Running Times and Asymptotic Analysis
Part 2

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CMSC 441 — Algorithms

Outline

More Asymptotic Sets

- $O(g(n))$ and $\Omega(g(n))$
- $o(g(n))$ and $\omega(g(n))$

The Set Θ

Recall from the previous lecture:

Definition of Θ

Let $g(n)$ be a function. The set $\Theta(g(n))$ consists of all functions $f(n)$ for which there exists positive constants c_1, c_2, and n_0 such that, if $n \geq n_0$, then

$$c_1 g(n) \leq f(n) \leq c_2 g(n).$$

The sets $O(g(n))$ and $\Omega(g(n))$ are closely related to $\Theta(g(n))$: O is the upper bound from the definition of Θ and Ω is the lower bound.

The Sets O and Ω

Definition of O

The set $O(g(n))$ consists of all functions $f(n)$ for which there exists positive constants c and n_0 such that if $n \geq n_0$, then

$$0 \leq f(n) \leq cg(n).$$

So $f(n) \in O(g(n))$ if it is bounded from *above* by a constant multiple of g, at least for n sufficiently large.

Definition of Ω

The set $\Omega(g(n))$ consists of all functions $f(n)$ for which there exists positive constants c and n_0 such that if $n \geq n_0$, then

$$0 \leq cg(n) \leq f(n).$$

So $f(n) \in \Omega(g(n))$ if it is bounded from *below* by a constant multiple of g, at least for n sufficiently large.

Example: Linear Search

Input: a numeric array A of length n; a value v to search for in A.

```
for i = 1 to n
    if v == A[i]
        return i
return Nil
```

Example: Linear Search

Input: a numeric array A of length n; a value v to search for in A.

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for i = 1 to n
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return NIL
```

Theorem

$f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ and *$f(n) \in \Omega(g(n))$.*

Example: Problem 3-4 (a)

Problem

Is it true that $f(n) = O(g(n))$ implies that $g(n) = O(f(n))$?

The Set $o(g(n))$

Definition of $o(g(n))$

$o(g(n))$ is the set of all functions $f(n)$ for which given *any* positive constant c there exists a positive constant n_0 such that if $n \geq n_0$,

$$0 \leq f(n) < cg(n).$$

This is almost the same as the definition of O, except that c can be any positive constant. In particular, we can make c very small, and it still has to be the case that $f(n)$ is smaller than $cg(n)$ for n sufficiently large.

Example: $n^\beta \in o(n^\alpha)$

Example

Show that if $\alpha > \beta > 0$ are real numbers, then $n^\beta \in o(n^\alpha)$.

Alternative Definition of $o(g(n))$

Observation

The definition of $o(g(n))$ implies that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

This leads to the following alternative definition.

Alternative Definition of $o(g(n))$

Let $g(n)$ be *asymptotically non-negative*; that is, $g(n) \geq 0$ for n sufficiently large. Then $o(g(n))$ is the set of all functions $f(n)$ for which

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Example: $(\lg n)^\alpha \in o(n^\beta)$

Problem

Let α and β be positive constants. Show that $(\lg n)^\alpha \in o(n^\beta)$.

The set $\omega(g(n))$

Definition of $\omega(g(n))$

$\omega(g(n))$ is the set of all functions $f(n)$ for which given *any* positive constant c there exists a positive constant n_0 such that if $n \geq n_0$,

$$0 \leq c(g(n)) < f(n).$$

The definition is similar to that of Ω except that inequality must hold for *any* c. In particular, c can be chosen to be very large, and it still must be the case that $f(n)$ is large than $cg(n)$ for n sufficiently large.

Example: $n \lg n \in \omega(n)$

Problem

Show that $n \lg n \in \omega(n)$.

Alternative Definition of $\omega(g(n))$

As with $o(g(n))$, we have an alternative definition using limits:

Alternative Definition of $\omega(g(n))$

Let $g(n)$ be *asymptotically non-negative*; that is, $g(n) \geq 0$ for n sufficiently large. Then $\omega(g(n))$ is the set of all functions $f(n)$ for which

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty.$$

Example: $n^n \in \omega(n!)$

Problem

Show that $n^n \in \omega(n!)$ using *Stirling's Formula*:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$