Running Times and Asymptotic Analysis

C.S. Marron
cmarron@umbc.edu

CMSC 441 — Algorithms

Outline

Running Times

- Random Access Machines
- The Set $\Theta(g(n))$
- Example: $b + a \lg \lceil n \rceil \in \Theta(\lg n)$

Properties of Θ

- Reflexivity, Symmetry, Transitivity
- Polynomials
- Logarithms

Random Access Machines

An important goal of analyzing an algorithm is to determine it's running time in a manner that is not tied to any specific computer architecture. To this end, we use an abstract computational model, the *Random Access Machine* (RAM).

- RAM can perform operations commonly found in real computers.
- Memory is flat: we do not concern ourselves with levels of cache or other details of real memory architectures.
- Data types are integers and floating point numbers.
- The model does not entail a fixed word size; however, for a particular algorithm, we typically assume some upper bound on word size to avoid unrealistic results.

Example: Binary Search

Input: a numeric array A of length n.

```
first = 1
last = n
while first < last
    middle = ⌊(first + last)/2⌋
    if v ≤ A[middle]
        last = middle
    else
        first = middle + 1
if v == A[first]
    return first
else
    return NIL
```

Running Time of Binary Search

Simplified Running Time

We have that the running time of BINARY-SEARCH as a function of the length n of the input array is

$$T(n) = b + a \lg \lceil n \rceil$$

where a and b are positive constants.

Further Simplification?

Intuitively, it seems that the important part of $T(n)$ is the logarithmic term, which determines the "shape" of the curve, whereas the constant term just shifts the curve up by a constant amount. We need to justify this further simplification.

The Set Θ

Definition of Θ

Let $g(n)$ be a function. The set $\Theta(g(n))$ consists of all funcitons $f(n)$ for which there exists positive constants c_1, c_2, and n_0 such that, if $n \geq n_0$, then

$$c_1 g(n) \leq f(n) \leq c_2 g(n).$$

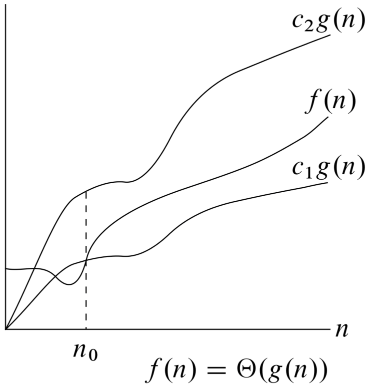
If $f(n)$ is in $\Theta(g(n))$, we may write

- "$f(n)$ *is* $\Theta(g(n))$"
- "$f(n) \in \Theta(g(n))$"
- "$f(n) = \Theta(g(n))$"

The Set Θ

Interpretation of Θ

For n sufficiently large, $f(n)$ is sandwiched between $c_1 g(n)$ and $c_2 g(n)$. As n becomes large, $f(n)$ has a similar "shape" as $g(n)$.



Example: $b + a \lg\lceil n \rceil \in \Theta(\lg n)$

Proof of Upper Bound.

We will suppose that some $c_2 > 0$ exists such that

$$b + a \lg\lceil n \rceil \leq c_2 \lg\lceil n \rceil$$

and try to derive a lower bound on n that makes this true. Rearranging the inequality gives the equivalent expression

$$\frac{b}{c_2 - a} \leq \lg\lceil n \rceil,$$

assuming that $c_2 - a > 0$. Exponentiating each side results in

$$2^{b/(c_2-a)} \leq \lceil n \rceil.$$

This last inequality certainly holds if $n \geq 2^{b/(c_2-a)}$. $\square$

Example: $b + a \lg\lceil n \rceil \in \Theta(\lg n)$

Proof of Lower Bound.

Similarly, suppose c_1 exists such that

$$b + a \lg\lceil n \rceil \geq c_1 \lg\lceil n \rceil$$

and, assuming that $c_1 - a < 0$, derive the equivalent expression

$$\lceil n \rceil \geq 2^{b/(c_1 - a)}.$$

This last inequality is certainly true if $n \geq 2^{b/(c_1 - a)}$. $\square$

Example: $b + a \lg\lceil n \rceil \in \Theta(\lg n)$

Completing the Proof.

Let n_0 be the larger of $2^{b/(c_2-a)}$ and $2^{b/(c_1-a)}$. Then if $n \geq n_0$, we may follow our previous arguments "backwards" to conclude that

$$c_1 \lg\lceil n \rceil \leq b + a \lg\lceil n \rceil \leq c_2 \lg\lceil n \rceil.$$

and so $b + a \lg\lceil n \rceil \in \Theta(\lg\lceil n \rceil)$. The last step is to deal with the ceiling functions. We can show that, for $n \geq 2$,

$$\lg n \leq \lg\lceil n \rceil < \lg(n+1) \leq 2 \lg n,$$

so $\lg\lceil n \rceil \in \Theta(\lg n)$. It follows that

$$b + a \lg\lceil n \rceil \in \Theta(\lg n).$$

$\square$

Some Properties of Θ

We will prove a few basic properties about Θ:

- Reflexivity. $f(n) \in \Theta(f(n))$
- Symmetry. $f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$
- Transitivity. If $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$, then $f(n) \in \Theta(h(n))$

Proof of Reflexivity.

The first property is really easy to prove. Since for all n,

$$1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n),$$

$f(n) \in \Theta(f(n))$ (with $c_1 = c_2 = 1$ and n_0 any value). $\square$

Some Properties of Θ

Proof of Symmetry.

Suppose $f(n) \in \Theta(g(n))$. Then there exists positive constants c_1, c_2, and n_0 such that if $n \geq n_0$, then

$$c_1 g(n) \leq f(n) \leq c_2 g(n).$$

It follows that

$$\frac{1}{c_2} f(n) \leq g(n) \leq \frac{1}{c_1} f(n),$$

so $g(n) \in \Theta(f(n))$. The converse is similar. $\square$

Some Properties of Θ

Proof of Transitivity.

Suppose $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$. Then there exists positive constants c_1, c_2,n_0 and c_1', c_2', n_0' such that if $n \geq n_0$, then

$$c_1 g(n) \leq f(n) \leq c_2 g(n).$$

and if $n \geq n_0'$, then

$$c_1' h(n) \leq g(n) \leq c_2' h(n).$$

So, if $n \geq \max(n_0, n_0')$ we have that

$$c_1' c_1 h(n) \leq f(n) \leq c_2' c_2 h(n)$$

and so $f(n) \in \Theta(h(n))$. $\square$

Polynomials and Θ

Theorem

Let $p(n)$ be a degree k polynomial with positive leading coefficient; that is,

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

where $a_0, a_1, \ldots, a_k$ are real coefficients with $a_k > 0$. Then $p(n) \in \Theta(n^k)$.

Examples

The theorem makes it much easier to determine the Θ class of polynomial functions.

1. $n^2 + n + 1 \in \Theta(n^2)$.
2. $\frac{n}{2} + 1 = \frac{1}{2} n + 1 \in \Theta(n)$.

Logarithms and Θ

The principle behind the previous Theorem is that when we have a sum of terms, the asymptotically largest (fastest growing) dominates and determines the Θ class.

Theorem

Any positive power of $\lg n$ *is asymptotically smaller than any positive power of* n*; that is, for any positive constants* α *and* β*,* $(\lg n)^\beta$ *is asymptotically smaller than* n^α*.*

Example

This tells us that $\lg n$ is *much smaller asymptotically* than n.

1. $(\lg n)^{700}$ is asymptotically smaller than $n^{0.00001}$.
2. $n^2 + \lg n \in \Theta(n^2)$. Apply the principle that the larger term determines the Θ class.
3. $n^2 + n \lg n \in \Theta(n^2)$.

Logarithms and Θ

Sketch of Proof.

Consider the limit

$$\lim_{n\to\infty} \frac{(\lg n)^\beta}{n^\alpha}.$$

Applying L'Hospital's rule some finite number of times, we show that the limit is zero. Therefore, for *any* $\epsilon > 0$ and n sufficiently large,

$$(\lg n)^\beta < \epsilon n^\alpha,$$

so $(\lg n)^\beta$ is dominated by n^α. $\square$