CMSC 341

Lecture 5 Asymptotic Analysis (Continued)

Based on slides from previous iterations at UMBC

Today's Topics

- More on Big O notation
  - Examples
  - Big Omega (Ω)
  - Big Theta (Θ)
- More detail on proof by induction
- In-class bonus question

First, an Explanation from Last Class…

Big O: Example 4

- Code:

```
sum2 = 0;
for (i = 1; i <= n; i++) {
  for (j = 1; j <= i; j++) {
    sum2++;
  }
}
```

- Complexity:
  - Quadratic – $O(n^2)$

But why???

Big O: Example 4

- Code:

```
sum2 = 0;
for (i = 1; i <= n; i++) {
   for (j = 1; j <= i; j++) {
      sum2++;
   }
}
```

how many times do we execute this statement?

1 + 2 + 3 + 4 + … + n-2 + n-1 + n

- Complexity:
  - Quadratic – $O(n^2)$

Expressing as a summation

- Can we express this as a summation?
  - Yes!

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Does this have a known formula?
  - Yes!
- What does this formula multiply out to?
  - $(n^2 + n) / 2$
  - or $O(n^2)$

Other Geometric Formulas

- $O(n^3)$ $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- $O(n^4)$ $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$
- $O(c^n)$ $$\sum_{i=0}^{n} r^i = \frac{1-r^{(n+1)}}{1-r}, \text{ where } r \neq 1$$

Big O: Example 5

- Code:

```
sum3 = 0;
for (i = 1; i <= n; i++) {
   for (j = 1; j <= i; j++) {
      sum3++; }
}
for (k = 0; k < n; k++) {
   a[ k ] = k;
}
```

- Complexity:
  - Quadratic – $O(n^2)$

Big O: Example 6

- Code:

```
sum4 = 0;
for (k = 1; k <= n; k *= 2)
   for (j = 1; j <= n; j++) {
      sum4++;
}
```

- Complexity:
  - O(n log n)

More Help with Summations

- CodeTimeComplexityNotes.pdf
- AsymptoticAnalysisNotes.pdf
- MathReviewNotes.pdf
- Simplifying functions for geometric series
- Solving summations quickly
- Mathematical series shortcuts
- Reading sigma notation
- And more!

Big Omega (Ω) and Big Theta(Θ)

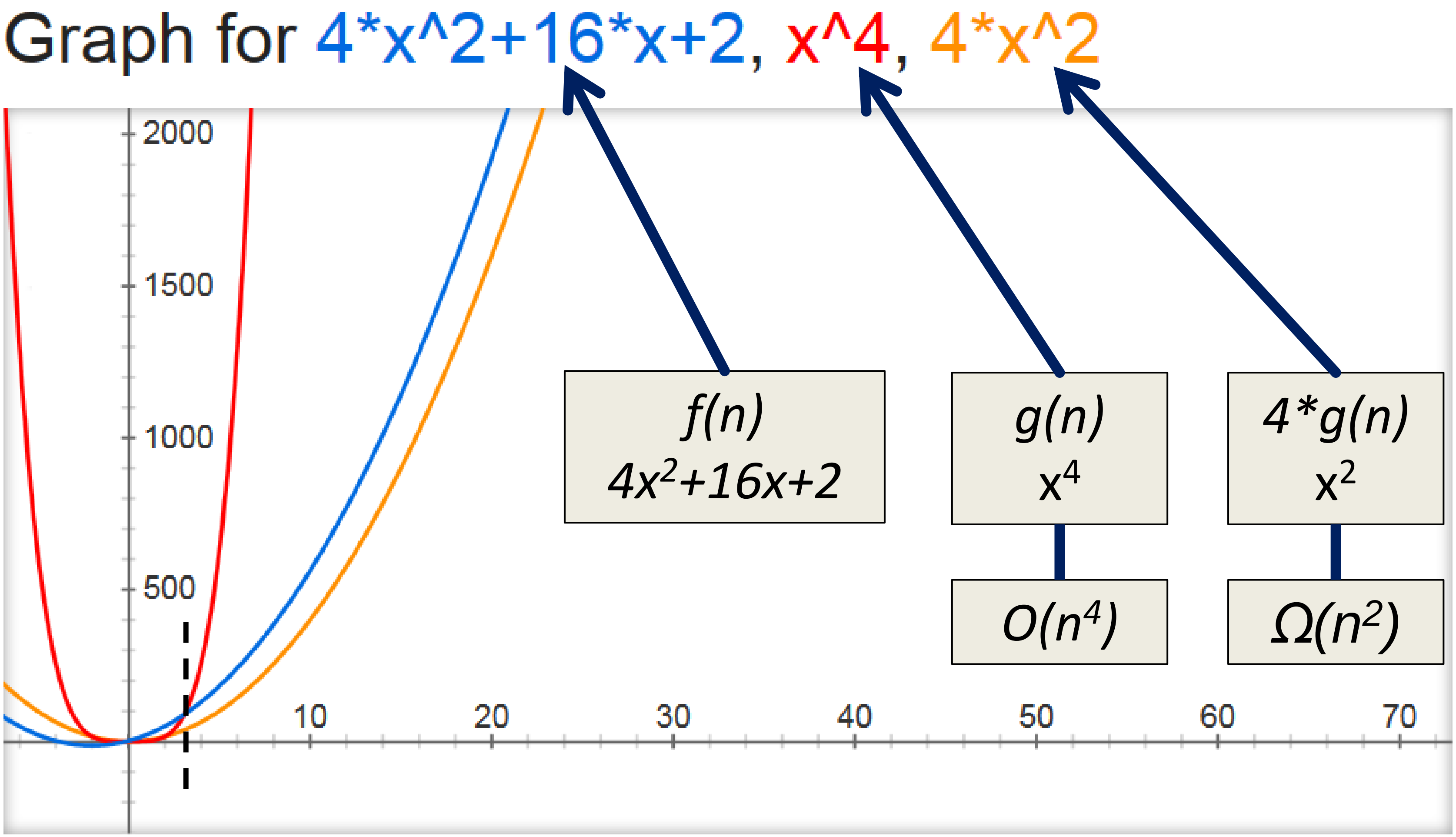
“Big” Notation (words)

- Big O describes an *asymptotic upper bound*
  - The worst possible performance we can expect
- Big Ω describes an *asymptotic lower bound*
  - The best possible performance we can expect
- Big Θ describes an *asymptotically tight bound*
  - The best and worst running times can be expressed with the same equation

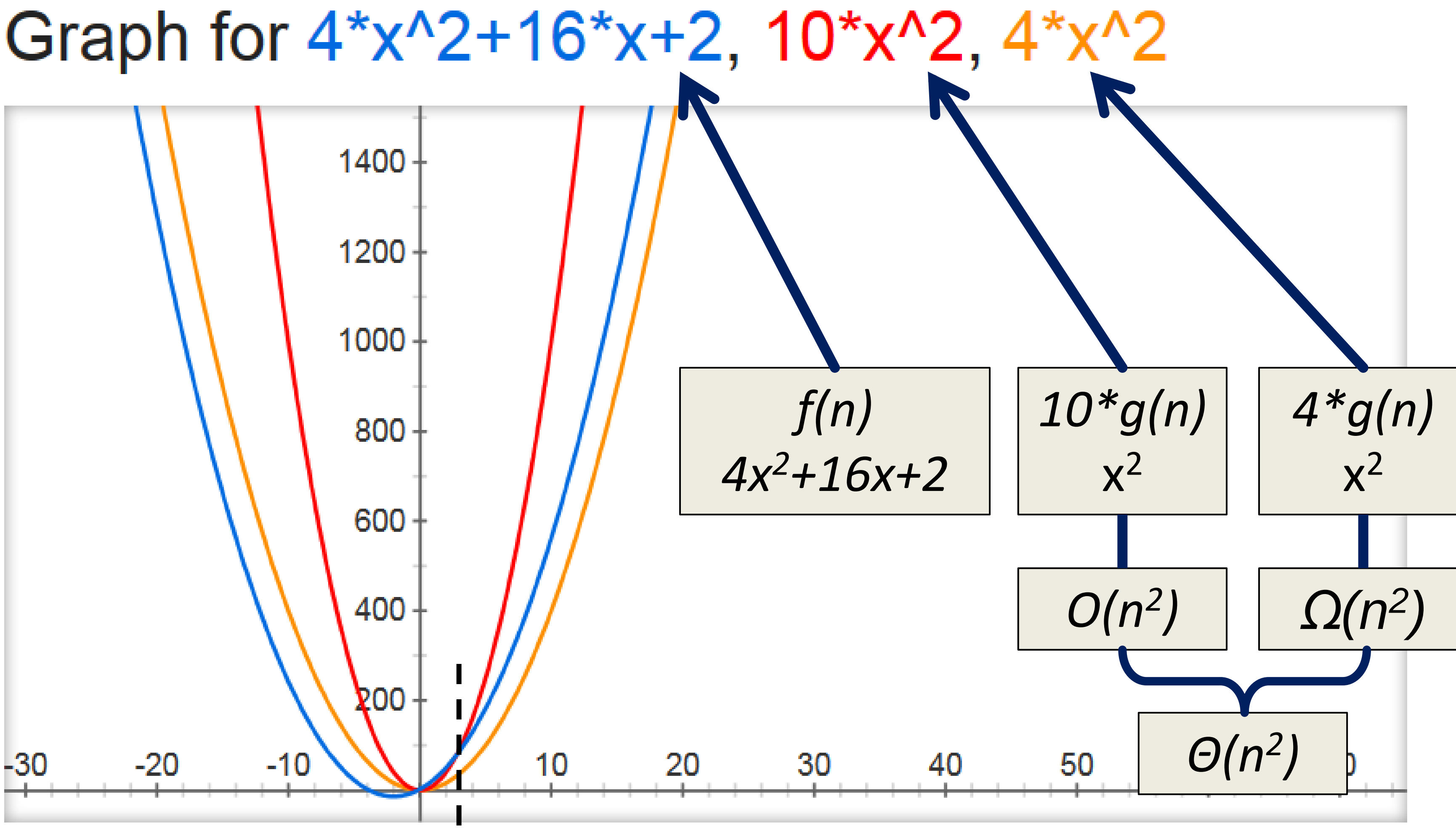
"Big" Notation (equations)

- Big O describes an *asymptotic upper bound*
  - *f(n)* is asymptotically **less than or equal to** *g(n)*
- Big Ω describes an *asymptotic lower bound*
  - *f(n)* is asymptotically **greater than or equal to** *g(n)*
- Big Θ describes an *asymptotically tight bound*
  - *f(n)* is asymptotically **equal to** *g(n)*

Big O and Big Omega Example



Big Theta Example



A Simple Example

- Say we write an algorithm that takes in an array of numbers and returns the highest one
  - What is the absolute fastest it can run?
    - Linear time – $\Omega(n)$
  - What is the absolute slowest it can run?
    - Linear time – $O(n)$
  - Can this algorithm be *tightly* asymptotically bound?
    - YES – so we can also say it's $\Theta(n)$

Proof by Induction

Proof by Induction

- The only way to prove that Big O will work
  - As n becomes larger and larger numbers

- To prove F(n) for any positive integer n
  1. Base case: prove F(1) is true
  2. Hypothesis: Assume F(k) is true for any k >= 1
  3. Inductive: Prove the if F(k) is true, then F(k+1) is true

Induction Example (Step 1)

- Show that for all $n \geq 1$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

1. Base case:
   - $n = 1$
   - (This is our n_0)

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2(1)+1)}{6}$$

$$\sum_{i=1}^{1} i^2 = \frac{1(2)(3)}{6}$$

$$\sum_{i=1}^{1} i^2 = \frac{6}{6}$$

$$\sum_{i=1}^{1} i^2 = 1$$

Induction Example (Step 2)

- Show that for all $n \geq 1$:

2. Hypothesis:
   - Assume that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

     holds for any $n \geq 1$

Induction Example (Step 3)

- Show that for all $n \geq 1$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3. <u>Inductive</u>:
   - Prove that if $F(k)$ is true (assumed), the $F(k+1)$ is also true
   - We've already proved $F(1)$ is true
   - So proving this step will prove $F(2)$ from $F(1)$, and $F(3)$ from $F(2)$, …, and $F(k+1)$ from $F(k)$

Induction Example (Step 3)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k(2k+1)+6(k+1))}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(2k^2+7k+6)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Bonus Question!

Announcements

- Homework 2 is out
  - Due Thursday, September 21st at 8:59:59 PM

- Project 1 is out this week
  - Due Tuesday, September 26th at 8:59:59 PM