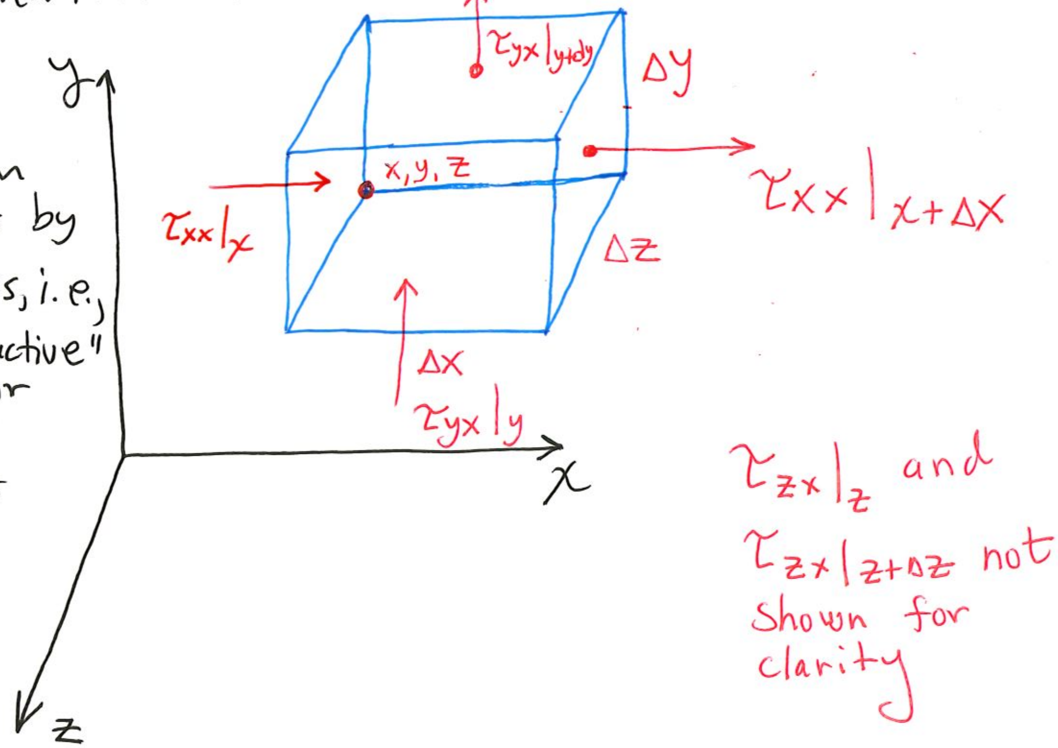
Part 4

Momentum Transfer in x direction

Momentum transport by Shear stress, i.e., the "Conductive" transport or "viscous" transport



Convective transport is given by:

$$\underbrace{(\rho v_i)}_{\substack{\text{mass flux} \\ \text{crossing } x, y, \text{or } z \\ \text{plane}}} v_x = \text{convective flow of x-direction momentum crossing } x, y, \text{or } z \text{ plane}$$

$i = x, y,$ or z

Replace τ_{ix} with $v_i v_x$ in above diagram to yield diagram for "convective" part of momentum transport

Viscous transport in terms of momentum/time:
(ins - outs parts)

$$\tau_{xx}|_x\,dz\,dy - \tau_{xx}|_{x+\Delta x}\,dz\,dy + \tau_{yx}|_y\,dx\,dz$$
$$- \tau_{yx}|_{y+\Delta y}\,dx\,dz + \tau_{zx}|_z\,dx\,dy - \tau_{zx}|_{z+\Delta z}\,dx\,dy$$

A similar equation applies for convective transport with $\rho v_i v_x$ replacing τ_{ix}

$i = x, y,$ or z

Pressure force and gravitational force

$$\Delta y \, \Delta z \left(P|_x - P|_{x+\Delta x} \right) + \rho g_x \, \Delta x \, \Delta y \, \Delta z$$

Accumulation: $(\partial \rho v_x / \partial t) \, \Delta x \, \Delta y \, \Delta z$

Equate and divide by $\Delta x \, \Delta y \, \Delta z$ so every term is on a per unit volume basis:

$$\underbrace{\frac{\partial \rho v_x}{\partial t}}_{\text{Accumulation}} = \underbrace{- \left(\frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) - \frac{\partial}{\partial x} \rho v_x v_x + \frac{\partial}{\partial y} \rho v_y v_x + \frac{\partial}{\partial z} \rho v_z v_x - \frac{\partial P}{\partial x} + \rho g_x}_{\text{ins} - \text{out} + \Sigma F}$$

More generally, for all three directions, we can write

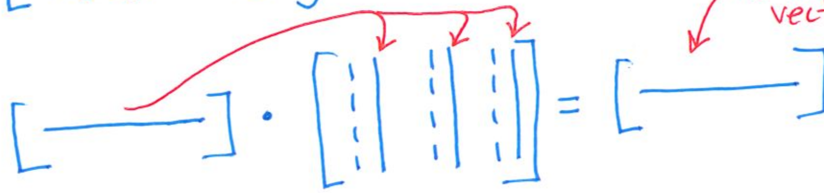
$$\underbrace{\frac{\partial \rho \vec{v}}{\partial t}}_{\substack{\text{Rate of} \\ \text{momentum} \\ \text{increase per} \\ \text{unit volume}}} = - \underbrace{\left(\vec{\nabla} \cdot \rho \vec{v} \vec{v} \right)}_{\substack{\text{Convective momentum} \\ \text{inflow - outflow per} \\ \text{unit volume}}} - \underbrace{\vec{\nabla} \cdot \overline{\overline{\tau}}}_{\substack{\text{inflow -} \\ \text{outflow of} \\ \text{momentum} \\ \text{by viscous} \\ \text{transfer}}} - \underbrace{\vec{\nabla} P}_{\substack{\text{Pressure} \\ \text{force} \\ \text{per} \\ \text{unit} \\ \text{volume}}} + \underbrace{\rho \vec{g}}_{\substack{\text{gravitational} \\ \text{force per} \\ \text{unit} \\ \text{volume}}}$$

Note: $\overline{\overline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$ $\tau_{\text{row, column}}$

$\vec{v} = [v_x, v_y, v_z]$

$\vec{v}\vec{v} = \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}$ dyadic product

Premultiply a matrix by a row vector:



Product is a row vector

Use continuity equation to get:

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}P - \vec{\nabla}\cdot\bar{\bar{\tau}} + \rho\vec{g}$$

Mass per unit volume times acceleration — Pressure force — Viscous force — gravity force — Equivalent to $F = ma$

General relation between τ_{ij} and velocity gradient:

$$\tau_{yx} = \tau_{xy} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{v_y}{\partial x}\right)$$

$\bar{\bar{\tau}}$ is symmetric

For ρ and μ constant, and using $\vec{\nabla}\cdot\vec{v} = 0$, the Navier-Stokes equation results:

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \mu\nabla^2\vec{v} + \rho\vec{g}$$

For $\vec{\nabla}\cdot\bar{\bar{\tau}} = 0$, Euler's equation results

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \rho\vec{g}$$

Forming the "dot" product of the Navier stokes equation and $\vec{v}$ yields the microscopic mechanical energy balance (see next page).

Microscopic mechanical energy balance

$$\frac{\partial}{\partial t} \frac{1}{2}\rho |\vec{v}|^2 = -\vec{\nabla}\cdot \frac{1}{2}\rho |\vec{v}|^2 \vec{v}$$

– efflux of kinetic energy per unit volume

$$-\vec{\nabla}\cdot P\vec{v} - P(-\vec{\nabla}\cdot\vec{v})$$

rate of work done by pressure

Reversible conversion to internal energy

$$-(\vec{\nabla}\cdot[\bar{\bar{\tau}}\cdot\vec{v}])$$

rate of work done by shear stress

$$-(-\bar{\bar{\tau}}:\vec{\nabla}\vec{v})$$

Irreversible conversion to internal energy

$$+\rho(\vec{v}\cdot\vec{g})$$

Rate of work done by gravity