A HARMONIC-TYPE MAXIMAL PRINCIPLE IN COMMUTANT LIFTING

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In this note, we prove a harmonic-type maximal principle for the Schur parametrization of all intertwining liftings of an intertwining contraction in the commutant lifting theorem.

0 Introduction

Commutant lifting theorems were first discovered by D. Sarason [8] and his work turned out to be one of the unifying principles in classical interpolation theory developed much earlier by Carathéodory, Schur, Pick and Nevanlinna. B. Sz.-Nagy and C. Foias formulated and proved in [10] the general geometric commutant lifting theorem which henceforth played a central role in operator theory as well as in several important problems in robust control theory. The set of all contractive intertwining liftings of a given intertwining contraction can be explicitly parametrized by the closed unit ball $H^\infty_1(G, G')$ of the space $H^\infty_1(G, G')$. Here $G$ and $G'$ are some adequate Hilbert spaces and $H^\infty_1(G, G')$ denotes the space of all bounded analytic functions defined on the open unit disc in the complex plane and with values operators from $G$ to $G'$ (see §1 below). This parametrization, which is quite analogous to the Schur parametrization of all solutions in the classical Carathéodory interpolation problem, has been obtained in [1], [4], [5] and plays an useful role in pure operator theory as well as its applications such as in inverse scattering problems and in robust control theory. In this paper we state and prove a maximal principle in the case of the commutant lifting theorem which closely resembles the classical maximum principle in analytic function theory. Precisely, we will show (our Theorem 1.1) that in case $\|A\| < 1$, if the norm of the intertwining lifting associated to some $\Gamma$ in the interior of $H^\infty_1(G, G')$ is equal to one, then that norm is constant on the whole interior $H^\infty_{1,0}(G, G')$ of $H^\infty_1(G, G')$. Note that this result is meaningful only when $G \neq \{0\}$ and $G' \neq \{0\}$. In this case, as we show in section 4 (iii), the supremum of the norms of the intertwining liftings associated to $\Gamma \in H^\infty_{1,0}(G, G')$ is equal to one.

The organization of this paper is as follows: in section 1 we describe the set up and state the main theorem, in section 2 we state and prove a few propositions and lemmas pertaining to the Schur representation in the commutant lifting theorem. In section 3 we
give the proof of our main result. In section 4 we make some concluding remarks on the connection between our work and an example provided by M. Bakonyi in [2].

1 Preliminaries

We follow the standard notation and terminologies as in [3], [9]. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. By $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ we denote the algebra of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. An operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is a contraction if $\|C\| \leq 1$. If $C$ is a contraction in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, then we denote by $D_C$ the positive square root of $I - C^*C$ and by $\mathcal{D}_C$ the closed range of $D_C$. The orthogonal projection onto a Hilbert space $\mathcal{H}$ is denoted by $P_\mathcal{H}$. The Hardy space of all analytic functions in the open unit disc $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$, with values in a Hilbert space $\mathcal{F}$ whose Taylor coefficients are square summable is denoted by $H^2(\mathcal{F})$. Finally let $H^\infty(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all norm bounded analytic functions on the unit disc taking values in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. The closed unit ball of $H^\infty(\mathcal{H}_1, \mathcal{H}_2)$ is denoted by $H^\infty_{1,0}(\mathcal{H}_1, \mathcal{H}_2)$ and let $H^\infty_{1,0}(\mathcal{H}_1, \mathcal{H}_2)$ be its interior, that is

$$H^\infty_{1,0}(\mathcal{H}_1, \mathcal{H}_2) = \{f \in H^\infty(\mathcal{H}_1, \mathcal{H}_2) : \|f\| < 1\}.$$  

Let $T'$ be a contraction on $\mathcal{H}'$. An operator $U'$ on $\mathcal{K}' \supseteq \mathcal{H}'$ is said to be a minimal isometric dilation of $T'$, if $U'$ is an isometry, $\mathcal{H}'$ is an invariant subspace of $U'^*$ satisfying $U'^*\mathcal{H}' = T'^*$ and $\mathcal{H}'$ is cyclic for $U'$, that is,

$$\mathcal{K}' = \bigoplus_{n=0}^{\infty} U'^n \mathcal{H}' .$$  

(1.1)

Any contraction $T'$ on $\mathcal{H}'$ admits a minimal isometric dilation and all minimal isometric dilations of $T'$ are isomorphic (see [3]). To be precise, if $U$ on $\mathcal{K}$ is another minimal isometric dilation of $T'$, then there exists an unique unitary operator $\Phi$ from $\mathcal{K}$ to $\mathcal{K}'$ satisfying $U'\Phi = \Phi U'$ and $\Phi|\mathcal{H} = I$. So without loss of generality, we can and shall always work with the Sz.-Nagy-Schaffer form of the minimal isometric dilation which is the isometry $U'$ defined by

$$U' = \begin{bmatrix} T' & 0 \\ D_T & S \end{bmatrix} \quad \text{on } \mathcal{K}' = \mathcal{H}' \oplus H^2(D_T)$$  

(1.2)

where $S$ is the unilateral (or forward ) shift on $H^2(D_T)$ that is $(Sf)(z) = zf(z)$, $z \in \mathbb{D}, f \in H^2(D_T)$.

Throughout this paper, $T$ on $\mathcal{H}$ is an isometry, $T'$ is a contraction on $\mathcal{H}'$ and $U'$ on $\mathcal{K}'$ is the Sz.-Nagy-Schaffer minimal isometric dilation of $T'$. Moreover, $A$ is a contraction from $\mathcal{H}$ into $\mathcal{H}'$ intertwining $T$ with $T'$, that is $T'A = AT$. An operator $B$ from $\mathcal{H}$ to $\mathcal{K}'$ is said to be an intertwining lift of $A$ if $P_{\mathcal{H}}B = A$ and $U'B = BT$. It has been proven in [1], that the set of all contractive intertwining liftings $B$ of $A$ is parametrized by the closed unit ball in some $H^\infty(\mathcal{G}, \mathcal{G}')$. We describe this parametrization below.
Let $\Pi'$ be the operator from $D_{T'} \oplus D_A$ to $D_{T'}$ that picks up the first component and $\Pi_A$ the operator from $D_{T'} \oplus D_A$ to $D_A$ that picks up the second component, that is
\[ \Pi'(x \oplus y) = x, \quad \Pi_A(x \oplus y) = y, \quad (x \oplus y) \in D_{T'} \oplus D_A. \] (1.3)

Let $F$ be the subspace of $D_A$ defined by $F = (D_A T'H)^{-}$ and $G = D_A \oplus F$. Let $\omega$ be the isometry from $F$ into $D_{T'} \oplus D_A$ defined by
\[ \omega D_A Th = D_{T'} Ah \oplus D_A h \quad (h \in H). \] (1.4)

It is easy to check that $\omega$ is an isometry ([4]). Finally we say $W(z)$ is a Schur contraction if $W(z) = \omega P_{T'} \oplus \Gamma(z) P_G$ where $\Gamma \in H_{10}^\infty(G, G')$ and $G' = (D_{T'} \oplus D_A) \ominus \omega F$. We call such $W$ a Schur contraction associated with $\Gamma$ and write $W = W(\Gamma)$. Then the set of all contractive intertwining lifting of $A$ is parametrized by the set of all Schur contractions or equivalently by $H_{10}^\infty(G, G')$ as follows ([5]) :
\[ \Gamma \rightarrow W(\Gamma) \rightarrow B_W = \left[ \Pi W(z)(I - z \Pi_A W(z))^{-1}D_A \right] \] (1.5)

The converse association, that is an explicit formula to compute $W$ from $B_W$ is obtained in ([4]). In our note we prove a maximal principle closely resembling the classical maximal principle in complex function theory. The analogous maximal principle in this case can be stated as follows:

**Theorem 1.1.** Let $\| A \| < 1$. Suppose there exists $\Gamma_0 \in H_{10}^\infty(G, G')$ such that $\| B_W(\Gamma_0) \| = 1$. Then for all $\Gamma \in H_{10}^\infty(G, G')$, the same conclusion holds true.

In other words, if for a $\Gamma$ in the interior of $H_{10}^\infty(G, G')$, the corresponding lifting $B$ attains its maximum norm ($=1$), then the same is true on the whole interior of $H_{10}^\infty(G, G')$. The proof, as will be seen below, goes via the central solution $B_c$ corresponding to the Schur contraction $W(z) = \omega P_{T'}$.

### 2 Some properties of the Schur representation in commutant lifting theorem

In this section, we assemble some properties of the Schur representation in the commutant lifting theorem which we need in the proof of our main theorem. Although the first proposition follows from [6], we provide a direct proof here for the sake of completeness as well as for its role in our work.

**Proposition 1.** The functions $\Phi(z) = P_G(1 - z \Pi_A \omega P_{T'})^{-1} \Pi_A|_{G'}$ and $\Psi(z) = z \Pi \omega P_{T'}(1 - z \Pi_A \omega P_{T'})^{-1} \Pi_A|_{G'} + \Pi|_{G'}$ defined on $\mathbb{D}$ satisfy:
\[ \Phi \in H_1^\infty(G', G) \] (2.1)
\[ \Psi \in H_1^\infty(G', D_{T'}). \] (2.2)
PROOF. We show (2.1) by a direct computation. For \( g' \in \mathcal{G} \) and \( z \in \mathbb{D} \) we have

\[
\|P_G(1 - z\Pi \omega P_{\mathcal{F}})^{-1}\Pi \omega g'\|^2
\]
\[
= \|P_G\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|\omega P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
\leq \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|z\omega P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \||(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|g'\|^2 - \|g'\|^2
\]
\[
= \|g'\|^2
\]

But \( (1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g', g' \) = \( (\sum_{n=0}^{\infty} z^n(\omega P_{\mathcal{F}}\Pi A)^n g', g' ) \) = \( (g', g') \) = \|g'\|^2 because \( g' \) is orthogonal to \( \mathcal{F}' = \ker \omega \). So we have \( \|P_G(1 - z\Pi \omega P_{\mathcal{F}})^{-1}\Pi \omega g'\| \leq \|g'\|^2 \). This establishes (2.1).

Before passing on to the proof of (2.2) we note that in the course of the proof, we obtained the following useful estimate for \( z \in \mathbb{D} \) and \( g' \in \mathcal{G} \):

\[
\|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|z\omega P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 \leq \|g'\|^2. \quad (2.3)
\]

We now pass to the proof of (2.2) which is very similar to that of (2.1). With \( g' \in \mathcal{G} \) and \( z \in \mathbb{D} \) we have

\[
\|\Pi'(1 + z\omega P_{\mathcal{F}}(1 - z\Pi \omega P_{\mathcal{F}})^{-1}\Pi A)g'\|^2
\]
\[
= \|\Pi'(1 + z\omega P_{\mathcal{F}}(1 - z\Pi \omega P_{\mathcal{F}})^{-1})g'\|^2
\]
\[
= \|\Pi'(1 + z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
= \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
- \|P_G\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
\leq \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|z\omega P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2
\]
\[
\leq \|(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2 - \|z\omega P_{\mathcal{F}}\Pi A(1 - z\omega P_{\mathcal{F}}\Pi A)^{-1}g'\|^2.
\]

Now the conclusion in (2.2) follows from (2.3).

Remark 1. The functions \( \Phi \) and \( \Psi \) define contractive multiplication operators between appropriate \( H^2 \) spaces.

Proposition 2. Let \( W = \omega P_{\mathcal{F}} \oplus \Gamma(z)P_G \) where \( \Gamma \in H^\infty(\mathcal{G}, \mathcal{G}') \). Then, for \( z \in \mathbb{D} \), \( P_G(1 - z\Pi A W)^{-1} : \mathcal{D}_A \to H^2(\mathcal{D}_A) \) as a multiplication operator satisfies

\[
\|P_G(1 - z\Pi A W)^{-1}\| \leq \frac{1}{\sqrt{1 - \|\Gamma\|^2}}.
\]

PROOF. Let \( D(z) = (1 - z\Pi A W)^{-1} = \sum_{n=0}^{\infty} z^n D_n \). Then we have

\[
Q_n \frac{D(z) - \frac{I}{z}}{h} = Q_n \Pi A W Q_n D(z)h \quad \forall h \in \mathcal{D}_A;
\]

where

\[
Q_n(A_0 + \cdots + z^n A_n + \cdots) = A_0 + \cdots + z^n A_n.
\]
Therefore, for $h \in \mathcal{D}_A$, we get
\[
\|Q_n z^{D(z)-1} h\|_{H^2}^2 \leq \|Q_n z^{D(z)} P \| Q_n h\|_{H^2}^2 \\
= \|Q_n z^{D(z)} P \| Q_n h\|_{H^2}^2 + \|Q_n z^{D(z)} P \| Q_n h\|_{H^2}^2 \\
\leq \|P z^{D(z)} P \| Q_n h\|_{H^2}^2 + \|Q_n z^{D(z)} P \| Q_n h\|_{H^2}^2.
\]

Consequently,
\[
\sum_{i=1}^{n+1} \|D_i h\|^2 \leq \|P h\|^2 + \|\Gamma\|^2 \|P h\|^2 + \sum_{i=1}^{n} \left( \|P D_i h\|^2 + \|\Gamma\|^2 \|P h\|^2 \right)
\]

(1 - \|\Gamma\|^2) \sum_{i=1}^{n} \|P D_i h\|^2 + \|D_i h\|^2 \leq \|P h\|^2 + \|\Gamma\|^2 \|P h\|^2
\]

Now, recalling that $D_0 = I$, we obtain
\[
(1 - \|\Gamma\|^2) \sum_{i=2}^{n} \|P D_i h\|^2 \leq \|P h\|^2 + \|\Gamma\|^2 \|P h\|^2 + (1 - \|\Gamma\|^2) \|P h\|^2,
\]

which implies the proposition.

**Proposition 3.** Let $B_W$ be the intertwining lifting corresponding to the Schur contraction $W = W(\Gamma)$ with $\Gamma \in H_0^{\infty}(\mathcal{G}, \mathcal{G}')$. We denote by $B_c$ the central lifting, that is, the intertwining lifting associated to $W(z) = W(\Gamma)$ or equivalently $\Gamma = 0$. Then
\[
\|B_W - B_c\| h \leq \|\Gamma\| \|P h\| (1 - z I A W) = 0 \forall h \in H.
\]  

**Proof.** We have
\[
(B_c - B_W) h = \begin{bmatrix} 0 \\
\Pi' \omega P \Gamma (1 - z I A W) = 0 \end{bmatrix} - \begin{bmatrix} 0 \\
\Pi' \omega P \Gamma (1 - z I A W) = 0 \end{bmatrix} D_A h
\]

We note that
\[
\Pi' \omega P \Gamma (1 - z I A W) = \Pi' \omega P \Gamma (1 - z I A W) = \Pi' \omega P \Gamma (1 - z I A W) = \Pi' \omega P \Gamma (1 - z I A W)
\]

(2.4)

The relation (2.4) now follows quite easily from Remark 1.

**Lemma 2.1.** Let $z \in \mathcal{D}$ and $W = W(\Gamma)$, with $\Gamma \in H_0^{\infty}(\mathcal{G}, \mathcal{G}')$.

i) If there exists $\{h_j\}$, $h_j \in H$ such that $\|P h\| (1 - z I A W) = 0 \Rightarrow \|B_W - B_c\| h \rightarrow 0$.

ii) Suppose there exists $\{h_j\}$, $h_j \in H$ such that $\|P h\| (1 - z I A W) = 0 \Rightarrow \|B_W - B_c\| h \rightarrow 0$.

**Proof.** The proof of (i) is immediate from (2.4). To see (ii), we notice that
\[
P h\| (1 - z I A W) = P h\| (1 - z I A W) P h\| (1 - z I A W) = P h\| (1 - z I A W) P h\| (1 - z I A W)
\]

Let $Q(z) : \mathcal{G} \rightarrow \mathcal{G}$, $Q(z) = \Phi(z) \Gamma(z) P$. Then, $\forall z \in \mathcal{D}$, $\|Q(z)\| \leq \|\Gamma\| < 1$ by (2.1).

Consequently, $[1 - z \Phi(z) \Gamma(z) P]^{-1} \in H_0^{\infty}(\mathcal{G}, \mathcal{G})$. Hence the Lemma follows.
3 Main Result

We now state and prove the main theorem.

**Theorem 3.1.** Let \( ||A|| < 1. \) Then \( ||B_c|| = 1 \) implies for all \( \Gamma \in H^\infty(G, G') \), \( ||B_W|| = 1 \) where \( W = W(\Gamma) \). Conversely, if there exists \( \Gamma_0 \) with \( \Gamma_0 \in H^\infty(G, G') \) and such that \( ||B_W|| = 1 \), where \( W = W(\Gamma_0) \), then \( ||B_c|| = 1. \)

**Proof.** Suppose \( ||B_c|| = 1 \), then there exists \( \{ h_j \} \), \( ||h_j|| = 1 \) and \( D_{B_c}h_j \rightarrow 0. \)

Now, a straightforward computation gives (see (2.1) in [7])

\[
||D_{B_W}h_j||^2 = ||P_0(1 - z\Pi_AW)^{-1}D_Ah_j||^2 + \lim_{n \rightarrow \infty} (\Pi_AW)^nD_Ah_j^2
\]

Therefore, \( ||P_0(1 - z\Pi_AW)^{-1}D_Ah_j|| \rightarrow 0. \) Now by Lemma (2.1 ii) we have

\[
||P_0(1 - z\Pi_AW)^{-1}D_Ah_j|| \rightarrow 0. \] This in turn, by virtue of Lemma (2.1 i), now implies that \( ||(B_W - B_c)h_j|| \rightarrow 0. \)

On the other hand, we see that

\[
||D_{B_W}h_j||^2 = ||h_j||^2 - ||B_Wh_j||^2
\]

\[
= ||h_j||^2 - ||B_ch_j||^2 + (||B_ch_j||^2 - ||B_Wh_j||^2)
\]

\[
\leq ||D_{B_c}h_j||^2 + 2(||B_ch_j|| - ||B_Wh_j||)
\]

\[
\leq ||D_{B_c}h_j||^2 + 2||B_ch_j - B_Wh_j||
\]

Each of the terms on the right-hand side of the last inequality goes to zero, which implies \( ||D_{B_W}h_j|| \rightarrow 0. \) Hence, the desired result, i.e. \( ||B_W|| = 1 \), now follows.

Conversely, suppose now that \( ||B_W|| = 1. \) Then there exists \( \{h_j\}, ||h_j|| = 1, h_j \in \mathcal{H} \) such that \( ||D_{B_W}h_j|| \rightarrow 0. \)

Let

\[
B_W = \left[ \begin{array}{c} \Pi'W(1 - z\Pi_AW)^{-1}D_A \\ \Pi W(1 - z\Pi_AW)^{-1}D_A \end{array} \right],
\]

\[
D(z) = (1 - z\Pi_AW)^{-1}; (WD)_n = W_nD_0 + \cdots + W_0D_n.
\]

\[
||D_{B_W}f||^2 = ||D_Af||^2 - \sum_{n=0}^{\infty} ||(\Pi'(WD)_nD_Af||^2
\]

\[
= ||D_Af||^2 - \sum_{n=0}^{\infty} (||WDf||^2 - ||\Pi_A(WD)_nD_Af||^2)
\]

\[
= \sum_{n=0}^{\infty} ||(WDf)(WD)_nD_Af||^2
\]

(3.1)

for all \( f \in \mathcal{H} \), where for any \( A \in H^\infty(\mathcal{H}_1, \mathcal{H}_2) \), we denote by \( (A)_n \) the \( n \)-th Fourier coefficient of \( A \).

Now, since \( z\Pi_AW(z)D(z) = D(z) - I \), we have by comparing Fourier coefficients

\[
D_0 = I; \quad D_n = \Pi_A(WD)_n, \quad n \geq 1.
\]

(3.2)

Let \( P_n(A_0 + A_1z + \cdots) = A_0 + A_1z + \cdots + A_nz^n \). Then

\[
P_n(WD)_n(z)D_Af = P_nW(z)P_nD(z)D_Af
\]

\[
||P_nD(z)D_Af||^2 = ||D_Af||^2 + \sum_{i=1}^{n} ||D_iD_Af||^2
\]

\[
||P_n(WD)_n(z)D_Af||^2 =||(WD)_0D_Af||^2 + \sum_{i=1}^{n} ||(WD)_iD_Af||^2.
\]
Then by (3.1) and (3.2) we have

\[ \|D_{B_w} f\|^2 = \|D_A f\|^2 - \lim_{n \to \infty} \sum_{i=0}^{n+1} (\|f(WD)D_A f\|^2 - \|D_iD_A f\|^2) \]
\[ \geq \|D_A f\|^2 - \|f(WD)D_A f\|^2 + \sum_{i=1}^{n+1} \|D_iD_A f\|^2 \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n WP_n DDAf\|_{H^2}^2) \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n \omega P_{F} DDAf\|_{H^2}^2) \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n \omega P_{F} DDAf\|_{H^2}^2) \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n \omega P_{F} DDAf\|_{H^2}^2) \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n \omega P_{F} DDAf\|_{H^2}^2) \]
\[ = \lim_{n \to \infty} (\|P_n DDAf\|_{H^2}^2 - \|P_n \omega P_{F} DDAf\|_{H^2}^2) \]

Note that \(\|P_G(1-z\Pi_A W)^{-1} DDAf\|_{H^2}^2 < \infty\) by Proposition 2. Hence, \(\lim_{n \to \infty} \|P_n P_G DDAf\|_{H^2}^2 = \|P_G(1-z\Pi_A W)^{-1} DDAf\|_{H^2}^2\) and

\[ \lim_{n \to \infty} \|P_n \omega P_{F} DDAf\|_{H^2}^2 = \|P_n \omega P_{F} DDAf\|_{H^2}^2. \]

Therefore, it follows that

\[ \|D_{B_w} f\|^2 \geq \|P_G(1-z\Pi_A W)^{-1} DDAf\|_{H^2}^2 - \|\Gamma\|^2 \|P_G(1-z\Pi_A W)^{-1} DDAf\|_{H^2}^2 \]
\[ = (1 - \|\Gamma\|^2) \|P_G(1-z\Pi_A W)^{-1} DDAf\|_{H^2}^2. \]

Now, with \(f = h_j\) and letting \(j\) go to infinity we obtain

\[ \|\Gamma\|^2 \|P_G(1-z\Pi_A W)^{-1} DDAh_j\|_{H^2}^2 \to 0 \]

By virtue of Lemma 2.1(i), \(\|(B_W - B)\| \to 0\) and consequently \(\|B_c\| = 1\). The proof of the theorem is now complete.

4 Concluding Remarks

i) We note that our main result implies that if \(\|A\| < 1\) and \(\|B_c\| = 1\) then for all Schur contractions \(W\) associated to \(\Gamma\) with \(\|\Gamma\| < 1\) we have \(\|B_W\| = 1\). However, the condition that \(\Gamma\) be a strict contraction cannot be relaxed. Indeed, M. Bakonyi has given an example (see [2]) in which \(A\) is a strict contraction and \(\|B_c\| = 1\). By virtue of the commutant lifting theorem, we know that there exists a contractive intertwining lifting \(B = B_W(\Gamma_1)\) of Bakonyi's operator \(A\) such that \(\|B\| = \|A\| < 1\). As a consequence of our theorem, \(\Gamma_1\) must have norm one.

ii) From the discussion in i), it also follows that, the association \(\Gamma \to B_W(\Gamma)\) from \(H_1^\infty(G, G')\) into \(L(H, K')\) may not be norm continuous. To see this, in Bakonyi's example, we let \(\Gamma_\alpha = \alpha \Gamma_1\), \(\alpha \in \mathbb{C}\), \(|\alpha| \leq 1\). By virtue of our result, \(\|B_W(\Gamma_\alpha)\| = 1\) for all \(|\alpha| < 1\). Thus,

\[ \lim_{|\alpha| \to 1, \alpha \to 1} \|\Gamma_\alpha - \Gamma_1\| = 0 \text{ and } \liminf_{|\alpha| \to 1, \alpha \to 1} \|B_W(\Gamma_\alpha) - B_W(\Gamma_1)\| \geq 1 - \|A\| > 0. \]
iii) Let \(|A| < 1\). To complete the analogy with the classical maximal principle, we show that in the case when both \(G\) and \(G'\) are nontrivial spaces,

\[
\sup \{ \|B_{W(\Gamma)}\| : \Gamma \in H_{10}^{\infty}(G, G') \} = 1. \tag{4.1}
\]

We now proceed to prove (4.1). Clearly, there exists \(0 \neq g_0 \in G\) and \(h' \oplus d_0 \in G'(\subseteq D_T \oplus D_A)\) such that \(\|g_0\| = \|h' \oplus d_0\|\). Let \(\gamma\) be a contraction from \(G\) to \(G'\) satisfying \(\gamma g_0 = h' \oplus d_0\) and let \(h = -TD_A^{-1}d_0 + D_A^{-1}g_0\). Then \(h \neq 0\) since obviously \(D_A h \neq 0\). On the other hand, it is easy to see that \(Xh = 0\) where

\[
X := \left[ \frac{\Pi_A(\omega P_T \oplus \gamma P_{D_T})D_A}{D_A P_T D_A} \right].
\]

The above discussion shows that if we let \(\Gamma(z) \equiv \gamma, z \in \mathbb{C}\), then the contractive one step intertwining lifting \(A_1 = P_{H_1}B_{W(\Gamma)}\) of \(A\) is of norm one, where \(H_1 = H' \oplus D_T\) and \(H_1' \subseteq K'\) by identifying \(D_T\) as the space of all constant functions in \(H^2(D_T)\). Indeed, by virtue of corollary 1.6 in Chap V of [3], we have \(X = WD_A\), for some adequate unitary operator \(W\), consequently \(\|A_1\| = 1\). Now if we take \(\Gamma_\alpha = \alpha \Gamma, \alpha \in \mathbb{D}\) then by (1.5) we can easily infer that \(\lim_{\alpha \to 1} \|A_{1, \alpha} - A_1\| = 0\), where \(A_{1, \alpha} = P_{H_1'} B_{W(\Gamma_\alpha)}\). Consequently, \(\sup \{ \|B_{W(\Gamma_\alpha)}\| : \alpha \in \mathbb{D} \} = 1\). Therefore, we have (4.1) by noting that \(\Gamma_\alpha \in H_{10}^{\infty}(G, G')\) for \(\alpha \in \mathbb{D}\).

In the other cases, that is when either \(G = \{0\}\) and/or \(G' = \{0\}\), clearly \(H_{10}^{\infty}(G, G') = \{0\}\) and hence the intertwining lifting of \(A\) is unique. The commutant lifting theorem implies that there is an intertwining lifting \(B_{opt}\) of \(A\) so that \(\|B_{opt}\| = \|A\|\). Uniqueness of the lifting now implies that \(B_{opt}\) must be the central lifting and hence the supremum in (4.1) is equal to \(\|A\|\).

**Acknowledgements**

This research was partially supported by grants from the National Science Foundation DMS-9400615. Also, thanks are due to Prof. C. Foias for suggesting the problem to me and for subsequent useful discussions.

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AMS subject classification (1991): 47A20, 47A57

Submitted: December 5, 1996