Math Review

Why are you showing mathematical properties? (again)

- You will be solving complex division/power problems
- You will need below as a tool to help you solve faster
- using these to solve Proofs!! (by Induction)

Exponents

- used as shorthand for multiplying a number by itself several times
- used in identifying sizes of memory
- determine the most efficient way to write a program

Exponent Identities	
$x^a x^b = x^{(a+b)}$ $x^a y^a = (xy)^a$ $(x^a)^b = x^{(ab)}$ $x^{(a+b)} = x^{(a+b)}$ $x^{(a-b)} = x^a / x^b$ $x^a x^b = x^{(a+b)}$	$x^a y^a = (xy)^a$ $(x^a)^b = x^{(ab)}$ $x^{(a/b)} = b^{th}$ root of $(x^a) = (b^{th} \surd(x))^a$ $x^{(-a)} = 1 / x^a$ $x^{(a-b)} = x^a / x^b$

Logarithms

- always based 2 in CS unless stated
- used for
  - conversion from one numbering system to another
  - determining the mathematical power needed
- base 10 to base 2, converted to a formula

Logarithmic Identities
$\log_b(1) = 0$
$\log_b(b) = 1$
$\log_b(x*y) = \log_b(x) + \log_b(y)$
$\log_b(x/y) = \log_b(x) - \log_b(y)$
$\log_b(x^n) = n \log_b(x)$
$\log_b(x) = \log_b(c) * \log_c(x) = \log_c(x) / \log_c(b)$

Summations

- an integral of a function from one variable to a closed interval

Reading Sigma Notation for Arithmetic

start at this value

go to this value

what to sum

$$\sum_{n=1}^{4} n = 1+2+3+4 = 10$$

$$\begin{aligned}\sum_{n=1}^{6} 4n &= 4(1)+4(2)+4(3)+4(4)+4(5)+4(6)\\ &= 4+8+12+16+20+24\\ &= 84\end{aligned}$$

- a function could be broken into several summations
  - makes it easier to match some of the shortcuts below

Breaking up Summations

$$\sum_{i=1}^{100}(4+3i) = \sum_{i=1}^{100} 4 + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 4 + 3\left(\sum_{i=1}^{100} i\right)$$

Mathematical Series (shortcuts)

- arithmetic series
  - $1 + 2 + 3 + 4 \ldots + N$

Reading Sigma Notation for Arithmetic

start at this value

go to this value

what to sum

$$\sum_{n=1}^{4} n = 1+2+3+4 = 10$$

$$\sum_{n=1}^{6} 4n = 4(1)+4(2)+4(3)+4(4)+4(5)+4(6)$$

$$= 4+8+12+16+20+24$$

$$= 84$$

- arithmetic series can be simplified into another formula

Simplifying function for Arithmetic Series

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

notice that “i” in the formula portion is alone

- geometric series (sequence)
  - is a series with a **constant** ratio between successive terms
    - exponent keeps increasing
    - called geometric growth

Reading Sigma Notation for Geometric

$$\sum_{i=1}^{10} 2^i = \mathbf{2, 4, 8, 16, 32, \ldots, 1024}$$

finite example

  - formula could really be anything, but the pattern is consistent in exponent
  - can be TWO limits
    - infinite
    - finite
  - ***this comes up in Theory of Induction!!***
  - called a geometric series because for any three consecutive terms the middle term is the geometric mean of the other two

Other geometric formulas

i. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$

ii. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$

iii. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4},$

iv. $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}, \quad r \neq 1$ (geometric sum).

- the formula in the geometric series ***may*** fit a given series below

Simplifying function for Geometric Series

$$\sum_{i=M}^{N} A^i = \begin{cases} \dfrac{A^{N+1} - A^M}{A-1} & \text{if } A > 1 \\[2ex] \dfrac{A^M - A^{N+1}}{1-A} & \text{if } A < 1 \end{cases}$$

this is finite

$$\sum_{i=0}^{\infty} A^i = \lim_{i\to\infty} \frac{1 - A^{i+1}}{1-A} = \frac{1}{1-A} \quad (A < 1)$$

this is infinite

prove that the finite works correctly with the original example

Solving Summations Quickly

$$\sum_{k=1}^{5}(\mathrm{k}-1) = ?$$

Long way

$= (1 - 1) + (2 - 1) + (3 - 1) + (4 - 1) + (5 - 1)$
$= 0 + 1 + 2 + 3 + 4$
$= 10$

Smart Way

$= \frac{(k-1)((k-1)+1)}{2}$ (using the Arithmetic Series equation)
$= \frac{(5-1)((5-1)+1)}{2}$ (5 came from the upper limit)
$= \frac{4(5)}{2}$
$= 10$

Smartest Way

$= \sum_{k=1}^{5} \mathrm{k} + \sum_{k=1}^{5} -1$
$= 1 + 2 + 3 + 4 + 5 + (5 * -1)$
$= 15 - 5$
$= 10$

Solve:

$$\sum_{i=1}^{200}(i-3)^2$$ **Answer$_b$:**	$$\sum_{k=1}^{6}(6-2k) = ?$$ **Answer$_b$:**	$$\sum_{i=1}^{10} 2^i = 2, 4, 8, 16, 32, \ldots, 1024$$ Do LONG and SHORT(equation above) way [(Proof it works video – volume is a little low)]()

Which equation (pattern) did you use to help you solve??

Proof by Induction

Three steps: to prove a theorem F(N) for any positive integer N

Step 1: Base case: prove F(1) is true

there may be different base cases (or more than one base)

Step 2: Hypothesis: assume F(k) is true for any k >= 1

(it is an assumption, don't try to prove it)

Step 3: Inductive proof:

prove that if F(k) is true (assumption) then F(k+1) is true

F(1) from base case

F(2) from F(1) and inductive proof

F(3) from F(2) and inductive proof

…

F(k+1) from F(k) and inductive proof

Overall Strategies

- when solving if the LHS = RHS
- Factor out
- Find common denominator
- solve to try and match LHS == RHS

Lupoli's over-the-top Proof by Induction Form

eq
Base Case (n = 1)
Induction Step Assume, Reduce, Factor, Common Denominator, Match Assume: true for n = k, show true for n = k+1 Assume: (eq) Show: Reduce: Match LHS and RHS/Solve:

Proof $4 + 9 + \ldots + (5n - 1) = \frac{n}{2}(3 + 5n)$

Base Case (n = 1)

$$5(1) - 1 \ ?= \frac{(1)}{2}(3 + 5(1))$$

$$4 \ ?= ½(8)$$

$$4 == 4 \ ✓$$

Induction Step Assume, Reduce, Factor, Common Denominator, Match

Assume: true for n = k, show true for n = k + 1

Assume: $4 + 9 + \ldots + 5(k)-1 == \frac{k}{2}(3 + 5k)$

Show: $4 + 9 \ldots + 5(k) - 1 + (5(k+1) - 1) \ ?= \frac{k+1}{2}(3 + 5(k + 1))$

<u>reduce:</u> $\frac{k}{2}(3 + 5k) + (5(k+1) - 1) \ ?= \frac{k+1}{2}(3 + 5(k + 1))$

<u>factor:</u> $\frac{3k}{2} + \frac{5k^2}{2} + 5k + 5 - 1 \ ?= \frac{k+1}{2}(8 + 5k)$

<u>LCD</u>: $(2 \times (\frac{3k}{2} + \frac{5k^2}{2} + 5k + 4 \ ?= \frac{k+1}{2}(8 + 5k)))$

$3k + 5k^2 + 10k + 8 \ ?= 8k + 5k^2 + 8 + 5k$

<u>match LHS and RHS:</u> $5k^2 + 13k + 8 == 5k^2 + 13k + 8$ ✓

Proof $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Prove: $\sum_{i=1}^{n} i^2$?= $\frac{n(n+1)(2n+1)}{6}$

Base Case: (n = 1)

1 ?= $\frac{1(1+1)(2(1)+1)}{6}$

1 ?= $\frac{(2)(3)}{6}$

1 ?= $\frac{6}{6}$

1 == 1 ✓

Assume: true for n = k, show true for n = k + 1

Assume: $\sum_{i=1}^{k} i^2$ == $\frac{k(k+1)(2k+1)}{6}$

Show: $\sum_{i=1}^{k} i^2$ + $(k + 1)^2$?= $\frac{k+1((k+1)+1)(2(k+1)+1)}{6}$

reduce: $\frac{k(k+1)(2k+1)}{6}$ + $(k + 1)^2$?= $\frac{k+1((k+1)+1)(2(k+1)+1)}{6}$

factor: $\frac{(k^2+k)(2k+1)}{6}$ + $(k + 1)^2$?= $\frac{(k+1)(k+2)(2k+3)}{6}$

$\frac{2k^3+k^2+2k^2+k}{6}$ + $(k + 1)^2$?= $\frac{k^2+2k+k+2(2k+3)}{6}$

… ?= $\frac{2k^3+4k^2+2k^2+4k+3k^2+6k+3k+6}{6}$

… ?= $\frac{2k^3+9k^2+13k+6}{6}$

LCD: (6 * $\frac{2k^3+k^2+2k^2+k}{6}$ + $(k + 1)^2$?= $\frac{2k^3+9k^2+13k+6}{6}$)

$2k^3 + k^2 + 2k^2 + k + 6(k+1)^2$?= $2k^3 + 9k^2 + 13k + 6$

$2k^3 + k^2 + 2k^2 + k + 6(k+1)(k+1)$?= …

$2k^3 + k^2 + 2k^2 + k + 6(k^2 + k + k + 1)$?= …

$2k^3 + k^2 + 2k^2 + k + 6k^2 + 12k + 6$?= $2k^3 + 9k^2 + 13k + 6$

$2k^3 + 9k^2 + 13k + 6$ == $2k^3 + 9k^2 + 13k + 6$ ✓

Steps for success in induction

1. Solve base case (n = 1)
2. Assume: true for n = k, show true for n = k + 1
3. **Assume:** $\sum_{i=1}^{k} i^2 == \frac{k(k+1)(2k+1)}{6}$

   or

   Assume: math nerd equation == $CS\ nerd\ equation$

4. Show that + (k + 1) will also work!!

   ex: $\sum_{i=1}^{k} i^2 + (k+1)^2 \mathrel{?=} \frac{k+1((k+1)+1)(2(k+1)+1)}{6}$

5. Reduce
6. Factor
7. LCD
8. Try to match LHS == RHS

Show that each of these are equivalent by Proof by Induction

#1 $1 + 3 + 5 + \ldots + 2n - 1 = n^2$

#2 $\sum_{k=1}^{n} 2k = n^2 + n$

#3 $-1 + 2 + 5 + 8 \ldots + 3(n-4) = \frac{n}{2}(3n - 5)$ // This one WILL have an issue

#4 $-1 + 2 + 5 + 8 \ldots + 3n-4 = \frac{n}{2}(3n - 5)$

#5 $1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Answer$_b$:

Answers:

Solving $\sum_{i=1}^{200}(i-3)^2$
$\sum_{i=1}^{200}(i-3)^2 = \sum_{i=1}^{200}(i^2-6i+9)$ **(This is just ONE way, there are others)** $= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9$ $= \sum_{i=1}^{200} i^2 - 6\left(\sum_{i=1}^{200} i\right) + \sum_{i=1}^{200} 9$ $= \frac{200(200+1)(400+1)}{6} - 6\left\{\frac{200(200+1)}{2}\right\} + 9(200)$ = 2,686,700 - 120,600 + 1800   = 2,567,900

Solving $\sum_{k=1}^{6}(6-2k)$
$= \sum_{k=1}^{6} 6 + \sum_{k=1}^{6} -2k$ (again, this is one way…) $= \sum_{k=1}^{6} 6 + -2\sum_{k=1}^{6} k$ $= \ldots$ = -6

Induction Exercises	
#1	Let $P(n): \quad 1+3+5+\cdots+2n-1=n^2$. • First we prove $P(1)$. $$\begin{aligned}\text{LHS of } P(1) &= 1\\ &= 1^2 = \text{RHS of } P(1).\end{aligned}$$ So $P(1)$ is true. • Now we prove that for any natural number k "*if $P(k)$ is true then $P(k+1)$ is true.*" So assume $P(k)$ is true, i.e. $$1+3+5+\cdots+2k-1=k^2.$$ Now try to deduce $P(k+1)$: $$\begin{aligned}\text{LHS of } P(k+1) &= 1+3+5+\cdots+2k-1+2(k+1)-1\\ &= \big(\text{LHS of } P(k)\big)+2(k+1)-1\\ &= \big(\text{RHS of } P(k)\big)+2k+1, \text{(by inductive assumption)}\\ &= k^2+2k+1\\ &= (k+1)^2\\ &= \text{RHS of } P(k+1).\end{aligned}$$ So $P(k+1)$ is true, *if $P(k)$ is true.* • Hence, by induction $P(n)$ is true for all natural numbers n. https://www.youtube.com/watch?v=J0zza185nVU (Ian Thompson F'14)
#2	http://youtu.be/fM0Pe0n2fTY **(Aparna Kaliappan F'14)** https://www.youtube.com/watch?v=5wx6pNgUblc&feature=youtu.be **(Matthew Landen F'14)** http://youtu.be/Beb8Rw97akE **(Anderson Chan F'14)** https://www.youtube.com/watch?v=mREg7mCpm78&list=UUbrYSyKJ_oTbnJT4gecwdpQ **(Kevin Nguyen F'14)** https://www.youtube.com/watch?v=9ZPyiiTAHTs&list=UUIzFbHeF4-czjtE8MbiWTrQ (Siqi Lin F'14) https://www.youtube.com/watch?v=ry3MqQ3vgXo&feature=youtu.be (James Guan F'14) https://www.youtube.com/watch?v=iAEgor3BFgc&list=UU9WrRJboIKSU0XV0uY4GSHA (Conor McCarthy F'14) https://www.youtube.com/watch?v=gH5-psuO1qA&feature=youtu.be (Anthony Stock F'14) https://www.youtube.com/watch?v=VxCT0EWpljo (Joseph Peterson F'14) https://www.youtube.com/watch?v=FILcqICDuwQ (Ian Thompson F'14)

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#3	http://youtu.be/aYJafQ_cgco https://www.youtube.com/watch?v=wgvBgRHSNW4&feature=youtu.be **(Ying Zhang F'14)**
#4	http://youtu.be/aYJafQ_cgco http://www.youtube.com/watch?v=suNIrHfDkxc&feature=youtu.be (Denmark Luceriaga F'14) https://www.youtube.com/watch?v=1E5FJ0FV5u4&feature=youtu.be (Neh Patel F'14) https://www.youtube.com/watch?v=TvZITOVt8EY (Abrielle Minor F'14) https://www.youtube.com/watch?v=74cfRNHT9L8&list=UU2U0VrhmNnkQRVfrhtpea6w&index=1 **(Joseph Wrobleski F'14)** https://www.youtube.com/watch?v=vU-KUc7IE-s (Ian Thompson F'14)
#5	Let $P(n): \quad 1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$. • Firstly, $$\begin{aligned}\text{LHS of } P(1) &= 1^2 = 1 \\ &= \tfrac{1}{6}(1+1)(2.1+1) = \text{RHS of } P(1).\end{aligned}$$ So $P(1)$ is true. • Now assume $P(k)$ is true, for some natural number k, i.e. $$1^2+2^2+3^2+\cdots+k^2=\tfrac{1}{6}k(k+1)(2k+1),$$ and deduce $P(k+1)$: $$\begin{aligned}\text{LHS of } P(k+1) &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2 \\ &= (\text{LHS of } P(k)) + (k+1)^2 \\ &= (\text{RHS of } P(k)) + (k+1)^2, \text{(by inductive assumption)} \\ &= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2 \\ &= \tfrac{1}{6}(k+1)\big(k(2k+1)+6(k+1)\big) \\ &= \tfrac{1}{6}(k+1)(2k^2+7k+6) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{6}(k+1)(k+1+1)\big(2(k+1)+1\big) \\ &= \text{RHS of } P(k+1).\end{aligned}$$ So $P(k+1)$ is true, *if $P(k)$ is true.* • Hence, by induction $P(n)$ is true for all natural numbers n.

Sources:

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