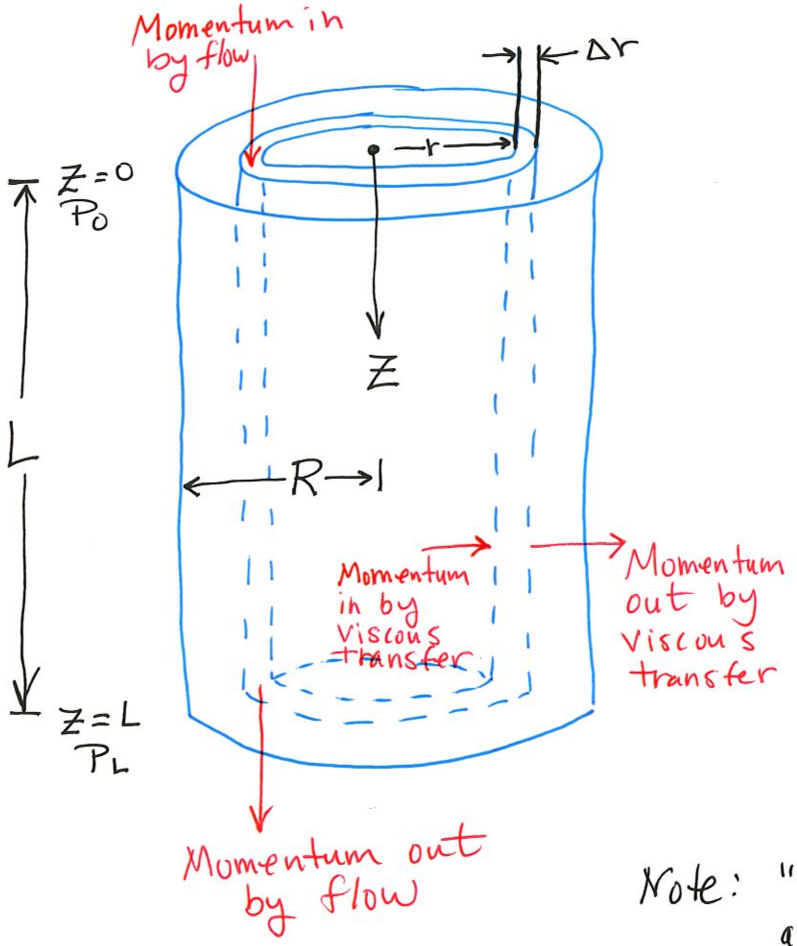
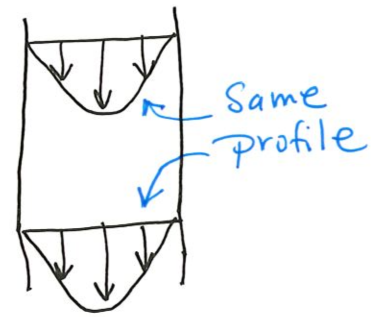
Lecture 3 : Momentum Balance

Consider flow of a fluid in a pipe



Assume:

1. ρ = constant
2. Laminar flow
3. Fully developed flow (no z dependence)



4. Steady state
5. No slip at wall

Note: "in" and "out" arrows are taken in the positive r & z directions

Production of momentum $= \sum F$

Inflow $-$ Outflow $+ \sum F = 0$ (accumulation $= 0$)

Momentum balance (momentum/time) : ?

Divide by $2\pi L \Delta r$, take limit $\Delta r \to 0$

$$\frac{d}{dr} r\, \tau_{rz} = \left(\frac{P_0 - P_L}{L} + \rho g\right) r = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right) r$$

$\mathcal{P} = P - \rho g z$

Integrate: $\tau_{rz} = \dfrac{\mathcal{P}_0 - \mathcal{P}_L}{2L} r + \cancel{\dfrac{C_1}{r}}^{\,0}$

$C_1 = 0$ so term does not go to ∞

Assume Newton's Law: $\tau_{rz} = -\mu \dfrac{dv_z}{dr}$

$$\frac{dv_z}{dr} = -\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu L}\right) r$$

Integrate to yield:

$$v_z = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L} r^2 + C_2$$

But $v_z = 0$ at $r = R$ so that

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{4\mu L} R^2 \left(1 - \left(\frac{r}{R}\right)^2\right)$$

Parabolic velocity profile

"Hagen Poiseuille" law

Other results

Average velocity

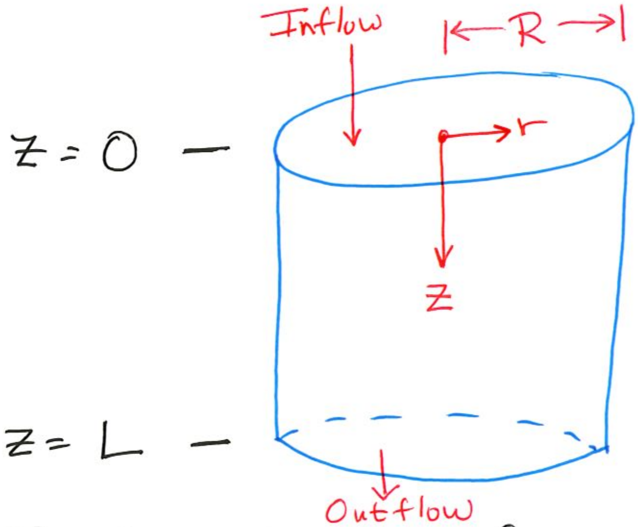
$$v_{z,max} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu L}\right) R^2 = 2\,\langle V_z \rangle$$

$$\langle V_z \rangle \pi R^2 = Q = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)}{8\mu L} R^4$$

volume flow

By definition: $\langle V_z \rangle = \int_0^R 2\pi r v_z\, dr / \pi R^2$

Macroscopic momentum balance on fluid in pipe
(inflow − outflow + $\Sigma F = 0$)



Result: Net pressure force + shear force + gravitational force = 0

Define friction factor f

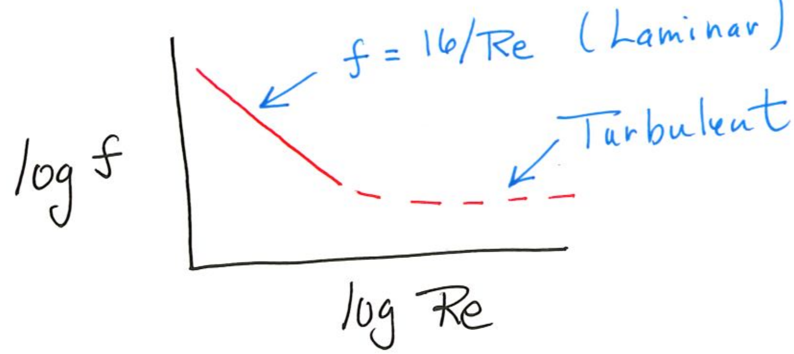
Substitute from previous page to eliminate $(\mathcal{P}_0 - \mathcal{P}_L)$

$$f = \frac{\tau_{rz}|_{r=R}}{\frac{1}{2}\rho \langle v_z \rangle^2} = \frac{(\mathcal{P}_0 - \mathcal{P}_L) R}{L \rho \langle v_z \rangle^2} = \frac{R \langle v_z \rangle 8 \mu L}{L \rho \langle v_z \rangle^2 R^2}$$

Kinetic energy per unit volume

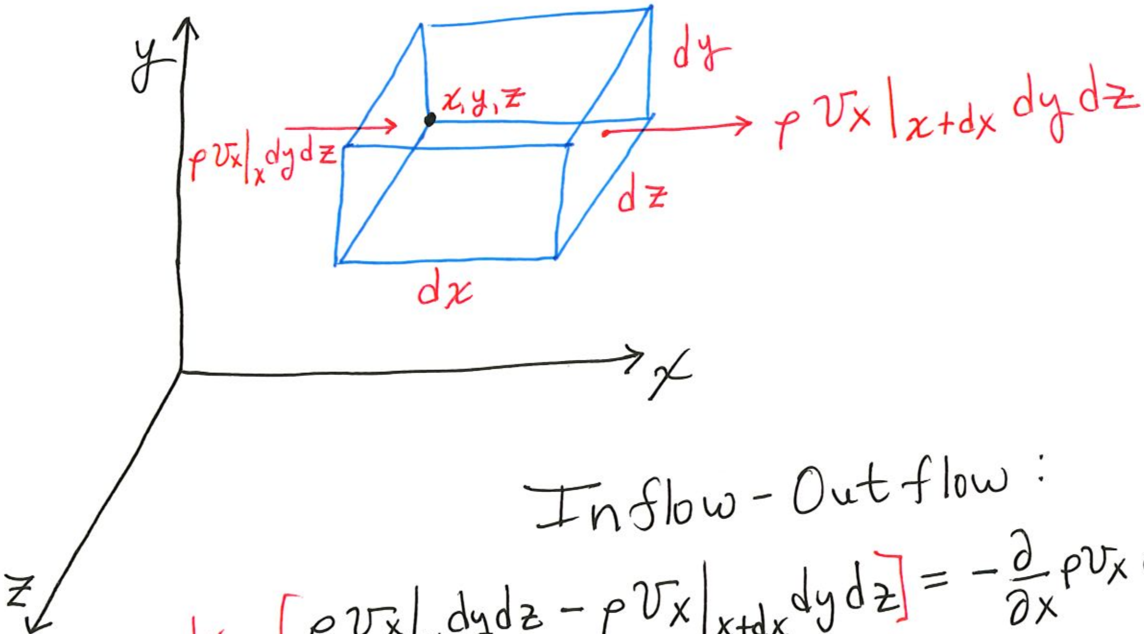
$$f = \frac{16}{\rho \langle v_z \rangle D / \mu} = \frac{16}{Re}$$

Friction factor chart



More general approach (for Cartesian coordinates)

Mass balance (in terms of mass/sec)



Inflow - Outflow:

$$dx \cdot \frac{\left[\rho v_x|_x dy\,dz - \rho v_x|_{x+dx} dy\,dz\right]}{dx} = -\frac{\partial}{\partial x}\rho v_x\, dx\,dy\,dz$$

Mass balance:

$$\frac{\partial \rho}{\partial t} dx\,dy\,dz = \overbrace{-\frac{\partial}{\partial x}\rho v_x\, dx\,dy\,dz}^{x\text{ direction}} \overbrace{-\frac{\partial}{\partial y}\rho v_y\, dx\,dy\,dz}^{y\text{ direction}}$$
$$\underbrace{-\frac{\partial}{\partial z}\rho v_z\, dx\,dy\,dz}_{z\text{ direction}}$$

Divide by dx dy dz:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$$
$$= \underbrace{-\vec{\nabla}\cdot\underbrace{\rho\vec{v}}_{\text{mass flux}}}_{\text{mass efflux/unit volume}}$$

Special forms: steady state $\Rightarrow$ $\vec{\nabla}\cdot\rho\vec{v} = 0$

ρ = constant $\Rightarrow$ $\vec{\nabla}\cdot\vec{v} = 0$

Can also be written as: $D\rho/Dt = -\rho(\vec{\nabla}\cdot\vec{v})$