Name: ____

Suppose we are given a set $\{x_1, x_2, \ldots, x_n\}$ of points on the real line, and we wish to determine the smallest set of unit-length closed intervals that contain all of the given points. Consider the following greedy algorithm. Sort the numbers so that $x_1 \leq x_2 \leq \cdots \leq x_n$. Place an interval with left endpoint x_1, i.e. $I_1 = [x_1, x_1 + 1]$; find the first element x_i that is not included in I_1 and place the interval $I_2 = [x_i, x_i + 1]$; continue in this fashion until all the x_j's are included in an interval. Prove that this algorithm has the Greedy Choice Property and Optimal Substructure.

Note: This problem is equivalent to the activity selection problem with $s_i = x_i$ and $f_i = x_i + 1$. Therefore, the analysis from the textbook can be applied directly.

Greedy Choice Property. We need to show that making the greedy choice can produce an optimal solution. One way to do this is to show that for any sub-problem, there exists an optimal solution that includes the greedy choice. Let $S_k = \{x_i : x_i > x_k + 1\}$ and let x_m be the smallest element of S_k. Then there exists a solution to the sub-problem S_k that includes the interval $I = [x_m, x_m + 1]$. To see this, consider any optimal solution A and let $I' = [a, a + 1] \in A$ contain x_m. If $a = x_m$, then I' is already constructed from the greedy choice and there is nothing to prove; if $a \neq x_m$, then it must be that $a < x_m \leq a + 1$, so I' can be "shifted right" to coincide with I. I contains at least as many elements x_i as I', so replacing I' with I in the optimal solution yields an optimal solution. Therefore, there is an optimal solution to the sub-problem that includes the greedy choice.

Optimal Substructure. We must show that an optimal solution to the full problem is built from optimal solutions to subproblems. Let $S = \{I_1, I_2, \ldots, I_m\}$ be an optimal solution for $A = \{x_1, x_2, \ldots, x_n\}$. Any x_k must be included in some interval $I_j = [a_j, a_j + 1]$. Define the subproblems $A' = \{x_i < a_j\}$ and $A'' = \{x_i > a_j + 1\}$. Then $S' = S \cap A'$ must be an optimal solution for A' and $S'' = S \cap A''$ must be an optimal solution for A''. Suppose not. Then either A' or A'' has a smaller solution. Suppose A' has the smaller solution; then there exists a solution R' on A' such that $|R'| < |S'|$. But then $R' \cup S'' \cup \{I_j\}$ is a solution for A but

$$|R' \cup S'' \cup \{I_j\}| = |R'| + |S''| + 1 < |S'| + |S''| + 1 = |S|,$$

contradicting the assumption that S is an optimal solution. The proof is similar if A'' has the smaller solution (or both A' and A'' have smaller solutions). Therefore the problem has optimal substructure.