde{L}_h(\gamma_y(y_h-y_d^h), \gamma_p(p_d^h-p_h))$, with the operators $\tilde{L}$ and $\tilde{L}_h$ as defined in Theorem 8. Thus,

$$\begin{aligned}|(\tilde{\mathcal{C}}(u)v - \tilde{\mathcal{C}}_h(u)v, v)| &= |2\tilde{c}(Lv; Lv, \tilde{z}) - 2\tilde{c}(L_hv; L_hv, \tilde{z}_h)| \\ &\leq |c(Lv; Lv, \tilde{z}) - c(L_hv; L_hv, \tilde{z}_h)| \\ &\quad + |c(Lv; \tilde{z}, Lv) - c(L_hv; \tilde{z}_h, L_hv)|.\end{aligned}$$

The first term in the inequality above can be bounded by

$$\begin{aligned}|c(L_hv; L_hv, \tilde{z}_h) - c(Lv; Lv, \tilde{z})| \leq |c((L_h-L)v; L_hv, \tilde{z}_h)| + |c(Lv; L_hv, \tilde{z}_h - \tilde{z})| \\ + |c(Lv; \tilde{z}, (L_h-L)v|,\end{aligned}$$

where we used $c(Lv; L_hv - Lv, \tilde{z}) = -c(Lv; \tilde{z}, L_hv - Lv)$ since $Lv \in V$. Thus, we have

$$\begin{aligned}&|c(L_hv; L_hv, \tilde{z}_h) - c(Lv; Lv, \tilde{z})| \\ &\leq C(\|Lv - L_hv\|_1\|L_hv\|_1\|\tilde{z}_h\|_1 + \|Lv\|_1\|L_hv\|_1\|\tilde{z}_h - \tilde{z}\|_1 + \|Lv\|_1\|\tilde{z}\|_1\|L_hv - Lv\|_1) \\ &\overset{(51),(64)}{\leq} Ch\|v\|^2(\gamma_y\|y-y_d\| + \gamma_p\|p-p_d\|_{W_0^{1,0}}).\end{aligned}$$

Note that we have used (64) also for $\|L_hv\|_1 \leq C\|v\|$, since $L_hv = \tilde{L}_h(v, 0)$. Also,

$$\begin{aligned}|c(L_hv; \tilde{z}_h, L_hv) - c(Lv; \tilde{z}, Lv)| &\leq |c((L_h-L)v; \tilde{z}_h, L_hv)| + |c(Lv; \tilde{z}_h, (L_h-L)v)| \\ &+ |c(Lv; Lv, \tilde{z} - \tilde{z}_h)| \\ &\leq \|L_hv - Lv\|_1\|\tilde{z}_h\|_1\|L_hv\|_1 + \|Lv\|_1\|\tilde{z}_h\|_1\|L_hv - Lv\|_1 \\ &+ \|Lv\|_1\|Lv\|_1\|\tilde{z} - \tilde{z}_h\|_1 \leq Ch\|v\|(\gamma_y\|y-y_d\| + \gamma_p\|p-p_d\|_{W_0^{1,0}}).\end{aligned}$$

Similarly, it can be shown that $\|(\tilde{\mathcal{C}}_{2h}(\pi_{2h}u) - \tilde{\mathcal{C}}(\pi_{2h}u)\pi_{2h}v\| \leq Ch\|v\|$. To estimate the third term in (105), let $\bar{z} = \tilde{L}(\gamma_y(Y(\pi_{2h}u) - y_d), \gamma_p(p_d - P(\pi_{2h}u)))$. Then

$$|(\tilde{\mathcal{C}}(\pi_{2h}u)v - \tilde{\mathcal{C}}(u)v, v)| = 2|c(Lv; Lv, \tilde{z}) - c(Lv; Lv, \bar{z})| = 2|c(Lv; Lv, \tilde{z} - \bar{z})|$$
$$\leq C\|Lv\|_1^2\|\tilde{z} - \bar{z}\|_1 \overset{(64)}{\leq} C\|v\|^2\|\tilde{z} - \bar{z}\|_1 \overset{(99)}{\leq} C(u, y_d, p_d, \gamma_y, \gamma_p)h\|v\|^2.$$

Finally, let $w = (\pi_{2h} - I)v$. With $L = L(\pi_{2h}u)$ we have (see (39))

$$\begin{aligned}
&|(\tilde{\mathcal{C}}(\pi_{2h}u)(\pi_{2h} - I)v, v)| = |((Lw \cdot \nabla)\bar{z} - (\nabla Lw)^T\bar{z}, Lv)| \\
&\leq |((Lw \cdot \nabla)\bar{z}), Lv)| + |(\nabla Lw)^T\bar{z}, Lv)| = |c(Lw; \bar{z}, Lv)| + |c(Lv; Lw, \bar{z})| \\
&= |c(Lw; \bar{z}, Lv)| + |c(Lv; \bar{z}, Lw)| \leq C\|Lw\|\|\bar{z}\|_2\|Lv\| \\
&\overset{(60)}{\leq} Ch^2\|v\|\|\bar{z}\|_2\|v\| \overset{(62)}{\leq} C(u, y_d, p_d, \gamma_y, \gamma_p)h^2\|v\|^2
\end{aligned}$$

which combined with the other estimates yields the conclusion. ∎

COROLLARY 2. *Let $u \in U \cap X_h$. If $\tilde{\mathcal{C}}_h(u)$ is symmetric positive definite then*

$$d(H_\beta^h(u), T_\beta^h(u)) \leq \frac{C}{\beta}h, \tag{106}$$

for $h < h_0(\beta, \Omega, L, M, \tilde{L})$.

REMARK 3. *The two-grid preconditioner can be extended to a multigrid preconditioner following essentially the same strategy as in [7], and the analysis is extended in a similar fashion to show that the multigrid preconditioner satisfies the estimates (97) and (106). Suffice it to say that the correct multigrid preconditioner has a W-cycle structure, while the associated V-cycle gives suboptimal results; furthemore, the coarsest level has to be sufficiently fine in order for the optimal quality to be preserved.*

5. Numerical results. We present a set of numerical results to showcase the behavior of our multigrid preconditioner in the Newton iteration of (43) on $\Omega = (0, 1)^2$. We consider uniform rectangular grids with mesh sizes $h = 1/32, 1/64, 1/128, 1/256$, and we use Taylor-Hood $\mathbf{Q}_2$-$\mathbf{Q}_1$ elements for velocity-pressure and $\mathbf{Q}_2$ elements for the controls. The data is given by $y_d^h = Y_h(u_h)$, $p_d^h = P_h(u_h)$, with u_h being the interpolant of the target control $u(x, y) = [10^3(\text{sign}(y - 0.9) + 1)(y - 0.9)^2, 0]$ (see Figure 1); the velocity field resembles one obtained from a lid-driven cavity flow. The Newton iteration is stopped when $\|\nabla \hat{J}_h\|_\infty \leq 10^{-10}$. On the coarsest grid at $h = 1/32$ we use a zero-initial guess for the Newton solve, while for subsequent grids we start the iteration using the solution from the coarser problem. The linear systems at each iteration are solved in two ways: first we use conjugate gradient preconditioned by the multigrid preconditioner (MGCG) (see Remark 3), with base cases $h_0 = 1/32$ or $1/64$, depending on necessity. Second, we solve the same systems using unpreconditioned conjugate gradient (CG). The reduced Hessian is applied matrix-free using (71)–(73). Obviously, the Hessian-vector multiplication (matvec) is the most expensive operation, as it essentially requires solving the linearized Navier-Stokes system twice. The goal is to show that, as a result of multigrid preconditioning, the number of matvecs at the highest resolution is relatively low compared to the unpreconditioned case.

We present in Table 1 results for low and in Table 2 for moderate Reynolds numbers, and we compare velocity control ($\gamma_p = 0$) with mixed velocity-pressure control with varying ratios of the two terms in $\hat{J}_h$ ($\gamma_y = 1$, $\gamma_p = 10^{-4}, 10^{-3}$). As for the regularization parameter we let $\beta = 10^{-4}, 10^{-5}$. For each of the twelve parameter choices and for each $h = 1/64, 1/128, 1/256$ we report the number of iterations of the MGCG/CG-based solves for each Newton iteration as well as the total (added) wall-clock time of the linear solves. For example, in the top left compartment of Table 2, we show the case $\nu = 0.01$, $\gamma_y = 1, \gamma_p = 0$ (velocity control only), $\beta = 10^{-4}$ with resolutions $1/64, 1/128, 1/256$. At resolution $h = 1/256$ two Newton iterations were required with CG necessitating 382 and 274 iterations, with a total time of linear solves of 11.4 hours, while the four-grid MGCG needed 6 and 4 iterations for a total of 0.58 hours, meaning almost twenty times faster. Note that at the coarsest level we actually build the Hessian at each Newton iteration and invert it using direct methods, the time of this operation being included in the reported wall-clock time. The relative tolerance for both CG and MGCG is set at 10^{-8}.

The tables indicate a behavior that is standard for the multigrid preconditioner presented in this work, and which is consistent with the analysis. First we notice that unpreconditioned CG is scalable, in the sense that for each case the number of CG iterations is bounded with respect to mesh-size (the wall-clock times suffer due to the fact that we used direct solvers for the linearized Navier-Stoles solves in the matvec). The MGCG instead shows an efficiency that increases over CG with decreasing h, measured both in terms of number of iterations and wall-clock time, and this can be seen for all the velocity control cases, and for the mixed control cases with base case $h_0 = 1/64$ at $\nu = 0.1$ (see Table 1). As usual with these types of algorithms, the lower order of approximation for the mixed/pressure control case leads not only to a slightly higher number of MGCG iterations, but also requires a finer base case; for all the mixed velocity-pressure control problems, the four-grid preconditioner at resolution $h = 1/256$ (base case $h_0 = 1/32$) led to a divergent iteration. However, the base case choice $h_0 = 1/64$ appears to be sufficient when $\nu = 0.1$. However, for the higher Reynolds number case, while we did not encounter divergence with base case $h_0 = 1/64$, it is conceivable that it may still be too coarse, that is, it will lead to divergence at higher resolutions. We should point out that we purposefully selected a set of parameters that exhibit a variety of behaviors expected from these types of algorithms. Yet we find it remarkable that whenever MGCG converges, it does so significantly faster than unpreconditioned CG, with significant wall-clock savings.

6. Conclusions. We have developed and analyzed a two-grid preconditioner to be used in the Newton iteration for the optimal control of the stationary Navier-Stokes equations. Under the natural assumption that the iteration starts sufficiently close to the solution it is shown that the preconditioner has a behavior that is similar to the optimal control of the stationary Stokes equations [7]. While the extension to multigrid is not explicitly discussed due to the similarity with the Stokes-control case, numerical results confirm that the behavior is consistent with the analysis, and can lead to significant savings over unpreconditioned CG-based solves.

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TABLE 1
Iteration counts and runtimes for MGCG vs. CG; $\nu = 0.1$, $Re = \nu^{-1}\|Y(u)\|_\infty \approx 1.1$.

	$\beta = 10^{-4}$			$\beta = 10^{-5}$		
h_j^{-1}	64	128	256	64	128	256
$\gamma_p = 0$ (velocity control only)						
# cg	44	42	41	107	103	103
$t_{\rm cg}$ (sec.)	113	497	3662	285	1284	9015
# mg ($h_0 = 1/32$)	3	3	2	5	4	3
$t_{\rm mg}$ (sec.)	56	149	637	49	166	735
eff$=t_{\rm cg}/t_{\rm mg}$	2.02	3.34	5.75	5.82	7.73	12.27
$\gamma_p = 10^{-4}$ (some pressure control)						
# cg	46, 39	46, 40	46, 41	116, 102	116, 103	117, 106
$t_{\rm cg}$ (sec.)	227	1016	7610	579	2754	19483
# mg ($h_0 = 1/32$)	5, 5	5, 5	nc	8, 6	8, 6	nc
$t_{\rm mg}$ (sec.)	122	348	–	141	389	–
eff$=t_{\rm cg}/t_{\rm mg}$	1.86	2.92	–	4.11	7.08	–
# mg ($h_0 = 1/64$)	n/a	4, 4	4, 4	n/a	5, 5	5, 5
$t_{\rm mg}$ (sec.)		4216	4981		4073	5792
eff$=t_{\rm cg}/t_{\rm mg}$		0.24	1.53		0.68	3.36
$\gamma_p = 10^{-3}$ (more pressure control)						
# cg	43, 46	44, 48	46, 48	104, 124	106, 120	109, 123
$t_{\rm cg}$ (sec.)	239	1087	8329	605	2679	20676
# mg ($h_0 = 1/32$)	5, 7	5, 7	nc	8, 9	8, 9	nc
$t_{\rm mg}$ (sec.)	129	359	–	140	419	–
eff$=t_{\rm cg}/t_{\rm mg}$	1.85	3.03	–	4.32	6.39	–
# mg ($h_0 = 1/64$)	n/a	5, 7	5, 7	n/a	7, 8	6, 8
$t_{\rm mg}$ (sec.)		4360	5573		4427	5768
eff$=t_{\rm cg}/t_{\rm mg}$		0.25	1.49		0.61	3.58

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TABLE 2
Iteration counts and runtimes for MGCG vs. CG; $\nu = 0.01$, $Re = \nu^{-1}\|Y(u)\|_\infty \approx 104$.

	$\beta = 10^{-4}$			$\beta = 10^{-5}$		
h_j^{-1}	64	128	256	64	128	256
$\gamma_p = 0$ (velocity control only)						
# cg	281, 289	259, 247	382, 274	819, 837, 459	777, 716	1136, 816
$t_{\rm cg}$ (hours)	0.42	1.73	11.40	1.54	5.15	33.63
# mg ($h_0 = 1/32$)	10, 10	9, 9	6, 4	27, 31, 15	25, 27	16, 11
$t_{\rm mg}$ (hours)	0.05	0.15	0.58	0.11	0.29	0.93
eff$=t_{\rm cg}/t_{\rm mg}$	8.81	11.30	19.82	14.70	18.07	36.11
$\gamma_p = 10^{-4}$ (some pressure control)						
# cg	299, 318	309, 306	500, 493	834, 909, 507	891, 965	1644, 1354
$t_{\rm cg}$ (hours)	0.43	2.12	16.92	1.45	6.36	50.57
# mg ($h_0 = 1/32$)	12, 13	13, 13	nc	36, 38, 23	35, 38	nc
$t_{\rm mg}$ (hours)	0.05	0.19	–	0.11	0.38	–
eff$=t_{\rm cg}/t_{\rm mg}$	7.94	10.99	–	12.60	16.83	–
# mg ($h_0 = 1/64$)	n/a	7, 7	9, 8	n/a	12, 12	19, 14
$t_{\rm mg}$ (hours)		1.27	1.62		1.23	1.98
eff$=t_{\rm cg}/t_{\rm mg}$		1.67	10.45		5.16	25.48
$\gamma_p = 10^{-3}$ (more pressure control)						
# cg	294, 332, 213	296, 336, 161	481, 597	922, 995, 728	802, 1028, 561	1321, 1818, 629
$t_{\rm cg}$ (hours)	0.62	2.70	18.62	1.62	8.18	64.39
# mg ($h_0 = 1/32$)	14, 16 12	15, 18, 11	nc	42, 51, 39	41, 57, 32	nc
$t_{\rm mg}$ (hours)	0.08	0.31	–	0.13	0.59	–
eff$=t_{\rm cg}/t_{\rm mg}$	7.99	8.97	–	12.17	13.78	–
# mg ($h_0 = 1/64$)	n/a	8, 11, 6	9, 17, 9	n/a	14, 18, 13	16, 23, 16
$t_{\rm mg}$ (hours)		1.78	2.84		1.97	3.19
eff$=t_{\rm cg}/t_{\rm mg}$		1.52	6.56		4.16	20.22

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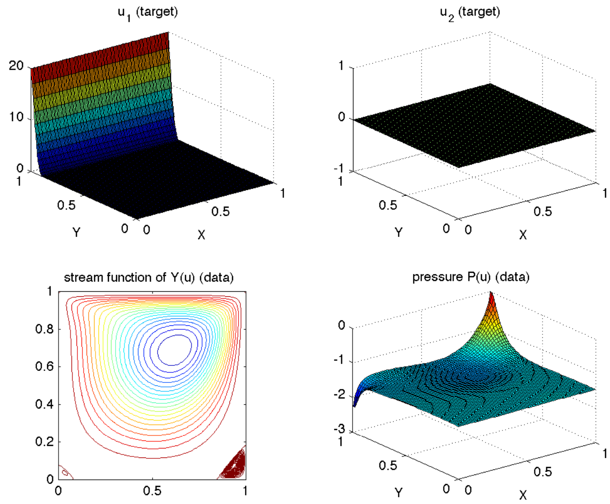

FIG. 1. *Top images: components of target control. Bottom images: velocity (stream function) and pressure data. Viscosity is $\nu = 0.01$ with $Re = \nu^{-1}\|Y(u)\|_\infty \approx 105$, and $h = 1/64$.*

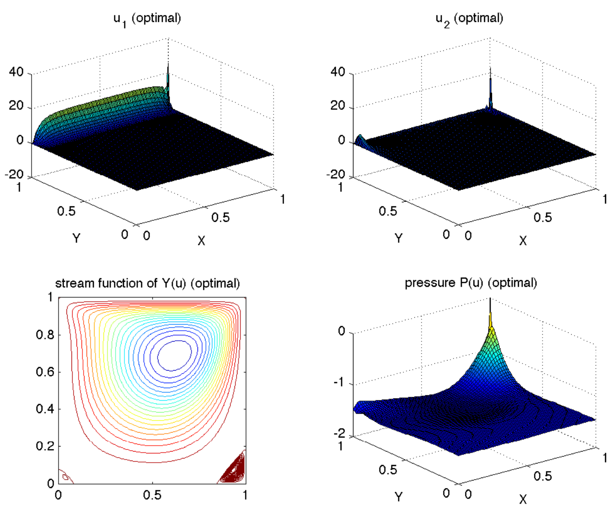

FIG. 2. *Top images: components of optimal control corresponding to data from Figure 1. Bottom images: optimal velocity (stream function) and pressure. Parameters values are: $\gamma_y = 1$, $\gamma_p = 10^{-3}$, and $\beta = 10^{-5}$.*

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