Name: ______________________________

1. Let $p(n) = a_d n^d + a_{d-1}n^{d-1} + \cdots + a_1 n + a_0$ be a polynomial of degree d with $a_d > 0$.

(a) (5 points) Prove that $p(n)$ is in $\Theta(n^d)$.

We need to show that there exists positive constants c_1, c_2, and n_0 such that whenever $n > n_0$, $c_1 n^d \leq p(n) \leq c_2 n^d$. I like to use the limit method. First note that

$$\frac{p(n)}{n^d} = a_d + \frac{a_{d-1}}{n} + \frac{a_{d-2}}{n^2} + \cdots + \frac{a_0}{n^d}.$$

Therefore, when we take the limit as $n \to \infty$, all but the first term go to zero, and we have

$$\lim_{n\to\infty} \frac{p(n)}{n^d} = a_d.$$

By the definition of the limit, given any $\epsilon > 0$, there is an n_0 such that $n > n_0$ implies that

$$\left| \frac{p(n)}{n^d} - a_d \right| < \epsilon,$$

which can be re-written as

$$-\epsilon < \frac{p(n)}{n^d} - a_d < \epsilon,$$

and isolating $p(n)$ gives

$$(a_d - \epsilon)n^d < p(n) < (a_d + \epsilon)N^d.$$

Let $c_1 = a_d - \epsilon$ and $c_2 = a_d + \epsilon$.

(b) (5 points) Prove that $p(n)$ is in $o(n^k)$ for $k > d$.

By definition, $f(n)$ is in $o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Since $k > d$, we have that

$$\frac{p(n)}{n^d} = \frac{a_d}{n^{k-d}} + \frac{a_{d-1}}{n^{k-d+1}} + \frac{a_{d-2}}{n^{k-d+2}} + \cdots + \frac{a_0}{n^k}.$$

and so $\lim_{n\to\infty} \frac{p(n)}{n^k} = 0$ and $p(n)$ is in $o(n^k)$.

(continued on other side)

2. (10 points) Suppose algorithm A has running time $T(n) = an \lg n + bn + c$, where a, b, and c are positive constants. Show that $T(n)$ is in $\Theta(n \lg n)$.

We need to show that there exists positive constants c_1, c_2, and n_0 such that whenever $n > n_0$,

$$c_1 n \lg n \leq an \lg n + bn + c \leq c_2 n \lg n.$$

Since a, b, and c are positive, $an \lg n \leq an \lg n + bn + c$ and so we can set $c_1 = a$. For $n \geq 2$, both $n \lg n$ and n are greater than or equal to one, so $bn + c \leq bn \lg n + cn \lg n$. Therefore,

$$an \lg n + bn + c \leq (a + b + c) n \lg n,$$

and we can set $c_2 = a + b + c$.