Part 6 : Creeping Flow

Consider the Navier-Stokes equation in 2D

x-direction:

Kinematic viscosity $= \mu/\rho$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right] + g_x$$

y-direction:

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right] + g_y$$

Ignore gravity terms, differentiate x-direction equation with respect to y and y-direction equation with respect to x, subtract to eliminate pressure terms, and substitute stream function, i.e.,

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}$$

Inertial forces term $(\vec{v}\cdot\vec{\nabla}\vec{v})$

Result is:

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = \nu\,\nabla^4(\psi)$$

zero at steady state

$$\nabla^4(\psi) = \nabla^2(\nabla^2\psi)$$

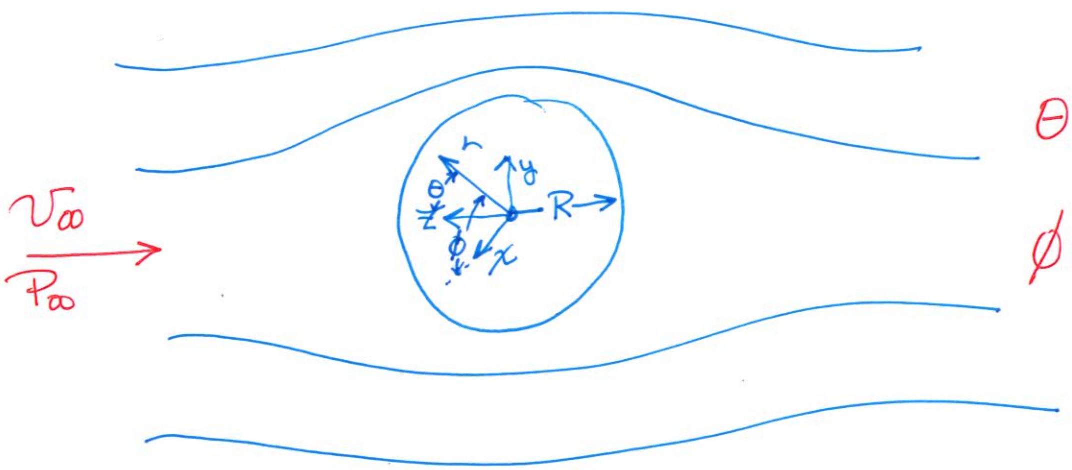
$$\frac{\partial(m,n)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial m}{\partial x} & \frac{\partial m}{\partial y} \\ \frac{\partial n}{\partial x} & \frac{\partial n}{\partial y} \end{vmatrix}$$

For steady state flow at low velocity where inertial terms can be neglected, the equation to solve is:

$$\nabla^4 \psi = 0$$

True in all coordinate systems (Cartesian, cylindrical, etc.)

Example: "Creeping" flow around a sphere (See BSL for details)



θ = polar angle

φ = azimuth angle

Boundary conditions: no slip at solid surface
$v = v_\infty$ far from sphere
$P = P_\infty$ far from sphere

Solution: Assume $\psi(r,\theta) = f(r)\, g(\theta)$
Substitute and separate variables

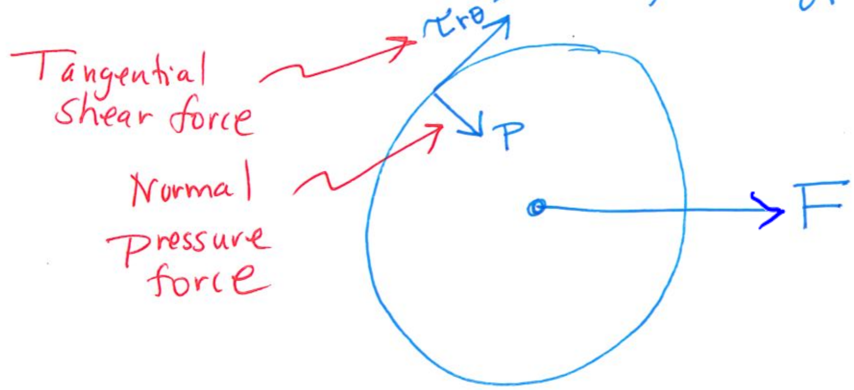
$$\psi(r,\theta) = \left(-\frac{v_\infty R^3}{4r} + \frac{3}{4} v_\infty R r - \frac{1}{2} v_\infty r^2 \right) \sin^2\theta$$

$$v_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} = v_\infty \left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3 \right] \cos\theta$$

$$v_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} = -v_\infty \left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3 \right] \sin\theta$$

Pressure field results from $\vec{\nabla} P = \mu \nabla^2 \vec{v}$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r \frac{\partial}{\partial r}\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$



$$F = \underbrace{2\pi \mu R v_\infty}_{\text{form drag from pressure}}$$

$$+\underbrace{4\pi \mu R v_\infty}_{\text{frictional drag from shear stress}}$$

Stokes law, valid if $Re = \dfrac{\rho v_\infty (2R)}{\mu} < 0.1$

$$F = 6\pi \mu R v_\infty$$

Coefficient of drag

Force / cross sectional area of sphere

$$C_D = \frac{F/\pi R^2}{\frac{1}{2}\rho v_\infty^2} = \frac{6\pi\mu R v_\infty / \pi R^2}{\frac{1}{2}\rho v_\infty^2}$$

Kinetic energy per unit volume

$$= \frac{24}{\rho v_\infty (2R)/\mu} = \frac{24}{Re}$$