# CMSC 341: Homework 4 Solutions

May 7, 2019

**Problem 1**

**Base Case:** For $n = 1$, we have $a_1 = 1 < \frac{7}{4}$. For $n = 2$, we have $a_2 = 2$ and $\left(\frac{7}{4}\right)^2 = \frac{49}{16} > 2$, so $a_2 < \left(\frac{7}{4}\right)^2$.

**Inductive Hypothesis:** Let $n > 2$ and suppose that for all $k$, $1 \leq k < n$, $a_k < \left(\frac{7}{4}\right)^k$.

**Inductive Step:** By the inductive hypothesis,

$$a_n = a_{n-1} + a_{n-2} < \left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2}.$$

Now

$$\left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2} = \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4} + 1\right)$$

and

$$\frac{7}{4} + 1 = \frac{11}{4} = \frac{44}{16} < \left(\frac{7}{4}\right)^2,$$

so

$$\left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2} < \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^n.$$

Therefore, $a_n < \left(\frac{7}{4}\right)^n$.

**Problem 2**

**Base Case:** For $n = 0$, $x^{2^0} - 1 = x - 1$ which is clearly divisible by $x - 1$.

**Inductive Hypothesis:** Let $n > 0$ and suppose that for all $k$, $0 \leq k < n$, $x^{2^n} - 1$ is divisible by $x - 1$.

**Inductive Step:**

$$x^{2^n} - 1 = (x^{2^{n-1}} - 1)(x^{2^{n-1}} + 1),$$

which can be verified by multiplying-out the right-hand side. By the inductive hypothesis, $x-1$ divides $(x^{2^{n-1}}-1)$, so $x-1$ divides $x^{2^n}-1$.

**Problem 3**

**Base Case:** For $n=1$, the sum is just $1^2=1$. The right-hand side is $1\cdot(1+1)\cdot(2\cdot 1+1)/6=1$.

**Inductive Hypothesis:** Suppose $n>1$ and for all $k$, $1\leq k<n$, $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$.

**Inductive Step:** Splitting the sum gives

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n-1} + n^2.$$

By the inductive hypothesis, this is

$$(n-1)(n-1+1)(2(n-1)+1)/6+n^2 = (n-1)n(2n-1)/6+n^2.$$

Multiplying out the last expression and combining terms with a common denominator gives:

$$(2n^2+3n^2+n)/6 = n(2n^2+3n+1)/6 = n(n+1)(2n+1)/6.$$

Therefore, $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

**Problem 4**

**Base Case:** For $n=2$, the king can make it to the eight squares adjacent to the starting square, as well as the 16 squares that are two steps away. In addition, it can make it back to the starting square. This is a total of $8+16+1=25$ squares that can be reached in two moves, which is equal to $(2\cdot 2+1)^2$.

**Inductive Hypothesis:** Suppose $n>2$ and for all $k$, $2\leq k<n$, $S(k)=(2k+1)^2$.

**Inductive Step:** It helps to draw a picture:

```
#######
#+++++#
#+++++#
#++X++#
#+++++#
#+++++#
#######
```

When $n = 2$, the king can reach all the "+" squares plus it's starting square. For n = 3, it can also reach the "#" squares. Note that the number of "#" squares (the new squares it can reach) is 24. In terms of the number of steps (3) this is just twice the size of the left and right side pieces $2 \cdot (2 \cdot 3 + 1)$ plus twice the size of the top and bottom pieces $2 \cdot (2 \cdot 3 - 1)$. It's not hard to see that the general pattern is: with $k$ moves, the number of *additional* squares that can be reached is:

$$2 \cdot (2 \cdot k + 1) + 2 \cdot (2 \cdot k - 1) = 8k.$$

Therefore, the number of squares reachable in $n$ moves is:

$$S(n) = S(n-1) + 8n = (2(n-1)+1)^2 + 8n = 4n^2 + 4n + 1 = (2n+1)^2.$$

**Problem 5**

First, derive the recurrence:

```
n = 0, [1,1] (one male, one female)
n = 1, [1,1] (pair can reproduce)
n = 2, [2,2] (only one pair can reproduce)
n = 3, [3,3] (only two pairs can reproduce)
n = 4, [5,5] (only three pairs can reproduce)
n = 5, [8,8] (only five pairs can reproduce)
```

The numbers in square brackets are the number of male and female rabbits at each time step. You can probably guess at this point that the number of pairs is a Fibonacci sequence: $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for n $\geq 2$.

**Base Case:** We have $a_0 = a_1 = 0$. Some arithmetic shows that the given expression $R(n)$ is 1 for $n = 0$ and $n = 1$.

**Inductive Hypothesis:** Let $n > 1$ and suppose that the formula is true for all $k$, $0 \leq k < n$.

**Inductive Step:** $a_n = a_{n-1} + a_{n-2}$ and by the inductive hypothesis, we can replace $a_{n-1}$ with the given expression (with powers $(n-1)+1 = n$) and replace

$a_{n-2}$ with the expression (with powers of $(n-2)+1 = n-1$), giving

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right\} \\
&= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right\} \\
&= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right\}.
\end{aligned}$$

Looking at this last epression, we can see what we *need* to be true:

$$\begin{aligned}
\frac{1+\sqrt{5}}{2}+1 &= \left(\frac{1+\sqrt{5}}{2}\right)^2, \text{ and} \\
\frac{1-\sqrt{5}}{2}+1 &= \left(\frac{1-\sqrt{5}}{2}\right)^2.
\end{aligned}$$

It is easy to verify that both these equalities are in fact true, which leaves us with

$$\begin{aligned}
a_n &= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right\} \\
&= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\},
\end{aligned}$$

which is what we wished to prove.