Prove $\displaystyle\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$ by induction

Intuition: $$\sum_{i=1}^{N} i = \overbrace{1+2+3}^{\frac{N}{2}\text{ pairs}} + \dots + N-2 + N-1 + N$$

$N+1$

$N+1$

$N+1$

Base Case

Prove $\displaystyle\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$, when $N=1$

$$\sum_{i=1}^{1} i = 1 = \frac{1}{2}(1+1) = \frac{N}{2}(N+1)$$

thus, $\displaystyle\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$ when $N=1$

Inductive Case

Given $\displaystyle\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$, Prove $\displaystyle\sum_{i=1}^{(N+1)} i = \frac{(N+1)}{2}((N+1)+1)$

i.e

Prove $\displaystyle\sum_{i=1}^{N+1} i = \frac{N+1}{2}(N+2)$

We know $$\sum_{i=1}^{N+1} i = \sum_{i=1}^{N} i + N+1$$

$$\underbrace{\overbrace{1 + 2 + 3 + \ldots + N-2 + N-1 + N}^{\sum_{i=1}^{N}} + N+1}_{\sum_{i=1}^{N+1}}$$

thus $$\sum_{i=1}^{N+1} i = \sum_{i=1}^{N} i + N+1$$

but we are "given" the value of this term

thus $$\sum_{i=1}^{N+1} i = \frac{N}{2}(N+1) + N+1$$

thus $$\sum_{i=1}^{N+1} i = \left(\frac{N}{2}+1\right)(N+1) = \frac{N+2}{2}(N+1) = \frac{N+1}{2}(N+2)$$

therefore $$\sum_{i=1}^{N+1} i = \frac{(N+1)}{2}\left((N+1)+1\right)$$

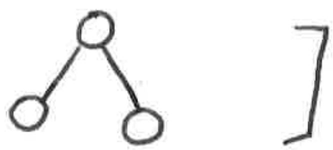
Prove that perfect binary trees have $O\log(N)$ height.

Intuition:

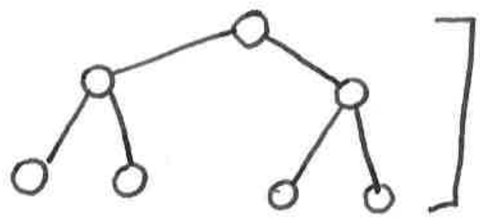
Height	# of Nodes
H:1	P(H): 1


Height 1

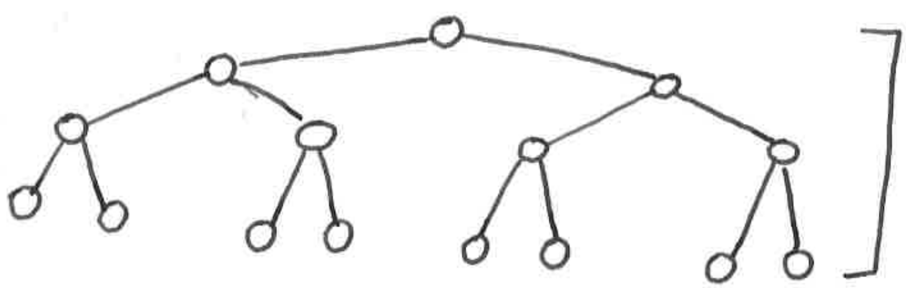
H:2	P(H): 3


Height 2

H:3	P(H): 7

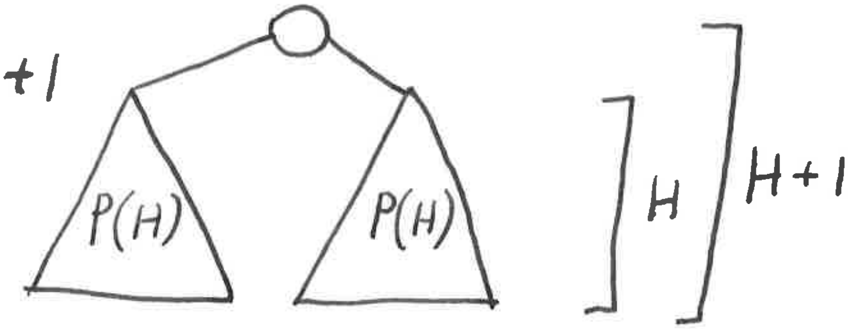

Height 3

H:4	P(H): 15


Height 4

We know

$P(H+1) = 2P(H) + 1$



Intuition

$H=1 \quad P(H)=1$

$H=2 \quad P(H)=3$

$H=3 \quad P(H)=7$

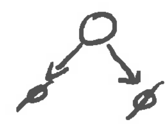
$H=4 \quad P(H)=15$

Prove by induction that

$$P(H)=2^H-1$$

Base Case

Prove $P(H)=2^H-1$ when $H=1$

we know when $H=1$, this is the tree: 

thus $P(1)=1$

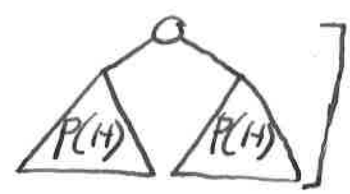
thus $P(1)=2^1-1$

thus $P(H)=2^H-1$, when $H=1$

Inductive case

Given $P(H)=2^H-1$ Prove $P(H+1)=2^{H+1}-1$

we know $P(H+1)=2\underline{P(H)}+1$



we're given this term

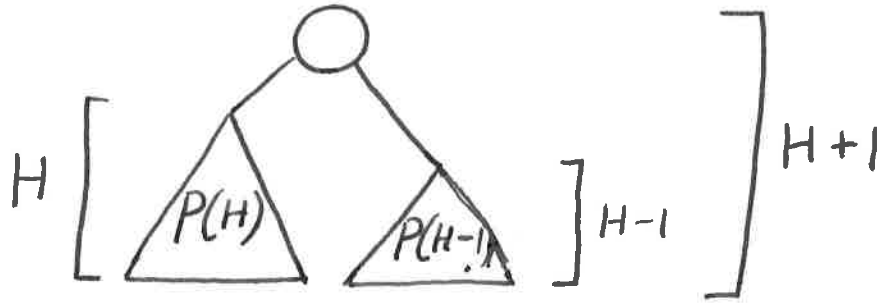
thus $P(H+1)=2(2^H-1)+1$

thus $P(H+1)=2^{H+1}-2+1$

therefore $P(H+1)=2^{H+1}-1$

Prove AVL trees have O lg N height

AVL property that left and right subtree heights differ by at most one:



\# of nodes in an AVL tree of height H+1:

we know $P(H+1) \geq P(H) + P(H-1) + 1$

Prove that $H \leq 2 \lg^{P(H)}$

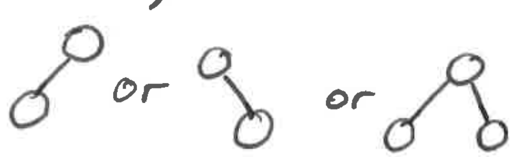
if and only if Prove that $\lg^{P(H)} \geq \frac{H}{2}$

if and only if Prove that $P(H) \geq 2^{\frac{H}{2}}$

Base Case

Prove that $P(H) \geq 2^{\frac{H}{2}}$ when $H=2$

we know that when $H=2$, these are the only trees:



thus $P(2) \geq 2$

thus $P(2) \geq 2^1$

thus $P(2) \geq 2^{\frac{2}{2}}$

therefore $P(2) \geq 2^{\frac{H}{2}}$ when $H=2$

Inductive Case

Given $P(H) \geq 2^{\frac{H}{2}}$ and $P(H-1) \geq 2^{\frac{H-1}{2}}$ Prove $P(H+1) \geq 2^{\frac{H+1}{2}}$

we know (from diagram) $P(H+1) \geq P(H) + P(H-1) + 1$

and we're given these terms

thus $P(H+1) \geq 2^{\frac{H}{2}} + 2^{\frac{H-1}{2}} + 1$

thus $P(H+1) \geq 2^{\frac{H+1-1}{2}} + 2^{\frac{H+1-2}{2}} + 1$

thus $P(H+1) \geq 2^{-\frac{1}{2}} 2^{\frac{H+1}{2}} + 2^{-\frac{2}{2}} 2^{\frac{H+1}{2}} + 1$

thus $P(H+1) \geq \left(2^{-\frac{1}{2}} + 2^{-\frac{2}{2}}\right) 2^{\frac{H+1}{2}} + 1$

thus $P(H+1) \geq \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) 2^{\frac{H+1}{2}} + 1$

But we know $\frac{1}{2} + \frac{1}{\sqrt{2}} \geq 1$

thus $P(H+1) \geq 1 \cdot 2^{\frac{H+1}{2}} + 1$

therefore $P(H+1) \geq 2^{\frac{H+1}{2}}$