Name: ____

1. (10 points) Use a recursion tree to determine a good asymptotic bound on the recurrence $T(n) = 2T(n-1)+1$. Use the substitution method to verify your answer.

The tree will branch by two at each level and will have n levels. The cost of each node will be 1. We estimate that the running time is

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

or $\Theta(2^n)$.

To prove this using substitution, suppose that $T(k) \le c2^k$ for $k < n$. Then

$$\begin{aligned} T(n) &= 2T(n-1)+1 \\ &\le 2c2^{n-1}+1 \\ &= c2^n + 1 \end{aligned}$$

Unfortunately, we are left with the constant term, so we will need to strengthen the inductive hypothesis. Suppose instead that $T(k) \le c2^k - d$ for $k < n$. Then

$$\begin{aligned} T(n) &= 2T(n-1)+1 \\ &\le 2(c2^{n-1} - d)+1 \\ &= c2^n - 2d + 1 \\ &\le c2^n - d \end{aligned}$$

so long as $d \ge 1$. We still need to consider the base case $T(1) \le 2c - d$. Since we can always choose a larger value of c, we can ensure that this case is satisfied. This proves that $T(n) = O(2^n)$.

We also need to prove that $T(n) = \Omega(2^n)$. Suppose $T(k) \ge c2^k$ for $k < n$. (Note that this is a different c). Then

$$\begin{aligned} T(n) &= 2T(n-1)+1 \\ &\ge 2c2^{n-1}+1 \\ &= c2^n + 1 \\ &\ge c2^n \end{aligned}$$

The base case $T(1) \ge 2c$ is easily satisfied by selecting c sufficiently small. Therefore, $T(n) = \Omega(2^n)$. We conclude that $T(n) = \Theta(2^n)$.

(continued on other side)

2. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Justify your answers.

(a) (5 points) $T(n) = 2T(n/2) + n^4$

$a = b = 2$, so $\log_b a = 1$ and $n^{\log_b a} = n$. $f(n) = n^4 = \Omega(n^{1+\epsilon})$. Lastly,

$$af(n/b) = 2\left(\frac{n}{2}\right)^4 = \frac{1}{8}n^4 = \frac{1}{8}f(n),$$

so $T(n) = \Theta(n^4)$ by part 3 of the Master Theorem.

(b) (5 points) $T(n) = 7T(n/2) + n^2$

$a = 7$, $b = 2$ and $\log_b a \approx 2.8074$ so $n^{\log_b a} \approx n^{2.8074}$. $f(n) = n^2 = O(n^{2.8074-\epsilon})$, so $T(n) = \Theta(n^{\log_2 7})$ by part 1 of the Master Theorem.

Theorem 1 (Master Theorem)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and $T(n)$ a recurrence defined for the non-negative integers by $T(n) = aT(n/b) + f(n)$. Then $T(n)$ has the following asymptotic bounds:*

1. *If $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.*
3. *If $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant epsilon > 0, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.*