PERIODIC LONGITUDINAL MOTIONS OF A VISCOELASTIC ROD

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Abstract. In this paper, we investigate the existence of a time-periodic solution for a quasilinear hyperbolic-parabolic equation arising in the study of longitudinal motion of a one-dimensional, viscoelastic rod. By employing the implicit function theorem, we show that when the specified forcing is sufficiently small, a nontrivial time-periodic solution exists.

1. Introduction

We consider a model of longitudinal motions of a viscoelastic rod

$$w_{tt} = \left[\varphi'(w_s) + \sigma(w_s, w_{st})\right]_s \tag{1.1}$$

Here $w(s,t)$ is the position at time t of the material point whose rest position is $w^*(s) = s$ ($s \in [0,1]$) so the velocity is $v = w_t$ and the strain (deformation) is $u = w_s - 1$ (whence $u \equiv 0$ for the static configuration $w \equiv w^*$) while $z = u_t = w_{st} = v_s$ is the strain rate. For simplicity we omit consideration of possible body forces, have restricted our attention to a homogeneous rod (with constant mass density equal to 1), and take the nonlinear constitutive function

$$n(u,z) = \varphi'(u) + \sigma(u,z) \tag{1.2}$$

not to be directly dependent on s, t. The potential energy density is φ and the term σ represents a viscous dissipation so n is the contact force. [Note that u, z are physical variables (components of the state) but also appear here as independent variables in defining n in (1.2); we will sometimes clarify this distinction by writing $N(s,t) := n(u(s,t),\, u_t(s,t))$.] Equations of quite similar form to (1.1) also apply to compressible gas in a pipe, shearing motion of a plate, etc.

We will always adjoin to (1.1) the boundary conditions

$$v\Big|_{s=0} \equiv 0 \qquad n\Big|_{s=1} = y(t). \tag{1.3}$$

Note that the nonautonomous boundary condition $N(1,t) = y(t)$ represents an external forcing acting at the endpoint of the rod, so we cannot expect conservation of energy

We will actually work with the equivalent coupled system

$$u_t = v_s, \quad v_t = N_s \qquad \text{with (1.3)}, \tag{1.4}$$

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noting that $v = w_t$ and $u_t = (w_s - 1)_t = w_{st} = w_{ts} = v_s$; we will also consistently write z for $v_s = u_t$. We then write the solution vectorially as

$$X(s,t) = (u(s,t),\, v(s,t)) \quad \text{for } (s,t) \in \mathcal{Q} = [0,1] \times [0,T].$$

We will always impose (1.3) so the weak form of the PDE $v_t = N_s$ will be

$$\langle \eta, v_t \rangle + \langle \eta_s, N \rangle = (\eta y)\Big|_{s=1} \tag{1.5}$$

for suitable test functions η.

For the initial/boundary value problem (IBVP) we additionally specify as data the initial state:

$$X(\cdot, 0) = x \qquad \text{with } x = (u_0(s),\, v_0(s)) \text{ for } s \in [0,1]. \tag{1.6}$$

We then designate the problem as IBVP(x, y) to indicate the dependence on this specification of data, correspondingly writing $X(t; x, y)$ for the solution.

This IBVP was considered in [3] under assumptions on n — see Section 2 — permitting nonlinear dependence on both the strain and the strain rate while obtaining well-posedness and ensuring that one never develops infinite compression, i.e., ensuring that $u > -1$. See [3] and [4] for more detailed comment on the physics and the use of other boundary conditions as well as related prior work on the IBVP (e.g., [9], [6], [11], [8], [1], [2], [12], etc.). Although the paper [3] considered the IBVP for various choices of inhomogeneous boundary conditions, for expository convenience we restrict our attention here to the specific configuration in which the rod is fixed at one end with a specified force applied at the other end (compare [7]).

In contrast to [3], the primary concern here is not directly with the IBVP. Instead, the conditions (1.3) are imposed without specifying any initial data and one asks whether, assuming the forcing $y(\cdot)$ is periodic, there must be some periodic solution[1] i.e., whether there will be some set of initial data $x = (u_0, v_0)$ for which the solution $t \mapsto (u, v)$ of the IBVP(x, y) is similarly periodic in t. Our principal result will be:

Theorem 1.1. *Fix $T > 0$ and let the constitutive function n satisfy the conditions (H0), (H1), (H2) of Subsection 3.1. Then for any sufficiently small y in $\mathcal{Y} = \{y \in \mathbb{H}^1[0,T] : y(T) = y(0)\}$ there exists a correspondingly periodic solution X of* (1.4) *with* (1.3) *such that*

$$X(t) = (u, v) \in \mathcal{X} = \mathbb{H}^1[0,1] \times \mathbb{H}^2[0,1]$$

with $X(T) = X(0)$.

1.1. Notation.

- We always implicitly assume that the rod is consistently oriented by its labeling and cannot penetrate itself: if $s_1 < s_2$, then we must have $w(s_1, t) < w(s_2, t)$ so the model would be physically meaningful only if $w_s > 0$ so $u > -1$ and we take this as an implied constraint. Thus,

[1]Note that, in the context of the IBVP, w, u, v, etc., are intended for all $t \geq 0$ but for the periodicity problem are only needed up to T and should then repeat.

if one were to have $u(s,t) \to -1$ (infinite compression) there must be a strongly opposing elastic response: $\varphi'(u) \to -\infty$; similarly we expect $\varphi'(u) \to \infty$ when $u \to \infty$. Thus, the physical variables (as w, u, v, etc.) are considered for $(s,t) \in \mathcal{Q} = [0,1] \times [0,T]$ while the constitutive function $n(\cdot,\cdot)$ should need to be determined only for $(u,z) \in \Omega = (-1,\infty) \times \mathbb{R}$.

- For a solution $X = (u,v)$ of (1.4), (1.3) it will be convenient for us to set $\quad N(s,t) = n(u(s,t), u_t(s,t)) = (\varphi'(u) + \sigma(u,u_t))\Big|_{(s,t)}$ for all $(s,t) \in \mathcal{Q}$, also extending to $t \in [0,\infty)$.
- For any function g, possibly of both space and time, we denote
$$\|g(\cdot,t)\| := \left(\int_0^1 g^2(s,t)\,ds\right)^{1/2},$$
i.e., we will always use $\|\cdot\|$ (without any subscript) to denote the L^2-norm with respect to the space variable $s \in [0,1]$. Further, we write
$$\langle f, g\rangle = \int_0^1 f(s)g(s)\,ds$$
for the inner product in $L^2(0,1)$ as well as related duality products.
- We consider $T > 0$ as fixed and set $\mathcal{Q} = [0,1] \times [0,T]$; we then set
$$\|g\|_{L^2(\mathcal{Q})} = \left(\int_{\mathcal{Q}} g^2(s,t)\,ds\,dt\right)^{1/2};$$
with $\|g\|_{L^\infty(\mathcal{Q})}$ defined correspondingly.
- For $k \in \mathbb{N}$, denote the Hilbert space $\mathbb{H}^k = \mathbb{H}^k[a,b]$ by
$$\mathbb{H}^k = \left\{ u(\cdot) : \|u\|_{\mathbb{H}^k} = \left(\sum_{i=0}^{k} \|u^{(i)}\|^2\right)^{1/2} < \infty \right\},$$
where $u^{(i)}$ denotes the i-th derivative; when $i = 1$, we may alternatively write u' for $u^{(1)}$. We will set
$$\mathcal{X} = \mathbb{H}^1[0,1] \times \mathbb{H}^2[0,1] \qquad \mathcal{Y} = \{y \in \mathbb{H}^1[0,T] : y(T) = y(0)\}$$

2. Strategy

Suppose the IBVP(x,y) specified by (1.4) would be solvable for initial data $x \in \mathcal{X}$ as in (1.6) and boundary data $y \in \mathcal{Y}$ as in (1.3) to get a unique solution $X = X(\cdot; x, y)$ on $[0,T]$. This may then be used to define a function

$$\mathcal{F}(x,y) = X(T) - x. \tag{2.1}$$

Suppose, further, that for some such y, there would be a solution, denoted by $x = \mathcal{G}(y)$, of the equation

$$\mathcal{F}(x,y) = 0 \tag{2.2}$$

so $\mathcal{F}(\mathcal{G}(y), y) \equiv 0$. By our definition (2.1) of $\mathcal{F}$, this means that $X(T) - x = 0$ for the solution of the IBVT(x, y). Now obtain $\hat{X}$ by considering the IBVP$(\hat{x}, \hat{y})$ on the time interval $[T, 2T]$ with initial data $\hat{x} = X(T)$ and boundary data $\hat{y}(t) = y(t)$. If we would know that y is time-periodic (so $y(t+kT) = y(t)$ for $t \geq 0$ and $k = 1, 2, \ldots$) then, except for the translation in t, IBVP$(\hat{x}, \hat{y})$ on $[T, 2T]$ is identical to IBVT(x, y) on $[0, T]$ so the respective solutions are correspondingly identical:

$$X(t; x, y) = \hat{X}(t; \hat{x}, \hat{y}) = X(t - T; x, y) \qquad \text{for } T \leq t \leq 2T.$$

In particular, we have $\hat{X}(2T) = X(2T - T) = X(T) = \hat{x}$ so $\mathcal{F}(\hat{x}, \hat{y}) = 0$ whence $\hat{x} = \mathcal{G}(\hat{y})$ Proceeding inductively in this manner shows that the solution $X(\cdot; \mathcal{G}(y), y)$, initially constructed only for $t \in [0, T]$, extends periodically to all $t \geq 0$ and, indeed, to all $t \in \mathbb{R}$. The argument sketched above is our strategy to attain the goal of the paper: showing existence of a periodic motion $X = X(\cdot; \mathcal{G}(y), y)$ for each periodic forcing function y in an appropriate domain $\mathcal{Y}_0$.

The key to our argument is use of the Implicit Function Theorem (IFT) – once one has verified its assumptions in the setting of the rod model – to see that $y \mapsto \mathcal{G}(y) = x$ is implicitly defined by (2.2). The setting for the IFT is a pair of Banach spaces $\mathcal{X}, \mathcal{Y}$, a pair of elements $\hat{x} \in \mathcal{X}, \hat{y} \in \mathcal{Y}$, and a function $f : \hat{\mathcal{X}} \times \hat{\mathcal{Y}} \to \mathcal{Y}$ where $\hat{\mathcal{X}}, \hat{\mathcal{Y}}$ are neighborhoods of $\hat{x}, \hat{y}$, respectively.

Theorem 2.1. (IFT) *Assume that:*

- (A0) *The function f is defined and continuous: $\hat{\mathcal{X}} \times \hat{\mathcal{Y}} \to \mathcal{X}$,*
- (A1) *f is Fréchet differentiable on $\hat{\mathcal{X}} \times \hat{\mathcal{Y}}$,*
- (A2) *$f(\hat{x}, \hat{y}) = 0$,*
- (A3) *The partial derivative $\partial f / \partial x$ (evaluated at $(\hat{x}, \hat{y})$) is invertible,*

Then there exists a neighborhood $\mathcal{Y}_0$ of $\hat{y}$ (with $\mathcal{Y}_0 \subset \hat{\mathcal{Y}}$) and a differentiable function $g : \mathcal{Y}_0 \to \hat{\mathcal{Y}}$ such that $g(\hat{y}) = \hat{x}$ and $f(g(y), y) \equiv 0$ on $\mathcal{Y}_0$.

We are led to this strategy by the initial observation that without forcing (i.e., if we were to use $y = \hat{y} \equiv 0$ in (1.3)) we could have the trivial solution $w = w^*$ satisfying (1.1) so $u = (w - w^*)_s \equiv 0$ and $v = (w - w^*)_t \equiv 0$ giving $(u, v) = (0, 0)$ at $t = 0$ – which we write simply as '$\hat{x} = 0$'. As this is a steady state solution it is obviously periodic for arbitrary period length T, as is y. Thus we begin knowing the assumption (A2) and follow this strategy by proceeding to justify the remaining assumptions: (A0), (A1), and (A3) on the basis of the hypotheses (H0), (H1), (H2) we will impose on the constitutive function n.

For our application we will take $\mathcal{X} = \mathbb{H}^1(0, 1)) \times \mathbb{H}^2(0, 1)$ and $\mathcal{Y} = \mathbb{H}^1(0, T)$. Since we will be working with the trivial forcing $\hat{y} = 0$, all the forcing functions y in the neighborhood $\mathcal{Y}_0$ may be viewed as "small data" as in the abstract. This "smallness" requirement is certainly essential to the IFT and so to our use of this strategy, but at this point it is not clear whether it is essential to the physics of our application.

3. Hypotheses and Estimates

In this section we address the verification of the assumptions (A0) and (A1) — i.e., showing that the function $\mathcal{F}$ of (2.1), defined by the IBVP for (1.4), (1.3), is indeed well-defined and differentiable. Throughout we will assume without further mention that, as in (2.1), we have fixed $T > 0$ and $n(\cdot,\cdot)$ satisfying (H0), (H1), (H2); further, we assume that $X = (u, v)$ satisfies (1.4), (1.3) for some $y \in \mathcal{Y}$ and some suitable initial data $X(\cdot, 0) = x$.

Although (A0), the existence of solutions for the IBVP, was already treated in the paper [3], our major effort in this section is to provide (for the particular boundary conditions considered here) the relevant *a priori* estimates bounding various Sobolev norms which are key to obtaining well-posedness of (1.4), as well as differentiability of (2.1) Our arguments here for these estimates largely follow the presentation in [7].

3.1. **Hypotheses on the Constitutive Function.** Besides the form of the system, we will impose throughout the hypotheses (H0), (H1), (H2) on the constitutive function n:

(H0) We assume $n(\cdot,\cdot)$ is smooth on Ω with φ minimized at $u = 0$, so that

$$\varphi'(0) = 0 \text{ and } \sigma(u, 0) \equiv 0.$$

(H1) Uniform ellipticity: There exists $m > 0$ such that

$$n_z(u, z) = \sigma_z(u, z) \geq m.$$

This in particular implies

$$\sigma(u, z)\, z \geq mz^2 \quad \text{and } \sigma(u, z)/z \geq m.$$

(H2) The dissipative damping dominates effectively: for some κ

$$(i) \quad |n_u| \leq \kappa n_z \qquad \text{and} \qquad (ii) \quad |n_u| \leq \kappa \sqrt{n_z}\,\sqrt{\sigma/z} \text{ on } \Omega.$$

3.2. **Principal Estimates.**

Lemma 3.1. *Suppose n satisfies* (H0), (H1), (H2) *and assume that*

$$E_x = E(0) := \tfrac{1}{2}\|v_0\|^2 + \textstyle\int_0^1 \varphi(u_0)\, ds < \infty \qquad \text{and}$$
$$\mathcal{N}_x = \sup_{s\in[0,1]} |n_z((u_0)_s,\, (v_0)_s))| \; < \infty. \tag{3.1}$$

Now let $X(t) = (u(\cdot, t), v(\cdot, t))$ be a solution of IBVP(x, y), that is (1.4) *with initial data $x = (u_0, v_0)$ and satisfying boundary condition* (1.3)*. Let $x = (u_0, v_0) \in \mathcal{X} = \mathbb{H}^1(0, 1) \times \mathbb{H}^2(0, 1)$ and $y \in \mathbb{H}^1(0, T)$.*

Then

$$u \in L^\infty((0, T] \to \mathbb{H}^1), \quad v \in L^\infty((0, T] \to \mathbb{H}^2) \quad \text{and}$$
$$\sup_{t\in[0,T]} \left(\frac{1}{2}\|v(\cdot, t)\|^2 + \int_0^1 \varphi(u(s, t))\, ds\right) \quad \leq E(0) + \frac{4}{m}\|y\|^2_{L^2[0,T]}.$$

If we set

$$G(T) := E(0) + \tfrac{4}{m}\int_0^T y^2(\tau)\,d\tau, \qquad G_0(T) := 2\|u_0\|^2 + \tfrac{8T}{m}G_T,$$
$$G_1(T) := (\mathcal{N}_x^2\left(\kappa^2+1\right)\|^2 + \|v_0\|^2) + \frac{2\kappa^2}{4m}G_T + \frac{4}{m}\|y'\|^2_{L^2[0,T]} \tag{3.2}$$
$$G_2(T) := Te^{\kappa T}\left(\|(u_0)_s\| + \tfrac{1}{m}G_1(T)\right).$$

then the solution $X = (u,v)$ also satisfies the bounds

$$\|v(\cdot,t)\|^2 \le 2G(T), \quad \|u(\cdot,t)\|^2 \le G_0(T), \quad \|u_s(\cdot,t)\| \le G_2(T),$$
$$\|v_t(\cdot,t)\| \le G(T), \qquad \|v_{ss}(\cdot,t)\| \le \frac{\sqrt{G_1(T)}}{m} + \kappa G_2(T). \tag{3.3}$$

for all $0 < t \le T$.

Proof. The classical energy (kinetic plus potential) is

$$E(t) := \frac{1}{2}\|v(\cdot,t)\|^2 + \int_0^1 \varphi(u(s,t))\,ds \tag{3.4}$$

which we differentiate and use (1.5) with $\eta = v$ (integrating by parts and using the boundary conditions) for all $t > 0$; we readily obtain

$$E(t) + \int_0^t\int_0^1 \sigma(u,u_t)u_t\,ds\,d\tau = E(0) + \int_0^t y(\tau)v\Big|_{x=1}\,d\tau. \tag{3.5}$$

Observe that for any function $f : [0,1] \to \mathbb{R}$, with $f(0) = 0$, we have

$$|f(y)| \le \|f_x\| \quad \text{for } y \in (0,1]. \tag{3.6}$$

so $|v(1,t)| \le \|v_s(\cdot,t)\|$. Using this estimate, together with (H1) in (3.5), along with the fact that $u_t = v_s$ (see (1.4)) and Young's inequality, we readily obtain

$$\begin{aligned} E(t) + \frac{1}{2}\int_0^t\int_0^1 \sigma(u,u_t)u_t\,ds\,d\tau + \frac{m}{4}\int_0^t \|v_s\|^2 d\tau \\ \le E(0) + \frac{4}{m}\int_0^t y^2(\tau)\,d\tau \quad = G(T). \end{aligned} \tag{3.7}$$

From (3.7), the Gronwall Inequality gives

$$\|v(\cdot,t)\|^2 \le 2G(T) \quad \text{for all } 0 < t \le T. \tag{3.8}$$

Moreover,

$$\int_0^T \|v_s\|^2\,d\tau \le \frac{4}{m}\left(E(0) + \frac{4}{m}\int_0^T y^2(\tau)\,d\tau\right) = \frac{4}{m}G_T. \tag{3.9}$$

Observe that

$$u(s,t) = u_0(s) + \int_0^t u_t(s,\tau)\,d\tau = u_0(s) + \int_0^t v_s(s,\tau)\,d\tau. \tag{3.10}$$

Applying Cauchy-Schwartz, we readily have

$$u^2(s,t) \le 2u_0^2(s) + 2t\int_0^t v_s^2(s,\tau)\,d\tau.$$

Integrating in s from 0 to 1, we obtain for each $0 < t \le T$

$$\|u(\cdot,t)\|^2 \le 2\|u_0\|^2 + \frac{8T}{m}G(T) \ = G_0(T). \tag{3.11}$$

To obtain higher order estimates, we use the Chain Rule, observing that (1.4) yields

$$v_t = \partial_s\, n(u,u_t) = n_u\Big|_{(u,z)} u_s + n_z\Big|_{(u,z)} v_{ss} \tag{3.12}$$

Note, in particular, that the initial data should satisfy

$$v_t\Big|_{t=0} = n_u(u_0,(v_0)_s)\,(u_0)_s + n_z(u_0,(v_0)_s)\,(v_0)_{ss}. \tag{3.13}$$

Thus, in assuming $u_0 \in \mathbb{H}^1$ and $v_0 \in \mathbb{H}^2$, we also assume that $\|v_t(\cdot,0)\| < \infty$.

Much as in (3.12), we get from (1.4)

$$v_{tt} = \partial_s\partial_t \left[n(u,z)\Big|_{(u(s,t),u_t(s,t))}\right] = [n_u u_t + n_z v_s]_s \tag{3.14}$$

Observe that $n(u,u_t)\Big|_{s=1} = y(t)$ which implies $\partial_t n(u,u_t)\Big|_{s=1} = y'(t)$. Also, $v\Big|_{s=0} \equiv 0$ implies $v_t\Big|_{s=0} = 0$. Multiplying (3.14) by v_t, integrating by parts in s and using the boundary conditions yields

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|v_t\|^2 \quad &= -\int_0^1 \partial_t\, n(u,u_t)\, v_{ts}\, ds + y'(t) v_t\Big|_{s=1} \\ &\le -\int_0^1 [n_y(u,u_t)u_t + n_z(u,u_t)u_{tt}]\, v_{ts}\, ds + y'(t)\|v_{ts}\|,\end{aligned} \tag{3.15}$$

where we have already used (3.6) in (3.15). Rearranging the above inequality and using (H1), Young's inequality, the fact that $v_{ts} = u_{tt}$, and (H2), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_t\|^2 + \frac{m}{2}\|u_{tt}\|^2\, ds + \frac{m}{4}\int_0^1 n_z(u,u_t)u_{tt}^2\, ds \\ &\le \int_0^1 n_y(u,u_t)u_t u_{tt}\, ds + \frac{4}{m}(y'(t))^2 \\ &\le \kappa \int_0^1 \sqrt{n_z(u,u_t)}\sqrt{|\sigma(u,u_t)|\,|u_t|}\ |u_{tt}|\, ds + \frac{4}{m}(y'(t))^2 \\ &\le \frac{m}{4}\int_0^1 n_z(u,u_t)u_{tt}^2\, ds + \frac{2\kappa^2}{m}\int_0^1 \sigma(u,u_t)u_t\, ds + \frac{4}{m}(y'(t))^2,\end{aligned} \tag{3.16}$$

where in (3.16), we also used the fact that $\sigma(u,z)z \ge m > 0$ where defined. Integrating the resulting inequality and using (3.7), we obtain

$$\|v_t\|^2 + \frac{m}{2}\int_0^T \|u_{tt}\|^2\, dt \le \|v_t(\cdot,0)\|^2 + \frac{2\kappa^2}{4m}G(T) + \frac{4}{m}\|y'\|^2_{L^2[0,T]}. \tag{3.17}$$

From (1.4), we have

$$v_t\Big|_{(s,0)} = n_u(u_0),(v_0)_s)\,u_0 + n_z(u_0,(v_0)_s)\,(v_0)_{ss}$$

Using (H2), we have

$$|v_t\Big|_{(s,0)}|^2 \le (\kappa^2+1)|n_z(u_0(s)m(v_0)_s(s))|^2(|(u_0)_s(s)|^2 + |(v_0)_{ss}(s)|^2).$$

Due to (3.17), with $G_1(T)$ as in (3.2), we immediately obtain

$$\|v_t\|^2 + \frac{m}{2}\int_0^T \|u_{tt}\|^2 \le G_1(T). \tag{3.18}$$

Note that from the equation (1.4), we readily obtain

$$v_{ss} = u_{st} = \frac{v_t}{n_z(u,u_t)} - \left(\frac{n_u(u,u_t)}{n_z(u,u_t)}\right)u_s. \tag{3.19}$$

This implies

$$u_s(s,t) = u_s(s,0) + \int_0^t \frac{v_t}{n_z(u,u_t)}\,d\tau - \int_0^t \left(\frac{n_u(u,u_t)}{n_z(u,u_t)}\right)u_s\,d\tau.$$

Now taking the L^2- norm in the space variable in this equation, using (H1), (H2)(i), and the Minkowski inequality, we readily obtain

$$\|u_s\| \le t(\|u_s(\cdot,0)\| + \frac{1}{m}G_1(T) + \kappa\int_0^t \|u_s\|\,d\tau.$$

Applying the (integral) Gronwall inequality, we obtain

$$\|u_s\| \le Te^{\kappa T}\left(\|(u_0)_s\| + \frac{1}{m}G_1(T)\right) = G_2(T). \tag{3.20}$$

In view of (H2) and (3.19), we also have

$$|v_{ss}| \le \frac{|v_t|}{m} + \kappa|u_s|.$$

Consequently, from (3.20) and (3.18) and interpolation, we get

$$\|v_{ss}\| \le \frac{\sqrt{G_1(T)}}{m} + \kappa G_2(T). \tag{3.21}$$

Thus we have shown, given $y \in \mathcal{Y} = \mathbb{H}^1([0,T])$, that if a solution X of IBVP(x,y) starts in $\mathcal{X} = \mathbb{H}^1(0,1)) \times \mathbb{H}^2(0,1)$ at $t=0$, then it stays in $\mathcal{X}$ with a uniform bound. □

3.3. **Pointwise Lower Bound for u.** An important step in our analysis of the IBVP is showing, for some suitable set of solutions, that one can restrict the domain of n to a compact subset $\Omega_0 \subset \Omega$, i.e., restricting the range of $[u, z]$ over $\mathcal{Q}$. This essentially requires pointwise bounds for u, z and, in view of the embedding of $\mathbb{H}^1[0,1]$ into $C([0,1])$ and the estimates (3.20) and (3.21) for $\|u_s\|$ and $\|v_{ss}\| = \|(u_t)_s\|$, we already have such pointwise bounds:

$$|u(s,t)| \le M_u \qquad |u_t(s,t)| \le M_z$$

Since $\Omega = (-1,\infty) \times \mathbb{R}$, any upper and lower bounds for u_t and any upper bound for u will serve our purpose, but a lower bound $\underline{u} \le u$ is useful precisely if $\underline{u} > -1$.

It is not difficult to show that the estimates M we have obtained from (3.20), (3.21) can be made arbitrarily close to 0 by taking the data x, y to be small enough in $\mathcal{X} \times \mathcal{Y}$. If the force y is suitably small, then the solution can stay near the (trivial) equilibrium in sup norm. With such a bound on the data y for Theorem 1.1 we have an arbitrarily small uniform bound M_u so $\underline{u} = -M_u$ is a lower bound for u we can take

$$\Omega_0 = [\underline{u}, M_u] \times [-M_v, M_v] \quad \subset \ \Omega.$$

The significance of having a compact domain Ω_0 for the constitutive function n is that we have assumed (H0), so n – and all its derivatives of the form ${\partial_u}^j {\partial_z}^k n$ which have the same domain – will necessarily be bounded. In particular, n must be Lipschitzian where relevant with a constant given by the bounds on $|n_u|$ and n_z over Ω_0. Similarly, using the Chain Rule, we see that if one had two sets of arguments $(\hat{u}, \hat{z})$ and $(\tilde{u}, \tilde{z})$ with difference $(\bar{u}, \bar{z})$, then

$$\bar{n} = n(\hat{u}, \hat{z}) - n(\tilde{u}, \tilde{z}) = n_u \bar{u} + n_z \bar{z} + R$$

where the remainder (error term) is uniformly $\mathcal{O}(\bar{u}^2 + \bar{z}^2)$ since the functions n_{uu}, n_{zz}, n_{uz} are bounded uniformly on Ω_0. It is this uniformity which immediately justifies the formal differentiation of n in IBVP(x, y) as a Fréchet derivative.

Formally, to linearize around a particular solution $X_* = (u_*, v_*)$ of (1.4) we compute the coefficient functions

$$c_*^2 = \frac{\partial n}{\partial u}(u_*, (u_*)_t) \qquad \alpha_* = \frac{\partial n}{\partial z}(u_*, (u_*)_t) \tag{3.22}$$

and then consider the linear system

$$\mathfrak{u}_t = \mathfrak{v}_s \qquad \mathfrak{v}_t = \left[c_*^2 \mathfrak{u} + \alpha_* \mathfrak{u}_t\right]_s \tag{3.23}$$

for the corresponding variations; we adjoin the boundary conditions

$$\mathfrak{v}\Big|_{s=0} = 0 \qquad \left[c_*^2 \mathfrak{u} + \alpha_* \mathfrak{v}_s\right]\Big|_{s=1} = \mathfrak{y} \tag{3.24}$$

and initial condition

$$(\mathfrak{u}, \mathfrak{v})\Big|_{t=0} = \mathfrak{x} = (\mathfrak{u}_0, \mathfrak{v}_0) \tag{3.25}$$

so $(\mathfrak{x}, \mathfrak{y})$ are the variations (perturbations) of the data (x, y).

It is not difficult to make this argument a rigorous verification of assumption (A1) of the IFT (Theorem 2.1) by filling in the details of the error computation in integral form.

3.4. **Wellposedness.** In this final subsection we turn to the verification of the assumption (A0) for the rod model IBVP(x, y). We note that the arguments for uniqueness and Lipschitzian dependence of solutions on data would use essentially the same methods as were used above for differentiability so we can concentrate our attention on showing existence.

Our strategy for this is the use of Faedo-Galerkin approximation: introducing a sequence of finite-dimensional subspaces $\mathcal{X}_K = \mathcal{U}_K \times \mathcal{V}_K \subset \mathcal{X}$ in which we seek $(U, V) = (U_K, V_K) \in \mathcal{X}_K$ determined by the same weak formulation as in (1.5). It is convenient to take $\mathcal{V}_K$ to be the continuous piecewise linear functions V on $[0, 1]$ with nodes $\{s_k = k/K : k = 0, \ldots, K\}$ and fixing $V\Big|_{s=0} = V_0 = 0$ and then to take $\mathcal{U}_K$ to be functions piecewise constant on the subintervals $\{(s_k, s_{k+1})\}$

We can then require $Z := \dot{U} = \partial_s V \in \mathcal{U}_K$ for each $V \in \mathcal{V}_K$ and will compute $n(U, Z) \in \mathcal{U}_K$ pointwise. The variational form of our system, corresponding to (1.5), is now

$$\langle H, \dot{V}\rangle_{\mathcal{V}} + \langle \partial_s H, n(U, Z)\rangle_{\mathcal{U}} = [H(1)]\, y(t) \tag{3.26}$$

for all $H \in \mathcal{V} = \mathcal{V}_K$. We must show that this weak formulation implicitly determines ODEs for the time evolution of the state U, V, that (a subsequence) converges to some limit $X = (u, v)$, and that this limit is a solution of IBVP(x, y). Note that $y \in \mathcal{Y}$ as before and for each K we take the initial data $(U, V)_{t=0}$ to be the projection of $x = (u_0, v_0)$ to $\mathcal{X}_K$.

The same computation as for the first energy estimate in Lemma 3.1 — i.e., taking $H = V$ in (3.26) — shows that (U, V) remains bounded on each $[0, t)$ so the ODE is well-defined and, indeed, is bounded uniformly in K so (U_K, V_K) converges subsequentially weakly in L^2 to some limit (u, v). This weak convergence does not show that $(U_K, Z_K) \hookrightarrow (u, z)$ implies $wlimn(U_K, Z_K) = n(u, z)$ and so is insufficient to show that the limit (u, v) is a solution of the limit problem. Fortunately, however, (with some minor modification) the remaining estimates in Lemma 3.1 continue to hold if one takes initial data $x \in \mathcal{X}$. Thus, noting the compactness of the embedding of $\mathbb{H}^1$ into the continuous functions, we actually have (for a subsequence) convergence $(U, Z) \to (u, z)$ uniformly on $\mathcal{Q}$ so $N_K = n(U_K, Z_K)$ converges to the correct limit.

4. Spectral Expansion for the Linearized model

In this section, we study the linearization at the equilibrium solution.

To this end, we introduce a (small) parameter $\varepsilon \geq 0$ — replacing the forcing y by εy — and linearize (1.4) around $\varepsilon = 0$ for which $u = w_s \equiv 0$,

$v = w_t \equiv 0$ to get the autonomous linear equation

$$w_{tt} = [c^2 w_s + \alpha w_{st}]_s \quad \text{on } \mathcal{Q} = \mathbb{R}_+ \times (0, \ell) \tag{4.1}$$

where $c^2 = \varphi''(0)$, $\alpha = \partial\sigma/\partial z\,(0,0)$ assuming each of these is positive. We remark that this is just the wave equation if $\alpha = 0$ and is the usual diffusion equation (for w_t) if $c = 0$.

We can write (4.1) as a vector equation

$$X' = \mathbf{A}X \quad \text{with } \mathbf{A} : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} v_s \\ c^2 u_s + \alpha v_{ss} \end{pmatrix} \tag{4.2}$$

for which we will begin by considering the initial/boundary value problem in the state space $\mathcal{X} = \{X : u, v \in L^2(0,1)\}$. Note that specification of the domain of the operator $\mathbf{A}$ includes the homogeneous boundary conditions: $v(0) = 0$ and $c^2 u + \alpha v_s = 0$ at $s = 1$.

We then compute the eigenvalues and eigenfunctions of $-\mathbf{A}$, i.e.,

$$v_s = -\lambda u \qquad c^2 u_s + \alpha v_{ss} = -\lambda v \tag{4.3}$$

We first observe that $\lambda \neq 0$. Indeed, in case $\lambda = 0$, it follows that $v_s = 0$. The condition $v(0) = 0$ then implies $v \equiv 0$. From the second equation in (4.3), it then immediately follows that $u_s = 0$. The boundary condition $c^2 u(1) + \alpha v_s(1) = 0$, together with $v \equiv 0$ implies $u(1) = 0$ which immediately yields $u \equiv 0$. In other words, the boundary conditions ensure that $\mathbf{A}$ is injective and $\lambda = 0$ cannot be an eigenvalue.

We will now reformulate the boundary condition at $s = 1$ for the eigenvalue problem. From the first equation in (4.3), noting $\lambda \neq 0$ we get $u = -\frac{1}{\lambda} v_s$. So from the boundary condition at $s = 1$ we get $(\alpha - \frac{c^2}{\lambda}) v_s(1) = 0$. Thus if $\alpha - \frac{c^2}{\lambda} \neq 0$, then $v_s(1) = 0$. We now claim that $\alpha \neq \frac{c^2}{\lambda}$. From (4.3), we readily obtain

$$\left(\alpha - \frac{c^2}{\lambda}\right) v_{ss} = -\lambda v. \tag{4.4}$$

If $\alpha = \frac{c^2}{\lambda}$ and $\lambda \neq 0$, then $v \equiv 0$. Thus from (4.3), $u \equiv 0$. However, $[u\ v]^t$ is an eigenvector and therefore this is impossible. Thus, we see that the following conditions hold for the eigenvalue λ,

$$\lambda \neq 0, \qquad \alpha \neq \frac{c^2}{\lambda}.$$

Thus from (4.4), we obtain the eigenvalue problem for v, namely,

$$-v_{ss} = \mu^2 v; \qquad v(0) = 0, v_s(1) = 0 \tag{4.5}$$

with $\mu^2 = \lambda^2/(\alpha\lambda - c^2)$. We recognize (4.5) as a familiar Sturm-Liouville eigenvalue problem for which the eigenvalues μ^2 are strictly positive, given by

$$\mu_k = \left(k - \frac{1}{2}\right)\pi, \qquad k = 1, 2, 3, \cdots.$$

The corresponding eigenvectors are

$$v_k = v_k(s) = C_k \sin(\mu_k s)$$

where C_k are normalizing constants.

From (4.4), it follows that the eigenvalues λ of $-\mathbf{A}$ must be roots of the quadratics

$$\lambda^2 - \mu_k^2 \alpha\lambda + \mu_k^2 c^2 = 0 \tag{4.6}$$

Thus[2] for each $k = 1, 2, 3, \cdots$, we obtain two eigenvalues of $-\mathbf{A}$, namely,

$$\lambda_k^{\pm} = \frac{1}{2}\mu_k^2\,\alpha\left[1 \pm \sqrt{1 - \frac{4\,c^2}{\mu_k^2\,\alpha}}\right] \qquad \text{for } k = 1, 2, \cdots \tag{4.7}$$

Observe that

$$\lim_{k\to\infty} \frac{\lambda_k^+}{\mu_k^2\,\alpha} = 1 \quad \text{and} \quad \lim_{k\to\infty} \lambda_k^- = c^2. \tag{4.8}$$

The eigenvectors corresponding to $\lambda_k^{\pm}$ are

$$w_k^{\pm} := C_k\left[\left(\frac{\mu_k}{\lambda_k^{\pm}}\right)\cos(\mu_k s)\ \ \sin(\mu_k s)\right]^t, \tag{4.9}$$

where C_k is again a normalizing constant. Setting $\mathcal{W}_k = \text{Span}_{\pm}\{w_k^{\pm}\}$, we note that for $k \neq l$, we have $\mathcal{W}_k \perp \mathcal{W}_l$. By our nondegeneracy assumption, we have $\dim.\mathcal{W}_k = 2$ for each k. We compare this with $\{\sqrt{2}[\cos(\mu_k s)\ \ 0]^t, \sqrt{2}[0\ \ \sin(\mu_k s)]^t,\}$ which is an orthonormal basis of $\mathcal{X}$. With respect to the orthogonal decomposition $\mathcal{X} = \bigoplus_{k=1}^{\infty}\mathcal{W}_k$, and the orthogonal basis $\{\sqrt{2}[\cos(\mu_k s)\ \ 0]^t, \sqrt{2}[0\ \ \sin(\mu_k s)]^t\}$ for $\mathcal{W}_k$, the matrix of $\mathbf{A}$ is block diagonal with the $k-$th diagonal block given by the matrix

$$\mathbf{A}_k = \begin{bmatrix} 0 & \mu_k \\ -c^2\mu_k & -\alpha\mu_k^2 \end{bmatrix},\ k = 1, 2, \cdots.$$

A direct computation shows that $\lambda_k^{\pm}$ are the eigenvalues of $-\mathbf{A}_k$. In view of (4.8), it follows that the spectrum of $-\mathbf{A}$ is

$$\sigma(-\mathbf{A}) = \{\lambda_k^{\pm} : k = 1, 2, \cdots\} \cup \{c^2\} \tag{4.10}$$

so $\Re(\lambda) > 0$ for all $\lambda \in \sigma(-\mathbf{A})$.

5. Conclusions

In this section we complete the proof of our principal result, Theorem 1.1 by applying Theorem 2.1.

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[2]We will assume that $4c/\pi\sqrt{\alpha}$ is not an odd integer to avoid the degenerate possibility that the quadratic (4.6) has a double root.

Proof. We need the assumptions (A0), (A1), (A2), (A3) of Theorem 2.1 in the context of Theorem 1.1. We have treated the verification of (A0) and (A1) in Subsection 3.4 and Subsection 3.3 and had already verified (A2) in Section 2. Thus, it remains only to use the Separation of Variables computations of Section 4 for verification of the invertibility assumption (A3).

Note that the spectral decomposition for (4.2) was purely formal without specific determination of the domain of AA. This was given in Section 4 as $\mathcal{X} = L^2((0,1) \to \mathbb{R}^2)$, which is justified once we observe that, although not orthonormal, the eigenvector sequence $\{w_k^\pm\}$ of (4.9) is a Riesz basis for $\mathcal{X}$.

To obtain a periodic solution with period T, we need to find $x = \mathbf{X}_0 \in \mathcal{X}$ such that

$$\mathbf{X}_0 = \mathbf{X}(T) = e^{T\mathbf{A}}\mathbf{X}_0 + \int_0^T e^{(T-\tau)\mathbf{A}}\boldsymbol{y}(\tau)\, d\tau. \tag{5.1}$$

However, due to (4.10), our computation of the spectrum of the operator $e^{T\mathbf{A}}$ yields that $\sigma(e^{T\mathbf{A}}) = \left\{e^{-\lambda_k^\pm T}, k = 1, 2, \cdots\right\} \cup \{e^{c^2T}\}$ Thus, $1 \notin \sigma(e^{T\mathbf{A}})$. It follows that the operator $(I - e^{T\mathbf{A}})$ is invertible and the inverse is a bounded operator on the phase space $\mathcal{X}$. Therefore a unique $\mathbf{X}_0$ satisfying (5.1) is given by

$$\mathbf{X}_0 = (I - e^{T\mathbf{A}})^{-1} \int_0^T e^{(T-\tau)\mathbf{A}}\boldsymbol{y}(\tau)\, d\tau.$$

In particular, this also establishes the invertibility of $\partial_x \mathcal{F}$ evaluated at the trivial steady state. This completes the proof of Theorem 1.1. □

Acknowledgement

The work of Animikh Biswas was supported in part by the NSF grant DMS-1517027.

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