Definition

An \( n \)-permutation is a bijection \( p : [n] \rightarrow [n] \).
The set of all \( n \)-permutations is denoted by \( \mathcal{S}_n \).

Two/One-Line Notation

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 5 & 1 & 4 & 2 \\
\end{array}
\]
Definition

If $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a permutation written in one line notation, then the *plot* of $\pi$ is the set of points

$$\{ (1, \pi_1), (2, \pi_2), \cdots (n, \pi_n) \} \subset \mathbb{R}^2$$
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$\pi = 35142$
Definition
Let $A$ and $B$ be two sets of $n$ points in $\mathbb{R}^2$, each with the property that no two points lie on the same horizontal or vertical line. Say that $A \sim B$ if $A$ can be transformed into $B$ by stretching, contracting, and translating the axes horizontally and vertically.
Dots on a Plane

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Example
Dots on a Plane

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\[ \pi = 35142 \]
Permutation Patterns

Definition

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ be two permutations written in one-line notation. $\pi$ contains $\sigma$ as a pattern (written $\sigma \prec \pi$) if there is some subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ which is in the same relative order as the entries of $\sigma$ (i.e., $\pi_{i_j} < \pi_{i_k}$ if and only if $\sigma_j < \sigma_k$).
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\[
35142 \succ 213
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1  21
The Pattern Poset

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The Pattern Poset

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12 21

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The Pattern Poset
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1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321
The Pattern Poset
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Posets

Definition

Let $P$ be a poset, and $A \subseteq P$. $A$ is a downset if it is closed downwards (i.e., $x \in A$ and $y < x$ implies $y \in A$).

An upset is a subset which is closed upwards.

Fact

The complement of a downset is an upset.

Definition

A downset in the permutation pattern poset is called a permutation class.
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A downset in the permutation pattern poset is called a permutation class.
Permutation Classes - Growth Rates

Diagram showing relationships between permutations.
Permutation Classes - Growth Rates
The class $\text{Av}(132)$

**Definition**
Let $c_n$ be the number of permutations of length $n$ which *avoid* the pattern $132$, and $C(x) = \sum_{n \geq 0} c_n x^n$. 
The class $\text{Av}(132)$

**Definition**
Let $c_n$ be the number of permutations of length $n$ which *avoid* the pattern 132, and $C(x) = \sum_{n \geq 0} c_n x^n$.

**Question**
What does a 132-avoiding permutation look like?
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C(x) = xC(x)^2 + 1
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\]

\[
C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}
\]
Lattice Paths

Definition

A \( NS \) Lattice path of length \( 2n \) (or semilength \( n \)) is a sequence of vectors from the set \{\langle 1, 1 \rangle, \langle 1, -1 \rangle\} such that their sum is \( \langle 2n, 0 \rangle \).
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Questions

Q1) How many NS lattice paths are there of semilength $n$?

A1) $\binom{2n}{n}$.

Q2) How many NS lattice paths are there of semilength $n$ which never pass below the line $y = 0$?
Lattice Paths

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Dyck Paths
Dyck Paths

1

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Dyck Paths
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\[ P = xP^2 + 1 \quad 0 = xP^2 - P + 1 \quad P = \frac{1 \pm \sqrt{1 - 4x}}{2x} \]
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P = \frac{1 \pm \sqrt{1 - 4x}}{2x}
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\[
P = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots
\]
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$P = x^3 P^3$
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\[ P = 1 + xP + x^2P^2 + x^3P^3 + x^4P^4 + x^5P^5 + \cdots \]
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$$P(1 - xP) = 1 \quad \quad P = xP^2 + 1$$
Catalan Numbers

Fact

\[ C(x) = 1 - \sqrt{1 - 4x} \]

\[ C_n = \sum_{n \geq 0} \frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right) x^n. \]

The numbers \( C_n \) are called the Catalan numbers. The first few numbers in the sequence are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots
Catalan Numbers

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\[ C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} \frac{1}{n + 1} \binom{2n}{n} x^n. \]
Catalan Numbers

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Two Catalan Recurrences

\[ C(x) = xC(x)^2 + 1 \]

\[ C(x) = 1 + xC(x) + x^2C(x) + \cdots \]

\[ c_0 = 1, \quad c_i = 0 \quad \text{for} \quad i < 0 \]

\[ c_n = \sum_{i+j=n-1} c_i c_j \]

\[ c_n = c_{n-1} + \sum_{i+j+k=n-3} c_i c_j c_k + \cdots \]
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The Class $Av(123)$

**Question**

What does a 123-avoiding permutation look like?
The Class $\text{Av}(123)$

$$\pi = 4 \ 7 \ 6 \ 2 \ 5 \ 1 \ 3$$
The Class $\text{Av}(123)$

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Counting Patterns
Introduction

Example

The permutation 3 5 1 2 4 contains the pattern 3 1 2 three times.

Example

The set 
\{2341, 1234, 4321\}
contains the pattern 123 exactly 5 times.

Notation

Let \( f_\sigma(S) \) denote the number of occurrences of \( \sigma \) within the set \( S \).
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The set \{2341, 1234, 4321\} contains the pattern 123 exactly 5 times.

Notation
Let $f_\sigma(S)$ denote the number of occurrences of $\sigma$ within the set $S$. 
Question
How many times does the pattern 1324 occur within the set of all $n$-permutations? That is, what is $f_{1324}(\mathfrak{S}_n)$?
An Example

Answer

Let $X$ be a random variable denoting the number of 1324 patterns in a random $n$-permutation. Then $\mathbb{E}[X] = f_{1324}(\mathcal{S}_n)/n!$. 
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Answer
Let $X$ be a random variable denoting the number of 1324 patterns in a random $n$-permutation. Then $\mathbb{E}[X] = f_{1324}(\mathfrak{S}_n) / n!$.
Let $X_{i,j,k,l}$ be the random variable equal to 0 or 1, which indicates whether the $i$th, $j$th, $k$th, and $l$th entries form a 1324 pattern. Then

$$X = \sum_{1 \leq i < j < k < l \leq n} X_{i,j,k,l}$$
An Example

Answer
Let \( X \) be a random variable denoting the number of 1324 patterns in a random \( n \)-permutation. Then \( \mathbb{E}[X] = f_{1324}(S_n)/n! \).

Let \( X_{i,j,k,l} \) be the random variable equal to 0 or 1, which indicates whether the \( i \)th, \( j \)th, \( k \)th, and \( l \)th entries form a 1324 pattern.

Then

\[
X = \sum_{1 \leq i < j < k < l \leq n} X_{i,j,k,l}
\]

\[
\mathbb{E}[X] = \sum_{1 \leq i < j < k < l \leq n} \mathbb{E}[X_{i,j,k,l}]
\]
An Example

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Let $X$ be a random variable denoting the number of 1324 patterns in a random $n$-permutation. Then $\mathbb{E} [X] = f_{1324}(\mathcal{S}_n) / n!$. Let $X_{i,j,k,l}$ be the random variable equal to 0 or 1, which indicates whether the $i$th, $j$th, $k$th, and $l$th entries form a 1324 pattern. Then

$$X = \sum_{1 \leq i < j < k < l \leq n} X_{i,j,k,l}$$

$$\mathbb{E} [X] = \sum_{1 \leq i < j < k < l \leq n} \mathbb{E} [X_{i,j,k,l}]$$

$$= \sum_{1 \leq i < j < k < l \leq n} \frac{1}{4!}$$

Therefore $f_{1324}(\mathcal{S}_n) = \frac{n^4}{n! \cdot 4!}$. 


An Example

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$$= \sum_{1 \leq i < j < k < l \leq n} \frac{1}{4!} = \binom{n}{4} \frac{1}{4!}.$$

Therefore

$$f_{1324}(\mathcal{S}_n) = \binom{n}{4} \frac{n!}{4!}.$$
Motivation

Fact
In $\mathcal{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern.
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In $\mathcal{S}_n$, the number of occurrences of a specific pattern depends only on the length of the pattern.

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Previous Results

Theorem (Bóna)

In $Av_n 132$, the pattern 123 is the least common, 321 is the most common, and $f_{213} = f_{231} = f_{312}$. 
Data
### Data

#### Av 132

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### Av 123

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Patterns Within Av 123
Theorem (Cheng, Eu, Fu)

$$f_{12}(A^n_{123}) = 4n - 1 - (2n - 1)n.$$ 

Fact

$$(f_{12} + f_{21})(A^n_{123}) = n^2c_n.$$
Patterns of Length 2

Theorem (Cheng, Eu, Fu)

\[ f_{12}(\text{Av}_n 123) = 4^{n-1} - \binom{2n-1}{n}. \]
Patterns of Length 2

**Theorem (Cheng, Eu, Fu)**

\[ f_{12}(\text{Av}_n 123) = 4^{n-1} - \binom{2n - 1}{n}. \]

**Fact**

\[ (f_{12} + f_{21})(\text{Av}_n(123)) = \binom{n}{2} c_n. \]
Patterns of Length 3

Fact

\[ f_{132} + f_{231} + f_{321} = (n^3)c_n. \]

Proof.

Rewrite the left hand side as

\[ f_{132} + f_{213} + f_{231} + f_{312} + f_{321}. \]
Fact

\[ 2f_{132} + 2f_{231} + f_{321} = \binom{n}{3} c_n. \]
Patterns of Length 3

Fact

\[ 2f_{132} + 2f_{231} + f_{321} = \binom{n}{3} c_n. \]

Proof.
Rewrite the left hand side as

\[ f_{132} + f_{213} + f_{231} + f_{312} + f_{321} \]
Proposition

\[(4f_{132} + 2f_{231})(\text{Av}_n(123)) = (n - 2)f_{12}(\text{Av}_n(123)).\]
Patterns of Length 3

Proposition

$$(4f_{132} + 2f_{231})(Av_n(123)) = (n - 2)f_{12}(Av_n(123)).$$

Proof.
Rewrite as

$$(n - 2)f_{12} - f_{132} - f_{213} = f_{231} + f_{312} + f_{132} + f_{213}.$$

Both sides count the number of length three patterns with at least one non-inversion.
Indecomposable Permutations

Definition
We say that a permutation \( p = p_1 p_2 \ldots p_n \) is decomposable if there exists an integer \( k \) so that each of the entries \( p_1, \ldots, p_k \) is greater than each of the entries \( p_{k+1}, \ldots, p_n \). Otherwise, we say that \( p \) is indecomposable.

Example
The permutation 356412 is decomposable, as the entries 3564 are larger than the entries 12.

Definition
Denote by Av\(^*\)\(n\)(123) the set of indecomposable \( n \)-permutations which avoid 123.
Indecomposable Permutations

Definition
We say that a permutation $p = p_1 p_2 \ldots p_n$ is decomposable if there exists an integer $k$ so that each of the entries $p_1, \ldots, p_k$ is greater than each of the entries $p_{k+1}, \ldots, p_n$. Otherwise, we say that $p$ is indecomposable.
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Example
The permutation 356412 is decomposable, as the entries 3564 are larger than the entries 12.

Definition
Denote by $\text{Av}_n^*(123)$ the set of indecomposable $n$-permutations which avoid 123.
Indecomposable Permutations

Fact
\[ |Av_n^*(123)| = c_{n-1}. \]
Indecomposable Permutations

Fact
\[ |\text{Av}_n^*(123)| = c_{n-1}. \]

Proof.
\[
\text{Av}_n(123) = \begin{array}{c}
\text{Av}_n^*(123) \\
\text{Av}_n^*(123) \\
\vdots \\
\text{Av}_n^*(123)
\end{array}
\]
Indecomposable Permutations

Fact
| Avₙ*(123) | = cₙ−₁.

Proof.

\[ C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)} \]
Indecomposable Permutations

Fact

\[ |\text{Av}_n^*(123)| = c_{n-1}. \]

Proof.

\[ \text{Av}_n(123) = \text{Av}_n^*(123) \]

\[ C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)} \]

\[ C^*(x) = \frac{C(x) - 1}{C(x)} = \frac{(xC(x)^2 + 1) - 1}{C(x)} = xC(x). \]
Indecomposable Permutations

Fact

\[ |\text{Av}_n^*(123)| = c_{n-1}. \]

Alternate Proof.

\[ \text{Av}_n^*(123) \]

\[ = \]

\[ \text{Av}_n(123) \]

\[ C(x) = \frac{C(x) - 1}{C(x)} = \frac{(xC(x)^2 + 1) - 1}{C(x)} = xC(x). \]
Solving the System

Conjectures
Solving the System

Conjectures

\[
C(x)A(x) = xJ(x)C'(x)
\]

\[
A^*(x) + B^*(x) = \sum_{n \geq 0} f_{213}(A\nu_n^* 132)x^n
\]

\[
B^*(x)C(x) = 2xB(x)
\]

\[
A(x) + B(x) = 2 \sum_{n \geq 0} \left( f_{213}(A\nu_n^* 132) + f_{231}(A\nu_n^* 132) \right)x^n
\]

\[
A(x) + B(x) = xB^*(x)
\]

\[
J^*(x) = 2A^*(x)
\]
Corollary

\[ C(x)A(x) = xJ(x)C'(x) \]

\[ A^*(x) + B^*(x) = \sum_{n \geq 0} f_{213} \left( A\nu_n^* 132 \right) x^n \]

\[ B^*(x)C(x) = 2xB(x) \]

\[ A(x) + B(x) = 2 \sum_{n \geq 0} \left( f_{213} \left( A\nu_n^* 132 \right) + f_{231} \left( A\nu_n^* 132 \right) \right) x^n \]

\[ A(x) + B(x) = xB^*(x) \]

\[ J^*(x) = 2A^*(x) \]
The Lemma

\[ \text{Lemma} \]

\[ A^n(x) = x^3 - 4x^3/2 \]
The Lemma

Lemma

\[ A^*(x) = \frac{x^3 C(x)}{(1 - 4x)^{3/2}} \]
Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.
Sketch of proof

Let \( p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2 \), and count 213 patterns.
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2}$$
Sketch of proof

Let $p = 4 3 7 6 1 5 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2}$$
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2}$$
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2}$$
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

\[
f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5
\]
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$
Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$
Sketch of proof

Let $p = 4 \ 3 \ 7 \ 6 \ 1 \ 5 \ 2$, and count 213 patterns.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{1}{2} = 5$$
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$. 

\[ H(x, u) = \]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n, k \geq 0} h_{n,k} x^n u^k$.

\[ H(x, u) = ux(H(x, u) + 1)C(x) + xC(x)H(x, u) \]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[ H(x, u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}. \]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[
H(x, u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}.
\]

\[
A^*(x) = \sum_{n \geq 0} \binom{k}{2} h_{n-1,k} x^n
\]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[
H(x, u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}.
\]

\[
A^*(x) = \sum_{n \geq 0} \binom{k}{2} h_{n-1,k} x^n
\]

\[
A^*(x) = \frac{x \partial^2_u H(x)}{2} \bigg|_{u=1}
\]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[
H(x, u) = \frac{u x C(x)}{1 - u x C(x) - x C(x)}.
\]

\[
A^*(x) = \sum_{n \geq 0} \binom{k}{2} h_{n-1,k} x^n
\]

\[
A^*(x) = \left. x \frac{\partial^2}{\partial u^2} H(x) \right|_{u=1}
= \frac{2}{2} x^3 C(x)
= \frac{x^3 C(x)}{(1 - 4x)^{3/2}}.
\]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[
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A^*(x) = \sum_{n \geq 0} \binom{k}{2} h_{n-1,k} x^n
\]

\[
A^*(x) = \frac{x \partial^2_u H(x) \bigg|_{u=1}}{2}
\]

\[
= \frac{x^3 C(x)}{(1 - 4x)^{3/2}}
\]

\[
= x^3 + 7x^4 + 38x^5 + 1876 + \ldots
\]
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height $k$ in all Dyck paths of semilength $n$. Let $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$.

\[
H(x, u) = \frac{uxC(x)}{1 - uxC(x) -xC(x)}.
\]

\[
A^*(x) = \sum_{n \geq 0} \binom{k}{2} h_{n-1,k} x^n
\]

\[
A^*(x) = \frac{x \partial_x^2 H(x)}{2} \Big|_{u=1}
\]

\[
= \frac{x^3 C(x)}{(1 - 4x)^{3/2}}
\]

\[
= x^3 + 7x^4 + 38x^5 + 1876 + \ldots
\]
Corollaries
Corollaries

Corollary

\[ f_{231}(A \nu_n 123) = f_{231}(A \nu_n 132) \]
Corollaries

\[ A(x) = \frac{x - 1}{2(1 - 4x)} - \frac{3x - 1}{2(1 - 4x)^{3/2}} \]

\[ B(x) = \frac{3x - 1}{(1 - 4x)^2} - \frac{4x^2 - 5x + 1}{(1 - 4x)^{5/2}} \]

\[ D(x) = \frac{8x^3 - 20x^2 + 8x - 1}{(1 - 4x)^2} - \frac{36x^3 - 34x^2 + 10x - 1}{(1 - 4x)^{5/2}} \]
Corollaries

\[ a_n = \frac{n + 2}{4} \binom{2n}{n} - 3 \cdot 2^{2n-3} \]

\[ b_n = (2n - 1) \binom{2n-3}{n-2} - (2n + 1) \binom{2n-1}{n-1} + (n + 4) \cdot 2^{2n-3} \]

\[ d_n = \frac{1}{6} \binom{2n+5}{n+1} \binom{n+4}{2} - \frac{5}{3} \binom{2n+3}{n} \binom{n+3}{2} \\
+ \frac{17}{3} \binom{2n+1}{n-1} \binom{n+2}{2} - 6 \binom{2n-1}{n-2} \binom{n+1}{2} - (n + 1) \cdot 4^{n-1}. \]
Corollaries

\[ a_n \sim \sqrt[n]{\frac{n}{\pi}} 4^n \]

\[ b_n \sim \frac{n}{2} 4^n \]

\[ d_n \sim \frac{8}{3} \sqrt[n]{\frac{n^3}{\pi}} 4^n. \]
Larger patterns

\[ \text{Lemma} \]

\[ A(x) + B(x) + D(x) = x^3 \]

\[ 4A(x) + 2B(x) = x^3 \]

Theorem

For large enough \( n \), the descending pattern of length \( k \) occurs more often than any other length \( k \) pattern in \( A_v^n \).
Larger patterns

Lemma

\[
\begin{align*}
2A(x) + 2B(x) + D(x) &= \frac{x^3}{6}(C(x))''' \\
4A(x) + 2B(x) &= x^3(J(x)/x^2)'
\end{align*}
\]
Larger patterns

Lemma

\[
2A(x) + 2B(x) + D(x) = \frac{x^3}{6}(C(x))'''
\]

\[
4A(x) + 2B(x) = x^3(J(x)/x^2)'
\]

Theorem

For large enough \( n \), the descending pattern of length \( k \) occurs more often than any other length \( k \) pattern in \( Av_n(123) \).
Further Directions

Question
Are there any other 'surprising' symmetries across or within permutation classes?

Note
The increasing and decreasing patterns are not always the extremities of the class:

\[
\begin{align*}
123 & \quad (Av_{2413}) = 321 \\
132 & \quad (Av_{2413}) = 213 \\
213 & \quad (Av_{2413}) = 231 \\
231 & \quad (Av_{2413}) = 312 \\
312 & \quad (Av_{2413}) = 132
\end{align*}
\]

Relative Occurrences
Further Directions

Question

Are there any other ‘surprising’ symmetries across or within permutation classes?
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The Pattern Poset
The Pattern Poset
The Pattern Poset

12

21

1
The Pattern Poset
The Pattern Poset
The Pattern Poset
The Pattern Poset

1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

123

132

213

231

312

321

12

13

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