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Navier–Stokes Equations and Weighted Convolution Inequalities in Groups

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We obtain Gevrey regular mild solutions to the incompressible Navier–Stokes equations in $\mathbf{R}^n$ *with periodic boundary condition in a subset of the variables. The method is based on an extension of Young's convolution inequality in weighted Lebesgue spaces of measurable functions defined on locally compact abelian groups. This generalizes and provides a unified treatment of the Gevrey regularity result of Foias and Temam in the space periodic case and those of Le Jan and Sznitman and Lemarié–Rieusset in the whole space with no boundary.*

Keywords Convolution inequlities; Gevrey regularity; Navier–Stokes equations.

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1. Introduction

Many important evolutionary partial differential equations may be cast as dynamical systems with values in a Banach space X consisting of real or complex valued functions defined on a locally compact abelian group G, where typically the group G is a product of copies of $\mathbf{R}$ and $\mathbf{Z}$. Important equations such as the Navier–Stokes and Kuramoto–Sivashinsky equations take the form

$$\frac{d}{dt}\mathbf{u} + A\mathbf{u} + B(\mathbf{u}) = \mathbf{g}, \tag{1}$$

where $\mathbf{u}(\cdot) \in C([0, T]; X)$, $\mathbf{g}(\cdot) \in L^1([0, T]; X)$ is given, A is a densely defined linear operator on X, and B is a nonlinear operator defined on a dense subspace of X

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having the form

$$B(\mathbf{u}) = \widetilde{B}[\mathbf{u}, \mathbf{u}]$$

where $\widetilde{B}$ is a bilinear form on X satisfying a convolution inequality of the form

$$|\omega_3 \widetilde{B}[\mathbf{u}, \mathbf{v}]| \leq |\omega_1 \mathbf{u}| * |\omega_2 \mathbf{v}| \tag{2}$$

for $\mathbf{u}, \mathbf{v} \in X$. Here ω_1, ω_2, and ω_3 are nonnegative functions defined on G. Other examples, where in fact a combination of such terms occur, include reaction diffusion equations, convection equations, and the Cahn–Hilliard equation. A complete treatment of such systems requires a careful analysis of the behavior of convolutions with respect to various weighted norms.

Let G be a locally compact abelian (LCA) group with a Haar measure μ (cf. [6]). The convolution of two real or complex valued functions f and g defined on G is given by

$$f * g(x) = \int_G f(y) g(x-y) d\mu(y)$$

at all points $x \in G$ where the integral exists. The Young convolution theorem states that if $1 \leq p, q, r \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

then $f * g \in L^r(G)$ whenever $f \in L^p(G)$ and $g \in L^q(G)$, and moreover

$$\|f * g\|_{L^r(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^q(G)}.$$

In this paper we consider convolution inequalities in weighted Lebesgue spaces. A weight is a function $\omega : G \to (0, \infty)$. Given a weight ω and $1 \leq p \leq \infty$, the weighted Lebesgue space $L^p_\omega(G)$ consists of all complex-valued functions f defined on G with the property that

$$\|f\|_{\omega,p} = \|\omega f\|_{L^p(G)} < \infty.$$

We determine general sufficient conditions for the validity of inequalities of the form

$$\|f * g\|_{\omega_3, r} \leq C \|f\|_{\omega_1, p} \|g\|_{\omega_2, q}. \tag{3}$$

In the case when $G = \mathbf{Z}^n$ or $\mathbf{R}^n$ we consider various necessary conditions. In the case $G = \mathbf{R}^n$ and for weights of the form $\omega(x) = |x|^\alpha$ such inequalities have been previously considered by various authors (cf. [4, 13, 14, 19] and the references therein) while the case $G = \mathbb{Z}$, $\omega_1 = \omega_2 = \omega_3$, and $p = q = r$ has been considered in [17].

The organization of the paper is as follows. In Section 2 we discuss a sufficient condition which guarantees the validity of convolution inequalities of the type (3) in a general LCA group G, and in Section 3 we provide several examples of such weights when the group G has the form $\mathbf{R}^\ell \times \mathbf{Z}^m$. These examples form the basis of the application of our results to dynamical systems of the form (1) mentioned above.

In Section 4 we apply these techniques to the particular case of the Navier–Stokes equations, and obtain local Gevrey regular solutions when the space periodic boundary condition is imposed only on a subset of the variables. Foias and Temam [10] first initiated the use of Gevrey norms as a tool to explicitly estimate the radius of analyticity for the NSE with space periodic boundary condition. It was subsequently used by many authors for other equations [2, 8, 9, 11]. In [18], Gevrey estimates were used to derive upper bounds for the decay of higher order derivatives of solutions to the NSE. The use of Gevrey norms was subsequently extended to the study of the NSE in $\mathbb{R}^n$ for initial data in certain Besov spaces by Le Jan and Sznitman [15], and to certain Sobolev spaces by Lemarié–Rieusset [16]. The case where the initial data is in L^p, $p > n$ was addressed by Grujic and Kukavica [11] using a slightly different technique. In our work we unify the treatments in [1, 10] in the space periodic case with those in [11, 15, 16] with no boundary, and obtain generalizations of these results (see Remark 4.2). Unlike in [15] or [16], our method does not rely on results from harmonic analysis and thus applies to the more general case of periodic boundary condition on a subset of the variables. Moreover, for certain parameter values, we can take very rough initial data with a negative order of smoothness. Additionally, this method easily adapts to treat other equations of the form (1) with nonlinear terms B satisfying (2) or other more complicated convolution estimates, and also to the case of generalized Gevrey norms the study of which was initiated by Constantin [7].

From the point of view of applications, it is important to know whether estimates of the form (3) are sharp. In general, however, the sufficient condition in Section 2 does not characterize the weights satisfying the convolution inequality (3). Necessary conditions implied by (3) for power weights have been considered previously in [13, 14]. In Section 5 we establish various necessary conditions with general weights and obtain characterizations in certain special cases.

2. General Inequalities in Groups

Let G be a LCA group with a Haar measure μ. Recall that the essential supremum of a real-valued function f defined on G, denoted $\operatorname{ess\,sup}_{x\in G} f(x)$, is the infimum of all numbers $C > 0$ with the property that

$$\mu(\{x : f(x) > C\}) = 0.$$

Given weights ω_1, ω_2, ω_3 defined on G and $1 \leq t \leq \infty$ we will consider the condition

$$\operatorname*{ess\,sup}_{x\in G}\left(\left\|\frac{1}{\omega_1(\cdot)\omega_2(x-\cdot)}\right\|_{L^{t'}(G)}\omega_3(x)\right) < \infty, \tag{4}$$

where t' denotes the Hölder conjugate of t. When $1 \leq t' < \infty$, this condition states that

$$\omega_1^{-t'} * \omega_2^{-t'}(x) \leq C\omega_3(x)^{-t'} \tag{5}$$

μ almost everywhere, and when $t' = \infty$, it is interpreted to mean that

$$\omega_3(x) \leq C\omega_1(y)\omega_2(x-y) \tag{6}$$

for μ almost all $x, y \in G$. A similar condition was considered by Nikol'skiĭ [17] with $\omega_1 = \omega_2 = \omega_3$ in the context of sequence spaces; see also pp. 104–105 the paper of Kerman and Sawyer [14].

Definition 2.1. Let $1 \leq p, q, r \leq \infty$. We define

$$\mathscr{Y}_G(p, q, r) \tag{7}$$

as the set of all triples $(\omega_1, \omega_2, \omega_3)$ for which there exists a constant $C = C_{p,q,r,\omega_1,\omega_2,\omega_3}$ with the property that

$$\|f * g\|_{\omega_3,r} \leq C\|f\|_{\omega_1,p}\|g\|_{\omega_2,q}$$

for all $f \in L^p_{\omega_1}(G)$ and $g \in L^q_{\omega_2}(G)$.

The following proposition provides a criterion for membership in the class $\mathscr{Y}_G(p, q, r)$.

Proposition 2.2. *Assume that* $1 \leq t \leq \min\{p, q, r\} \leq \infty$ *and that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{t}. \tag{8}$$

If $\omega_1, \omega_2, \omega_3$ *are weights satisfying* (4), *then*

$$(\omega_1, \omega_2, \omega_3) \in \mathscr{Y}_G(p, q, r).$$

Proof. Assume that $f \in L^p_{\omega_1}(G)$ and that $g \in L^p_{\omega_2}(G)$. Then we have the basic estimate

$$\begin{aligned} |f * g(x)| &\leq \int_G |f(y)||g(x-y)|d\mu(y) \\ &= \int_G \frac{\omega_1(y)|f(y)| \cdot \omega_2(x-y)|g(x-y)|}{\omega_1(y)\omega_2(x-y)} d\mu(y). \end{aligned} \tag{9}$$

The stated assumptions on p, q, r, t imply that $r \geq \max\{p, q\}$, and so there are essentially four distinct cases to consider:

- $p = q = r = t = \infty$,
- $p = t < \infty$ and $q = r = \infty$,
- $p, q, t < \infty$ and $r = \infty$,
- $p, q, r, t < \infty$.

In case $p = q = r = t = \infty$ we have $t' = 1$, and (9) implies

$$|f * g(x)| \leq \|f\|_{\omega_1,\infty}\|g\|_{\omega_2,\infty} \int_G \frac{1}{\omega_1(y)\omega_2(x-y)} d\mu(y).$$

It follows from (4) that

$$\omega_3(x)|f * g(x)| \le C\|f\|_{\omega_1,\infty}\|g\|_{\omega_2,\infty}$$

for μ almost all $x \in G$. Therefore $f * g$ exists for μ almost all $x \in G$ and

$$\|f * g\|_{\omega_3,\infty} \le C\|f\|_{\omega_1,\infty}\|g\|_{\omega_2,\infty}.$$

In case $t = p < \infty$ and $q = r = \infty$, note that (9) leads to

$$\begin{aligned} |f * g(x)| &\le \|g\|_{\omega_2,\infty}\int_G \frac{\omega_1(y)|f(y)|}{\omega_1(y)\omega_2(x-y)}d\mu(y) \\ &\le \|g\|_{\omega_2,\infty}\|\omega_1 f\|_{L^t(G)}\left\|\frac{1}{\omega_1(\cdot)\omega_2(x-\cdot)}\right\|_{L^{t'}(G)}. \end{aligned}$$

Since $p = t$, we conclude from (4) that

$$\omega_3(x)|f * g(x)| \le C\|f\|_{\omega_1,p}\|g\|_{\omega_2,\infty}$$

for μ almost all $x \in G$. Therefore $f * g$ exists for μ almost all $x \in G$ and

$$\|f * g\|_{\omega_3,\infty} \le C\|f\|_{\omega_1,p}\|g\|_{\omega_2,\infty}.$$

For the remaining two cases we apply Hölder's inequality to (9) to see that $|f * g(x)|$ does not exceed

$$\left(\int_G \omega_1(y)^t|f(y)|^t \cdot \omega_2(x-y)^t|g(x-y)|^t d\mu(y)\right)^{1/t}\left\|\frac{1}{\omega_1(\cdot)\omega_2(x-\cdot)}\right\|_{L^{t'}(G)},$$

and refer to (4) to conclude

$$|f * g(x)|\omega_3(x) \le C\left(\int_G \omega_1(y)^t|f(y)|^t \cdot \omega_2(x-y)^t|g(x-y)|^t d\mu(y)\right)^{1/t} \tag{10}$$

for μ almost all $x \in G$. In case $p, q, t < \infty$ and $r = \infty$ we have $\frac{t}{p} + \frac{t}{q} = 1$ with $p/t, q/t > 1$. A second application of Hölder's inequality and the translation invariance of μ imply

$$\begin{aligned} |f * g(x)|\omega_3(x) &\le C\left(\int_G \omega_1(y)^p|f(y)|^p d\mu(y)\right)^{1/p}\left(\int_G \omega_2(x-y)^q|g(x-y)|^q d\mu(y)\right)^{1/q} \\ &= C\left(\int_G \omega_1(y)^p|f(y)|^p d\mu(y)\right)^{1/p}\left(\int_G \omega_2(y)^q|g(y)|^q d\mu(y)\right)^{1/q}. \end{aligned}$$

Thus $f * g$ exists for μ almost all $x \in G$ and

$$\|f * g\|_{\omega_3,\infty} \le C\|f\|_{\omega_1,p}\|g\|_{\omega_2,q}.$$

Finally consider the case $p, q, r, t < \infty$. Define

$$\tilde{f} = (\omega_1|f|)^t, \quad \tilde{g} = (\omega_2|g|)^t, \quad \tilde{p} = \frac{p}{t}, \quad \tilde{q} = \frac{q}{t}, \quad \tilde{r} = \frac{r}{t}.$$

With this notation we may express (10) in the form

$$\omega_3(x)|f * g|(x) \le C(\tilde{f} * \tilde{g})(x)^{1/t},$$

which implies

$$\|f * g\|_{\omega_3,r} = \|\omega_3(f * g)\|_{L^r(G)} \le C\|\tilde{f} * \tilde{g}\|_{L^{\tilde{r}}(G)}^{1/t}. \tag{11}$$

The stated assumption on p, q, r, t implies that $\tilde{p}, \tilde{q}, \tilde{r} \ge 1$ and that $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = \frac{1}{\tilde{r}} + 1$. Therefore the classical Young inequality implies

$$\|\tilde{f} * \tilde{g}\|_{L^{\tilde{r}}(G)} \le \|\tilde{f}\|_{L^{\tilde{p}}(G)}\|\tilde{g}\|_{L^{\tilde{q}}(G)} = \|\omega_1 f\|_{L^p(G)}^t\|\omega_2 g\|_{L^q(G)}^t = \|f\|_{\omega_1,p}^t\|g\|_{\omega_2,q}^t. \tag{12}$$

The desired inequality now follows from (11) and (12). ☐

3. Examples

In this section we provide several examples of triples $(\omega_1, \omega_2, \omega_3)$ belonging to the class $\mathscr{Y}_G(p, q, r)$ and determine the dependence of the corresponding constant C on any relevant parameters. Throughout this section we assume that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{t}$$

with $1 \le t \le \min\{p, q\}$ and $r \ge 1$.

Example 3.1. Let G be an arbitrary LCA group and let $t = 1$. If $\varphi_1, \varphi_2, \varphi_3$ are real-valued functions on G satisfying

$$\varphi_3(x) \le K + \varphi_1(y) + \varphi_2(x - y) \tag{13}$$

for all $x, y \in G$ and some constant $K \in \mathbf{R}$, then one may take

$$\omega_1(x) = e^{\varphi_1(x)}, \quad \omega_2(x) = e^{\varphi_2(x)}, \quad \omega_3(x) = e^{\varphi_3(x)}$$

with $C = e^K$.

Proof. In this case we have $t' = \infty$, and the result follows at once from Proposition 2.2. ☐

Example 3.2. Let $G = \mathbf{R}^n$ and let $t > 1$. Let $\varphi_1, \varphi_2, \varphi_3 : \mathbf{R}^n \to \mathbf{R}$ satisfy (13). If $0 < \alpha, \beta < \frac{n}{t'}$ and $\alpha + \beta > \frac{n}{t'}$, then one may take

$$\omega_1(x) = e^{\varphi_1(x)}|x|^\alpha, \quad \omega_2(x) = e^{\varphi_2(x)}|x|^\beta, \quad \omega_3(x) = e^{\varphi_3(x)}|x|^{\alpha+\beta-\frac{n}{t'}}.$$

with $C = C_{\alpha,\beta,t,n,K}$.

Proof. Under the stated hypotheses we have

$$\int_{\mathbf{R}^n} \frac{1}{|y|^{\alpha t'}|e-y|^{\beta t'}}\,dy = C_{\alpha,\beta,t,n}$$

for any vector $e \in \mathbf{R}^n$ with $|e| = 1$. The fact that this integral is finite follows from elementary potential estimates, cf. [12, 21, Lemma 2.8.3]. The invariance of the integral for $|e| = 1$ is immediate from the fact that the orthogonal group on $\mathbf{R}^n$ is transitive. A change of variable shows that

$$\int_{\mathbf{R}^n} \frac{1}{|y|^{\alpha t'}|x-y|^{\beta t'}}\,dy = C_{\alpha,\beta,t,n}|x|^{n-\alpha t'-\beta t'}$$

for all $x \in \mathbf{R}^n$, $x \neq 0$. Condition (13) implies that

$$e^{t'\varphi_3(x)} \le e^{t'K} e^{t'\varphi_1(y)} e^{t'\varphi_2(x-y)},$$

and therefore

$$\int_{\mathbf{R}^n} \frac{1}{e^{t'\varphi_1(y)}|y|^{\alpha t'} e^{t'\varphi_2(x-y)}|x-y|^{\beta t'}}\,dy \le C_{\alpha,\beta,t,n,K}\, e^{-t'\varphi_3(x)}|x|^{n-\alpha t'-\beta t'}.$$

Consequently

$$\left(\int_{\mathbf{R}^n} \frac{1}{e^{t'\varphi_1(y)}|y|^{\alpha t'} e^{t'\varphi_2(x-y)}|x-y|^{\beta t'}}\,dy\right)^{1/t'} \le C_{\alpha,\beta,t,n,K}\, e^{-\varphi_3(x)}|x|^{\frac{n}{t'}-\alpha-\beta}. \qquad \square$$

The hypotheses of Example 3.2 may be weakened somewhat; see [4, 13]. The proofs given in these papers rely on the Stein–Weiss fractional integration theorem [20] and are much more difficult. We next obtain extensions of the preceding example valid in more general groups. The following proposition is elementary and may be proved by switching each integral to spherical coordinates.

Proposition 3.3. *Let* $n \ge 1$. *Let* $\beta \in \mathbf{R}$ *and let* $r > 0$. *Then*

$$\int_{\{y\in\mathbf{R}^n:|y|<r\}} \frac{1}{(1+|y|)^{\beta}}\,dy \le C_{\beta,n} \begin{cases} (1+r)^{n-\beta} & \text{if } \beta < n, \\ \log(2+r) & \text{if } \beta = n, \\ 1 & \text{if } \beta > n, \end{cases}$$

and

$$\int_{\{y\in\mathbf{R}^n:|y|>r\}} \frac{1}{(1+|y|)^{\beta}}\,dy \le C_{\beta,n} r^{n-\beta} \quad \text{if } \beta > n.$$

Proposition 3.4. *Let* $n \ge 1$. *Let* $\beta, \sigma \in \mathbf{R}$, $\beta \le \sigma$, *and suppose that* $\beta + \sigma > n$. *Then*

$$\int_{\mathbf{R}^n} (1+|y|)^{-\beta}(1+|x-y|)^{-\sigma}\,dy \le C_{\beta,\sigma,n} \begin{cases} (1+|x|)^{n-\beta-\sigma} & \text{if } \sigma < n, \\ (1+|x|)^{-\beta}\log(2+|x|) & \text{if } \sigma = n, \\ (1+|x|)^{-\beta} & \text{if } \sigma > n, \end{cases} \tag{14}$$

for all $x \in \mathbf{R}^n$.

Proof. Let $x \in \mathbf{R}^n$. The proof consists of estimating the integral separately over the sets

- $E_1 = \{y \in \mathbf{R}^n : |y| < \frac{|x|}{2}\}$,
- $E_2 = \{y \in \mathbf{R}^n : \frac{|x|}{2} \leq |y| < 2|x|\}$, and
- $E_3 = \{y \in \mathbf{R}^n : |y| \geq 2|x|\}$.

For instance, if $y \in E_1$ then $|y| < \frac{|x|}{2}$, hence $1 + |x - y| \geq \frac{1}{2}(1 + |x|)$. It follows that

$$\int_{E_1} (1 + |y|)^{-\beta}(1 + |x - y|)^{-\sigma} dy \leq C_\sigma (1 + |x|)^{-\sigma} \int_{E_1} (1 + |y|)^{-\beta} dy,$$

in which case Proposition 3.3 implies

$$\int_{E_1} (1 + |y|)^{-\beta}(1 + |x - y|)^{-\sigma} dy \leq C_{\beta,\sigma,n} \begin{cases} (1 + |x|)^{n-\beta-\sigma} & \text{if } \beta < n, \\ (1 + |x|)^{-\beta} \log(2 + |k|) & \text{if } \beta = n, \\ (1 + |x|)^{-\beta} & \text{if } \beta > n. \end{cases}$$

Since $\beta \leq \sigma$, it is easily checked in all cases that this implies (14). The estimates over the sets E_2 and E_3 are similar and are omitted. □

Proposition 3.5. *Let* $n \geq 1$. *Let* $\beta, \sigma \in \mathbf{R}$, $\beta \leq \sigma$, *and suppose that* $\beta + \sigma > n$. *Then*

$$\sum_{h \in \mathbf{Z}^n} (1 + |h|)^{-\beta}(1 + |k - h|)^{-\sigma} \leq C_{\beta,\sigma,n} \begin{cases} (1 + |k|)^{n-\beta-\sigma} & \textit{if } \sigma < n, \\ (1 + |k|)^{-\beta} \log(2 + |k|) & \textit{if } \sigma = n, \\ (1 + |k|)^{-\beta} & \textit{if } \sigma > n, \end{cases} \tag{15}$$

for all $k \in \mathbf{Z}^n$.

Proof. Denote the floor of a real number x by $\lfloor x \rfloor$, and by the floor of a vector $x = (x_1, \ldots, x_n)$ by $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$. If $\psi : \mathbf{R}^n \to [0, \infty)$ has the property that $\psi(x) = \psi(\lfloor x \rfloor)$ for all $x \in \mathbf{R}^n$, then clearly

$$\int_{\mathbf{R}^n} \psi(x) dx = \sum_{k \in \mathbf{Z}^n} \psi(k).$$

It follows that

$$\sum_{h \in \mathbf{Z}^n} (1 + |h|)^{-\beta}(1 + |k - h|)^{-\sigma} = \int_{\mathbf{R}^n} (1 + |\lfloor y \rfloor|)^{-\beta}(1 + |k - \lfloor y \rfloor|)^{-\sigma} dy$$

for all $k \in \mathbf{Z}^n$. Since $|y - \lfloor y \rfloor| \leq \sqrt{n}$ we have

$$\frac{1}{1 + \sqrt{n}}(1 + |y|) \leq 1 + |\lfloor y \rfloor| \leq (1 + \sqrt{n})(1 + |y|)$$

and

$$\frac{1}{1 + \sqrt{n}}(1 + |k - y|) \leq 1 + |k - \lfloor y \rfloor| \leq (1 + \sqrt{n})(1 + |k - y|),$$

so that

$$\int_{\mathbf{R}^n} (1+|\lfloor y\rfloor|)^{-\beta}(1+|k-\lfloor y\rfloor|)^{-\sigma}dy \le C_{\beta,\sigma,n}\int_{\mathbf{R}^n}(1+|y|)^{-\beta}(1+|k-y|)^{-\sigma}dy.$$

The result now follows from Proposition 3.4. □

The following proposition for mixed continuous and discrete factors may be proved exactly as above.

Proposition 3.6. *Let $\ell, m \ge 1$. Let $\beta, \sigma \in \mathbf{R}$, $\beta \le \sigma$, and suppose that $\beta + \sigma > \ell + m$. Then*

$$\sum_{h\in\mathbf{Z}^m}\int_{\mathbf{R}^\ell}(1+|h|+|y|)^{-\beta}(1+|k-h|+|x-y|)^{-\sigma}dy$$
$$\le C_{n,\beta,\sigma}\begin{cases}(1+|k|+|x|)^{n-\beta-\sigma} & \text{if } \sigma < \ell+m,\\ (1+|k|+|x|)^{-\beta}\log(2+|k|+|x|) & \text{if } \sigma = \ell+m,\\ (1+|k|+|x|)^{-\beta} & \text{if } \sigma > \ell+m,\end{cases}$$

for all $x \in \mathbf{R}^\ell$ and $k \in \mathbf{Z}^m$.

For the remainder of this section we let $\ell, m \ge 0$ where $n = \ell + m \ge 1$ and consider the group $G = \mathbf{R}^\ell \times \mathbf{Z}^m$. In case ℓ or m equals zero, we will interpret G as $\mathbf{R}^\ell$ or $\mathbf{Z}^m$ respectively. Since $G \subset \mathbf{R}^n$, G inherits a modulus $|\cdot|$ from the Euclidean norm in $\mathbf{R}^n$. For $g = (x, k) \in G$, we will have occasion also to use the modulus $|g|_1 = |x| + |k|$. These moduli on G are equivalent and both satisfy the triangle inequality. G is an LCA group under addition, the topology being induced by the metric $d(g_1, g_2) = |g_1 - g_1|$ (or equivalently by $|g_1 - g_2|_1$) for $g_i = (x_i, k_i)$, $i = 1, 2 \in G$. The Haar measure μ on G is the Lebesgue measure × the counting measure.

Example 3.7. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ with $n = \ell + m \ge 1$. Assume that $t > 1$. Suppose that $\alpha, \beta \in \mathbf{R}$ and that $\alpha + \beta > \frac{n}{t'}$. Assume that $\varphi_1, \varphi_2, \varphi_3$ satisfy (13). Let $\omega_1(g) = e^{\varphi_1(g)}(1+|g|)^\alpha$ and let $\omega_2(g) = e^{\varphi_2(g)}(1+|g|)^\beta$.

(1) If $\max\{\alpha, \beta\} < \frac{n}{t'}$, then one may take

$$\omega_3(g) = e^{\varphi_3(g)}(1+|g|)^{\alpha+\beta-\frac{n}{t'}}.$$

(2) If $\max\{\alpha, \beta\} = \frac{n}{t'}$, then one may take

$$\omega_3(g) = e^{\varphi_3(g)}\frac{(1+|g|)^{\min\{\alpha,\beta\}}}{\log(2+|g|)^{1/t'}}.$$

(3) If $\max\{\alpha, \beta\} > \frac{n}{t'}$, then one may take

$$\omega_3(g) = e^{\varphi_3(g)}(1+|g|)^{\min\{\alpha,\beta\}}.$$

Proof. We will prove the first statement by verifying the hypotheses of Proposition 2.2. Assume without loss of generality that $\alpha \le \beta$ and that $\beta < \frac{n}{t'}$. Let $g \in G$. Equation (13) and the equivalence of $|\cdot|$, $|\cdot|_1$ lead to

$$\begin{aligned}\left(\omega_1^{-t'} * \omega_2^{-t'}\right)(g) &= \int_G e^{-t'\varphi_1(h)} e^{-t'\varphi_2(g-h)}(1+|h|)^{-\alpha t'}(1+|g-h|)^{-\beta t'}\,d\mu(h)\\ &\le Ce^{-t'\varphi_3(g)}\int_G (1+|h|_1)^{-\alpha t'}(1+|g-h|_1)^{-\beta t'}\,d\mu(h).\end{aligned}$$

Since $\alpha t' + \beta t' > n$, Proposition 3.6 implies that

$$\int_G (1+|h|_1)^{-\alpha t'}(1+|g-h|_1)^{-\beta t'}\,d\mu(h) \le Ce^{-t'\varphi_3(g)}(1+|g|_1)^{n-\alpha t'-\beta t'} \le C\omega_3(g)^{-t'},$$

and consequently

$$\omega_1^{-t'} * \omega_2^{-t'} \le C\omega_3^{-t'}.$$

Therefore the first statement holds. The proofs of the remaining parts are similar and follow from Proposition 3.6. □

Example 3.7 has a simple analogue when $t = 1$.

Example 3.8. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ with $\ell + m = n$, and let $t = 1$. Assume that $\varphi_1, \varphi_2, \varphi_3 : G \to \mathbf{R}$ satisfy (13). If $\alpha + \beta > 0$, then one may take

$$\omega_1(g) = e^{\varphi_1(g)}(1+|g|)^{\alpha}, \quad \omega_2(g) = e^{\varphi_2(g)}(1+|g|)^{\beta}, \quad \omega_3(g) = e^{\varphi_3(g)}(1+|g|)^{\min\{\alpha,\beta\}},$$

with $C = C_{\alpha,\beta,K}$.

Proof. Assume without loss of generality that $\alpha \le \beta$. If $\alpha \ge 0$ then the triangle inequality implies

$$(1+|g|)^{\alpha} \le (1+|h|)^{\alpha}(1+|g-h|)^{\beta}.$$

On the other hand, if $\alpha < 0$ we have

$$(1+|g|)^{\alpha} \le \left(\frac{1}{2}\right)^{\alpha}(1+|h|)^{\alpha}, \quad |g| \ge \frac{1}{2}|h|,$$

and

$$1 \le \left(\frac{1}{2}\right)^{\alpha}(1+|h|)^{\alpha}(1+|g-h|)^{\beta}, \quad |g| < \frac{1}{2}|h|.$$

Therefore

$$(1+|g|)^{\alpha} \le 2^{-\alpha}(1+|h|)^{\alpha}(1+|g-h|)^{\beta}$$

for all $g, h \in G$. Consequently we see that

$$(1+|g|)^{\alpha} \le 2^{-\min\{0,\alpha\}}(1+|h|)^{\alpha}(1+|g-h|)^{\beta},$$

hence

$$e^{\varphi_3(g)}(1+|g|)^{\alpha} \le C_{\alpha,K} e^{\varphi_1(h)}(1+|h|)^{\alpha} e^{\varphi_2(g-h)}(1+|g-h|)^{\beta}.$$

In other words, $\omega_3(g) \le C_{\alpha,K}\omega_1(h)\omega_2(g-h)$ for all $g, h \in G$, and we may appeal to Proposition 2.2. □

The following estimate will be used repeatedly.

Proposition 3.9. *Let $G = \mathbf{R}^{\ell} \times \mathbf{Z}^m$ with $n = \ell + m \ge 1$. Let $\theta > 0$ and $\lambda > 0$. Then*

$$\int_G e^{-\lambda|g|^{\theta}} d\mu(g) \le C_{\theta,n}\lambda^{-\frac{n}{\theta}}.$$

Proof. Switch to spherical coordinates and employ the change of variable $\tau = \lambda\rho^{\theta}$ to obtain

$$\begin{aligned}\int_{\mathbf{R}^{\ell}} e^{-\lambda|x|^{\theta}} dx &= \int_0^{\infty}\int_{|x|=\rho} e^{-\lambda|x|^{\theta}} dS(x)d\rho \\ &= C_{\ell}\int_0^{\infty} e^{-\lambda\rho^{\theta}}\rho^{\ell-1}d\rho \\ &= C_{\ell}\lambda^{-\frac{\ell}{\theta}}\theta^{-1}\int_0^{\infty} e^{-\tau}\tau^{\frac{\ell}{\theta}-1}d\tau \\ &= C_{\theta,\ell}\lambda^{-\frac{\ell}{\theta}}.\end{aligned}$$

Likewise, one may proceed as in the proof of Proposition 3.5 to obtain

$$\sum_{k\in\mathbf{Z}^m} e^{-\lambda|k|^{\theta}} \le C_{\theta,m}\lambda^{-\frac{m}{\theta}}.$$

The general inequality follows from Fubini's theorem and the basic fact that for $g = (x, k) \in G$ we have $|g|^{\theta} \ge C_{\theta}(|x|^{\theta} + |k|^{\theta})$. □

The following example is a variant of Example 3.7. The additional flexibility in the hypothesis is gained at the expense of the control over the constant C.

Example 3.10. Assume that $t > 1$. If $\lambda > 0$ and $\alpha + \beta > 0$, then one may take

$$\omega_1(g) = e^{\lambda|g|}(1+|g|)^{\alpha}, \quad \omega_2(g) = e^{\lambda|g|}(1+|g|)^{\beta}, \quad \omega_3(g) = e^{\lambda|g|}(1+|g|)^{\min\{\alpha,\beta\}-\frac{n}{t'}},$$

with $C = C_n(1+\lambda^{-n})^{1/t'}$.

Proof. Since $\alpha + \beta > 0$ we have that

$$(1+|g|)^{\min\{\alpha,\beta\}} \le C_{\alpha,\beta}(1+|h|)^{\alpha}(1+|g-h|)^{\beta}$$

for all $g, h \in G$, so the triangle inequality implies that

$$e^{-\lambda t'|y|}(1+|y|)^{-\alpha t'} e^{-\lambda t'|x-y|}(1+|x-y|)^{-\beta t'} \le e^{-\lambda t'|x|}(1+|x|)^{-\min\{\alpha t',\beta t'\}}$$

for all $g, h \in G$. In light of the basic growth estimate

$$\mu(\{|h| \le M\}) \le C_n M^n, \quad (M \ge 1)$$

we have

$$\begin{aligned}\int_{|h|\le 2|g|} & e^{-\lambda t'|h|}(1+|h|)^{-\alpha t'} e^{-\lambda t'|g-h|}(1+|g-h|)^{-\beta t'} d\mu(h) \\ &\le \mu(\{|h| \le 2|g|\}) e^{-\lambda t'|g|}(1+|g|)^{-\min\{\alpha t', \beta t'\}} \\ &\le C_n e^{-\lambda t'|g|}(1+|g|)^{n-\min\{\alpha t', \beta t'\}}\end{aligned}$$

for all $g \in G$. On the other hand, if $|h| \ge 2|g|$ then $|g-h| \ge |g|$ and

$$e^{\lambda t'|h|} e^{\lambda t'|g-h|} \ge e^{\lambda t'|h|} e^{\lambda t'|g|}.$$

In this case we have

$$e^{-\lambda t'|h|}(1+|h|)^{-\alpha t'} e^{-\lambda t'|g-h|}(1+|g-h|)^{-\beta t'} \le e^{-\lambda t'|h|} e^{-\lambda t'|g|}(1+|g|)^{-\min\{\alpha t', \beta t'\}}$$

and consequently

$$\begin{aligned}\int_{|h|\ge 2|g|} & e^{-\lambda t'|h|}(1+|h|)^{-\alpha t'} e^{-\lambda t'|g-h|}(1+|g-h|)^{-\beta t'} d\mu(h) \\ &\le e^{-\lambda t'|g|}(1+|g|)^{-\min\{\alpha t', \beta t'\}} \int_{|h|\ge 2|g|} e^{-\lambda t'|h|} d\mu(h).\end{aligned}$$

Proposition 3.9 states

$$\int_G e^{-\lambda t'|y|} dy \le C_{n,t} \lambda^{-n},$$

so we may add these estimates together to see that

$$\begin{aligned}\int_G & e^{-\lambda t'|h|}(1+|h|)^{-\alpha t'} e^{-\lambda t'|g-h|}(1+|g-h|)^{-\beta t'} d\mu(h) \\ &\le C_{n,t}(1+\lambda^{-n}) e^{-\lambda t'|g|}(1+|g|)^{n-\min\{\alpha t', \beta t'\}}\end{aligned}$$

and therefore

$$\begin{aligned}\bigg(\int_G & e^{-\lambda t'|h|}(1+|h|)^{-\alpha t'} e^{-\lambda t'|g-h|}(1+|g-h|)^{-\beta t'} d\mu(h)\bigg)^{1/t'} \\ &\le C_{n,t}(1+\lambda^{-n})^{1/t'} e^{-\lambda|g|}(1+|g|)^{\frac{n}{t'}-\min\{\alpha,\beta\}}.\end{aligned}$$

□

Example 3.11. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ with $n = \ell + m \ge 1$. If $\lambda > 0$, then one may take

$$\omega_1(g) = e^{\lambda|g|^2}, \quad \omega_2(g) = e^{\lambda|g|^2}, \quad \omega_3(g) = e^{\frac{\lambda}{2}|g|^2}.$$

Proof. The parallelogram identity states that

$$|h|^2 + |g-h|^2 = \frac{1}{2}|g|^2 + \frac{1}{2}|2h-g|^2 \tag{16}$$

for all $g, h \in G$. Thus Proposition 3.9 implies that

$$\begin{aligned}\int_G e^{-\lambda t'|h|^2} e^{-\lambda t'|g-h|^2}\,d\mu(h) &= \int_G e^{-\frac{\lambda t'}{2}|2h-g|^2} e^{-\frac{\lambda t'}{2}|g|^2}\,d\mu(h)\\ &= e^{-\frac{\lambda t'}{2}|g|^2}\int_G e^{-2\lambda t'|h|^2}\,d\mu(h)\\ &\le C_n e^{-\frac{\lambda t'}{2}|g|^2}(\lambda t')^{-\frac{n}{2}}.\end{aligned}$$

□

The preceding example may be extended to more general exponents via the Clarkson inequalities.

Example 3.12. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ with $n = \ell + m \ge 1$. Let $1 < \theta < \infty$ and let

$$\kappa(\theta) = \begin{cases} 2 & 1 < \theta \le 2,\\ 2^{\theta-1} & \theta > 2.\end{cases}$$

Then for any $\lambda > 0$ one may take

$$\omega_1(g) = e^{\lambda|g|^\theta}, \quad \omega_2(g) = e^{\lambda|g|^\theta}, \quad \omega_3(g) = e^{\frac{\lambda}{\kappa(\theta)}|g|^\theta}$$

with $C = C_{n,\theta}\lambda^{-\frac{n}{\theta t'}}$.

Proof. We have the inequality (see [5, Theorem 2])

$$|h|^\theta + |g-h|^\theta \ge \frac{1}{\kappa(\theta)}|g|^\theta + \frac{1}{\kappa(\theta)}|2h-g|^\theta$$

for all $g, h \in G$. Proceeding as above it follows that

$$\begin{aligned}\int_G e^{-\lambda t'|h|^\theta} e^{-\lambda t'|g-h|^\theta}\,d\mu(h) &\le \int_G e^{-\frac{\lambda t'}{\kappa(\theta)}|2h-g|^\theta} e^{-\frac{\lambda t'}{\kappa(\theta)}|g|^\theta}\,d\mu(h)\\ &= e^{-\frac{\lambda t'}{\kappa(\theta)}|g|^\theta}\int_G e^{-\frac{2^\theta\lambda t'}{\kappa(\theta)}|h|^\theta}\,d\mu(h)\\ &\le C_{n,\theta}\left(\frac{2^\theta\lambda t'}{\kappa(\theta)}\right)^{-\frac{n}{\theta}} e^{-\frac{\lambda t'}{\kappa(\theta)}|g|^\theta}.\end{aligned}$$

Therefore

$$\left(\int_G e^{-\lambda t'|h|^\theta} e^{-\lambda t'|g-h|^\theta}\,d\mu(h)\right)^{1/t'} \le C_{n,t,\theta}\lambda^{-\frac{n}{\theta t'}} e^{-\frac{\lambda}{\kappa(\theta)}|g|^\theta}$$

for all $g \in G$. □

Example 3.13. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ with $n = \ell + m \geq 1$. The results of Examples 3.10 and 3.12 may be combined to show that one may take

$$\omega_1(g) = e^{\lambda|g|^\theta}(1+|g|)^\alpha, \quad \omega_2(g) = e^{\lambda|g|^\theta}(1+|g|)^\beta, \quad \omega_3(g) = e^{\frac{\lambda}{\kappa(\theta)}|g|^\theta}(1+|g|)^{\min\{\alpha,\beta\}-\frac{n}{t'}}$$

whenever $\theta > 1$, $\lambda \geq 0$, and $\alpha + \beta > 0$. In this case the constant is $C = C_{n,\theta,\alpha,\beta}\lambda^{-\frac{n}{\theta t'}}$.

We now give a generalization of the convolution inequalities which have been used in [1, 2] to study equations of the form (23).

Definition 3.14. Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$, with μ being the Haar measure. Let $\lambda \geq 0$, $\alpha \in \mathbf{R}$ and $1 \leq p < \infty$. Given a real-valued function φ and a weight $\omega : G \to (0, \infty)$, the space $V_{\varphi,\omega,p}$ consists of all complex-valued μ measurable functions f defined on G with the property that

$$\|f\|_{\varphi,\omega,p} := \left(\int_G e^{p\varphi(x)}\omega(x)^p|f(x)|^p d\mu(x)\right)^{1/p} < \infty. \tag{17}$$

When $\varphi(x) = \lambda|x|$ and $\omega(x) = (1+|x|)^\alpha$ where $\alpha \in \mathbf{R}$ and $\lambda > 0$, we will write $\|f\|_{\lambda,\alpha,p}$ instead of $\|f\|_{\varphi,\alpha,p}$.

The following proposition gives a generalization of the convolution inequalities which have been used in [1, 2].

Proposition 3.15. *Let $G = \mathbf{R}^\ell \times \mathbf{Z}^m$. Assume that $t > 1$, $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta > \frac{n}{t'}$ and that $\varphi : G \to \mathbf{R}$ satisfies* (13) *with $\varphi_i = \varphi$, $i = 1, 2, 3$ and $K = K_\varphi$. Let $f \in V_{\varphi,\alpha,p}(G)$ and $g \in V_{\varphi,\beta,p}(G)$.*

(1) *If $\max\{\alpha, \beta\} < \frac{n}{t'}$ then $f * g \in V_{\varphi,\alpha+\beta-\frac{n}{t'},r}(G)$ and*

$$\|f * g\|_{\varphi,\alpha+\beta-\frac{n}{t'},r} \leq e^{K_\varphi}C_{\alpha,\beta,n,t}\|f\|_{\varphi,\alpha,p}\|g\|_{\varphi,\beta,q}.$$

(2) *If $\max\{\alpha, \beta\} > \frac{n}{t'}$ then $f * g \in V_{\varphi,\min\{\alpha,\beta\},r}(G)$ and*

$$\|f * g\|_{\varphi,\min\{\alpha,\beta\},r} \leq e^{K_\varphi}C_{\alpha,\beta,n,t}\|f\|_{\varphi,\alpha,p}\|g\|_{\varphi,\beta,q}.$$

(3) *If $\max\{\alpha, \beta\} = \frac{n}{t'}$ then letting $\omega(g) = \frac{(1+|g|)^{\min\{\alpha,\beta\}}}{\log(2+|g|)^{1/t'}}$, we have $f * g \in V_{\varphi,\omega,r}(G)$ and*

$$\|f * g\|_{\varphi,\omega,r} \leq e^{K_\varphi}C_{\alpha,\beta,n,t}\|f\|_{\varphi,\alpha,p}\|g\|_{\varphi,\beta,q}.$$

Proof. The proofs of these inequalities follow directly from Example 3.7. □

Finally we extend the preceding result to multi-fold convolutions. Estimates of this type are useful in the study of a class of equations including the Cahn–Hilliard equation and reaction-diffusion type equations.

Corollary 3.16. *Suppose that $1 < t \leq \min\{p_1, \ldots, p_k\} \leq \infty$ and that $r \geq 1$ satisfy*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r} + \frac{k-1}{t}.$$

Let $\alpha_1, \ldots, \alpha_k$ *be real numbers satisfying*

$$0 < \alpha_1, \ldots, \alpha_k < \frac{n}{t'} \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j > \frac{(m-1)n}{t'}, \quad m = 1 \ldots k. \tag{18}$$

If $f_j \in V_{\lambda,\alpha_j,p_j}$ *for each* j *then*

$$f_1 * \cdots * f_k \in V_{\lambda,\alpha_1+\cdots+\alpha_k-\frac{n(k-1)}{t'},r}$$

and

$$\|f_1 * \cdots * f_k\|_{\lambda,\alpha_1+\cdots+\alpha_k-\frac{n(k-1)}{t'},r} \le C \prod_{j=1}^{k} \|f_j\|_{\lambda,\alpha_j,p_j}$$

for an adequate constant $C = C_{\alpha_1,\ldots,\alpha_k,n,t,k}$.

Proof. The proof will use induction on k. The case $k = 2$ was established in Proposition 3.15. Assume that $k \ge 3$ and that the conclusion is valid with k replaced by $k-1$. Define $\tilde{p}$ by

$$\frac{1}{\tilde{p}} + \frac{1}{p_k} = \frac{1}{r} + \frac{1}{t}$$

so that $\tilde{p} \ge t$. According to (18) we have

$$0 < \alpha_1 + \cdots + \alpha_{k-1} - \frac{(k-2)n}{t'} < \frac{n}{t'}$$

and

$$\alpha_1 + \cdots + \alpha_{k-1} - \frac{(k-2)n}{t'} + \alpha_k > \frac{n}{t'},$$

and therefore Proposition 3.15 implies that

$$\begin{aligned} \|f_1 * \cdots * f_k\|_{\lambda,\alpha_1+\cdots+\alpha_k-\frac{n(k-1)}{t'},r} &= \|(f_1 * \cdots * f_{k-1}) * f_k\|_{\lambda,(\alpha_1+\cdots+\alpha_{k-1}-\frac{(k-2)n}{t'})+\alpha_k-\frac{n}{t'},r} \\ &\le C\|f_1 * \cdots * f_{k-1}\|_{\lambda,\alpha_1+\cdots+\alpha_{k-1}-\frac{(k-2)n}{t'},\tilde{p}} \|f_k\|_{\lambda,\alpha_k,p_k}. \end{aligned}$$

Writing

$$\frac{1}{p_1} + \cdots + \frac{1}{p_{k-1}} = \frac{1}{\tilde{p}} + \frac{k-2}{t}$$

we may apply the induction hypothesis to conclude that

$$\|f_1 * \cdots * f_{k-1}\|_{\lambda,\alpha_1+\cdots+\alpha_{k-1}-\frac{(k-2)n}{t'},\tilde{p}} \le C \prod_{j=1}^{k-1} \|f_j\|_{\lambda,\alpha_j,p_j}.$$

Therefore

$$\|f_1 * \cdots * f_k\|_{\lambda,\alpha_1+\cdots+\alpha_k-\frac{n(k-1)}{t'},r} \leq C\prod_{j=1}^{k}\|f_j\|_{\lambda,\alpha_j,p_j}$$

which completes the proof. □

4. Application: Navier–Stokes Equations

In this section, we will give an application of the inequalities developed thus far to ODE of the form (1), where $\mathbf{u}(\cdot) \in C([0,T];X)$ for suitable Banach space X, A is a densely defined linear operator on X, while B is a nonlinear operator defined on a (dense) subspace of X.

Although many equations have this general type, for the purpose of illustration we will consider the Navier–Stokes equations in $\mathbf{R}^n$. Recall that the Navier–Stokes initial-value problem for an n dimensional periodic homogeneous incompressible flow:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\mathbf{f} \tag{19}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{20}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x). \tag{21}$$

Here $\mathbf{u} : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}^n$ is the unknown velocity, $p : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ is the unknown pressure, $\mathbf{f} : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ is the given external body force, ν is the kinematic viscosity, ρ is the (constant) fluid density, and $\mathbf{u}_0$ is the initial velocity. To simplify the discussion we will henceforth take $\rho = 1$ and $\nu = 1$.

Let $0 \leq m \leq n$ and $\ell = n - m$. We assume that the functions $\mathbf{u}$, p, and $\mathbf{f}$ are periodic in last m space variables $x_{\ell+1}, \cdots, x_n$ with period $L = 2\pi$ at all times $t \geq 0$ and the relevant functions can therefore be viewed as $\mathbf{R}^n$-valued functions on $\mathbf{R}^\ell \times [0,2\pi]^m$. For such a function $\mathbf{u}$, we may take the Fourier transform $\vec{\mathbf{u}}$ to be the $\mathbf{C}^n$-valued function on $\mathbf{R}^\ell \times \mathbf{Z}^m$ defined by

$$\vec{\mathbf{u}}(g) = \frac{1}{(2\pi)^m}\int_{[0,2\pi]^m}\int_{\mathbf{R}^\ell}\mathbf{u}(s,t)e^{-\imath(s\cdot x+k\cdot t)}ds\,dt, \quad g = (x,k) \in \mathbf{R}^\ell \times \mathbf{Z}^m.$$

Define $G = \mathbf{R}^\ell \times \mathbf{Z}^m$ and let $\mathbb{V}$ denote the vector space of all $\mathbf{C}^n$-valued functions $\vec{\mathbf{v}}$ on G satisfying

$$g\cdot\vec{\mathbf{v}}(g) = 0, \quad \overline{\vec{\mathbf{v}}(g)} = \vec{\mathbf{v}}(-g) \quad (g \in G). \tag{22}$$

Here the dot product denotes the usual dot product where elements of G are viewed as elements in $\mathbf{R}^n$. Note that a function $\mathbf{u}$ on $\mathbf{R}^l \times (0,2\pi)^m$ is divergence free if and only if $g\cdot\vec{\mathbf{u}}(g) = 0$ and it is real-valued if and only if $\overline{\vec{\mathbf{u}}(g)} = \vec{\mathbf{u}}(-g)$ for all $g \in G$.

As is customary, we now apply the Stokes projection and project the Navier–Stokes equations on the space of divergence free vector fields in order to eliminate the pressure. For simplicity of exposition, we assume that the body force is potential, and consequently, its projection on the space of divergence free vectors is zero. Using the Fourier transform described above—noting that under our boundary conditions the Stokes projection and the Laplacian commute—it can now be verified that the system (19)–(21) may be cast as the following dynamical system with values in $\mathbb{V}$:

$$\frac{d}{dt}\vec{\mathbf{u}} + A\vec{\mathbf{u}} + B(\vec{\mathbf{u}}) = \vec{\mathbf{0}}, \quad \vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0, \tag{23}$$

where $\vec{\mathbf{u}}_0$ is a suitable initial data. Here $A : \mathbb{V} \to \mathbb{V}$ is the densely defined linear operator given by

$$(A\vec{\mathbf{u}})(g) = |g|^2\vec{\mathbf{u}}(g) \tag{24}$$

and B is the densely defined nonlinear operator given by $B(\vec{\mathbf{u}}) = B[\vec{\mathbf{u}}, \vec{\mathbf{u}}]$ where $B[\vec{\mathbf{u}}, \vec{\mathbf{v}}]$ is defined for $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{V}$ by

$$B[\vec{\mathbf{u}}, \vec{\mathbf{v}}](g) = \imath P_g \int_G ((g-h)\cdot\vec{\mathbf{u}}(h))\vec{\mathbf{v}}(g-h)d\mu(h), \tag{25}$$

where P_g denotes the orthogonal projection in $\mathbf{C}^n$ to the subspace in $\mathbf{C}^n$ orthogonal to $g \in G(\subset \mathbf{C}^n)$ and $\mu =$ Lebesgue measure $\times$ counting measure is the Haar measure on G. We will consider the existence of mild solutions to (23) which are solutions to the associated integral equation

$$\vec{\mathbf{u}}(t) = e^{-tA}\vec{\mathbf{u}}_0 - \int_0^t e^{-(t-s)A}B(\vec{\mathbf{u}}(s))ds, \quad \vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0. \tag{26}$$

Let V be the vector space of all complex valued functions on G. We define the (nonlinear) operator $\tau : \mathbb{V} \to V$ by

$$(\tau\vec{\mathbf{w}})(g) = |\vec{\mathbf{w}}(g)|, \quad g \in G, \ \ \vec{\mathbf{w}} \in \mathbb{V}.$$

The range of τ in fact lies in the positive cone of nonnegative real valued functions on G. In light of the fact that $h \cdot \vec{\mathbf{u}}(h) = 0$ for all $h \in G$ and $\vec{\mathbf{u}} \in \mathbb{V}$, we obtain the following crucial estimate for the structure of B, namely,

$$|B[\vec{\mathbf{u}}, \vec{\mathbf{v}}](g)| \le |g|((\tau\vec{\mathbf{u}})) * (\tau\vec{\mathbf{v}})(g). \tag{27}$$

For $\lambda > 0, \alpha \in \mathbf{R}$ and $1 \le p \le \infty$, we define

$$\mathbb{V}_{\lambda,\alpha,p} = \left\{\vec{\mathbf{u}} \in \mathbb{V} : \|\vec{\mathbf{u}}\|_{\lambda,\alpha,p} := \left(\int_G e^{\lambda p|x|}(1+|x|)^{\alpha p}|\vec{\mathbf{u}}(x)|^p d\mu(x)\right)^{1/p} < \infty\right\},$$

with the usual notational convention that when $p = \infty$, the integral over G is replaced with the essential supremum. The corresponding space $V_{\lambda,\alpha,p}$ of scalar-valued function is defined in a similar manner. Adopting the notation in Definition 3.14 we have that $\vec{\mathbf{u}} \in \mathbb{V}$ belongs to $\vec{\mathbf{u}} \in \mathbb{V}_{\lambda,\alpha,p}$ if and only if $\tau\vec{\mathbf{u}} \in V_{\lambda,\alpha,p}$ and in this case $\|\vec{\mathbf{u}}\|_{\lambda,\alpha,p} = \|(\tau\vec{\mathbf{u}})\|_{\lambda,\alpha,p}$. The following is one of our main results.

Theorem 4.1. *Assume that* $1 \le p \le \infty$ *and* $\alpha \ge \frac{n}{p'} - 1$. *If* $\vec{\mathbf{u}}_0 \in \mathbb{V}_{\alpha,p}$ *then there exist* $T > 0$ *and a solution* $\vec{\mathbf{u}}(\cdot) \in C([0, T]; \mathbb{V}_{\alpha,p})$ *of* (26) *satisfying*

$$\sup_{0 \le t \le T} \|\vec{\mathbf{u}}(t)\|_{\lambda\sqrt{t},\alpha,p} < \infty.$$

Moreover, there exists a constant $C_{\lambda,\alpha,p}$ *so that one may take*

$$T = \min\left\{C, \frac{C_{\lambda,\alpha,p}}{\|\vec{\mathbf{u}}_0\|_{\alpha,p}^{2p/(3-2p+\alpha p)}}\right\} \quad \textit{if } \alpha < \frac{n}{p'}$$

and

$$T = \min\left\{C, \frac{C_{\lambda,\alpha,p}}{\|\vec{\mathbf{u}}_0\|_{\alpha,p}^{2}}\right\} \quad \textit{if } \alpha > \frac{n}{p'}.$$

Furthermore, for every $0 < \theta < 2$ *there exists a constant* $C_{\lambda,\alpha,\theta,p}$ *so that one may take*

$$T = \min\left\{T', \frac{C_{\lambda,\alpha,\theta,p}}{\|\vec{\mathbf{u}}_0\|_{\alpha,p}^{2-\theta}}\right\} \quad \textit{if } \alpha = \frac{n}{p'}.$$

If $p = 1$, *then we may also take* $\theta = 0$ *for* $\alpha = \frac{n}{p'}$ *above.*

Remark 4.2. First we remark that Theorem 4.1 generalizes, and provides a unified approach, to several existing results. If we specialize to the case $G = \mathbb{Z}^n$ (i.e., NSE with space periodic boundary condition) we recover the result of Foias and Temam [10] when $p = 2$, $\alpha = 1$ and more generally, that in [1] for other values of α and p. For the NSE in the whole space $\mathbb{R}^n$ with no boundary, we recover the result in [16] for initial data in the space $\dot{H}^{n/2-1}$. This corresponds to the case $G = \mathbb{R}^n$, $p = 2$ and $\alpha = \frac{n}{2} - 1$ in our set up. A minor modification of our method also yields the result in [15] for the NSE in $\mathbb{R}^n$ which corresponds to $G = \mathbb{R}^n$, $p = \infty$ and $\alpha = n - 1$. We further mention that when $p < \frac{n}{n-1}$, the permissible range of α includes negative values. This corresponds to Besov type spaces with negative order of smoothness.

Remark 4.3. It should be noted that due to the Paley–Wiener theorem, the control on the Gevrey norm allows us to conclude that for all $t > 0$, the inverse Fourier transform of the solution obtained here is a classical solution of the NSE which admits an analytic extension on the domain $\{z = x + \imath y \in \mathbb{C}^n : |y| < \lambda\sqrt{t}\}$ (see [2, 10, 11]). We can also use generalized Gevrey norms as in [2] with no additional difficulty. Precisely, in the definition of the space $\mathbb{V}_{\lambda,\gamma,p}$, the term $e^{\lambda p|x|}$ may be replaced by $e^{\lambda p \Psi(x)}$ where $\Psi : G \to [0, \infty)$ satisfies a triangle inequality of the form $\Psi(x + h) \le C(\Psi(x) + \Psi(h))$.

The proof of Theorem 4.1 will be given below once we have established some necessary propositions. The following two propositions are generalizations of those in [1].

Proposition 4.4. *Let* $G = \mathbf{R}^\ell \times \mathbf{Z}^m$, $\alpha \in \mathbf{R}$, $\beta > 0$, $\lambda \ge 0$ *and* $1 \le p \le \infty$. *If* $\vec{\mathbf{u}} \in \mathbb{V}_{\lambda,\alpha,p}$ *then*

$$\|e^{-\eta A}\vec{\mathbf{u}}\|_{\lambda,\alpha+\beta,p} \le C_\beta e^{\eta} \eta^{-\frac{\beta}{2}} \|\vec{\mathbf{u}}\|_{\lambda,\alpha,p}$$

for all $\eta > 0$. *Moreover, if* $1 \le p < \infty$,

$$\lim_{\eta \to 0^+} \eta^{\frac{\beta}{2}} \|e^{-\eta A}\vec{\mathbf{u}}\|_{\lambda,\alpha+\beta,p} = 0.$$

Proof. For $\vec{\mathbf{u}} \in \mathbb{V}_{\lambda,\alpha,p}$ and $1 \le p < \infty$ we use the elementary inequality

$$e^{-\eta|g|^2}(1+|g|)^\beta = e^\eta e^{-\eta(1+|g|)^2}(1+|g|)^\beta \le C_\beta e^\eta \eta^{-\frac{\beta}{2}}, \quad \eta > 0, \;\; g \in G,$$

to obtain

$$\begin{aligned}\|e^{-\eta A}\vec{\mathbf{u}}\|^p_{\lambda,\alpha+\beta,p} &= \int_G e^{-\eta p|g|^2} e^{\lambda p|g|}(1+|g|)^{\alpha p+\beta p}|\vec{\mathbf{u}}(g)|^p d\mu(g)\\ &\le C_\beta e^{\eta p}\eta^{-\frac{\beta p}{2}} \int_G e^{\lambda p|g|}(1+|g|)^{\alpha p}|\vec{\mathbf{u}}(g)|^p d\mu(g).\end{aligned}$$

This proves the first part for $1 \le p < \infty$. The proof in case $p = \infty$ is similar.

To prove the second part, let $K \ge 1$ and set $\vec{\mathbf{u}}_K = \chi_{|g|\le K}\vec{\mathbf{u}}$, where χ_A is the characteristic function of a set A. Then, using the result just proved,

$$\begin{aligned}\|e^{-\eta A}\vec{\mathbf{u}}\|_{\lambda,\alpha+\beta,p} &\le \|e^{-\eta A}\vec{\mathbf{u}}_K\|_{\lambda,\alpha+\beta,p} + \|e^{-\eta A}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_K)\|_{\lambda,\alpha+\beta,p}\\ &\le \|e^{-\eta A}\vec{\mathbf{u}}_K\|_{\lambda,\alpha+\beta,p} + C_\beta e^\eta \eta^{-\frac{\beta}{2}}\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_K\|_{\lambda,\alpha,p}.\end{aligned} \tag{28}$$

It is easy to see that for any $\gamma \in \mathbf{R}$ and $1 \le q < \infty$, $e^{-\eta A}$ is a contraction on $\mathbb{V}_{\lambda,\gamma,q}$ and $\vec{\mathbf{u}}_K \in \mathbb{V}_{\lambda,\gamma,q}$. Consequently, $\lim_{\eta\to 0^+} \eta^{\frac{\beta}{2}}\|e^{-\eta A}\vec{\mathbf{u}}_K\|_{\lambda,\alpha+\beta,p} = 0$. It now follows from (28) that

$$\limsup_{\eta \to 0^+} \eta^{\frac{\beta}{2}}\|e^{-\eta A}\vec{\mathbf{u}}\|_{\lambda,\alpha+\beta,p} \le C_\beta\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_K\|_{\lambda,\alpha,p}.$$

The result now follows by noting that $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_K\|_{\lambda,\alpha,p} \to 0$ as $K \to \infty$. □

Proposition 4.5. *Let* $0 \le s \le t < \infty$, $\alpha \in \mathbf{R}$ *and* $0 \le \lambda < 1$. *Then for any* $\vec{\mathbf{u}} \in \mathbb{V}_{\lambda\sqrt{s},\alpha,p}$ *we have*

$$\|e^{-(t-s)A}\vec{\mathbf{u}}\|_{\lambda\sqrt{t},\alpha,p} \le e^\lambda \|e^{-(1-\lambda)(t-s)A}\vec{\mathbf{u}}\|_{\lambda\sqrt{s},\alpha,p}.$$

Proof. Assume that $0 \le s \le t < \infty$. Then,

$$\begin{aligned}\|e^{-(t-s)A}\vec{\mathbf{u}}\|^p_{\lambda\sqrt{t},\alpha,p} &= \int_G e^{-(t-s)p|g|^2} e^{\lambda p\sqrt{t}|g|}(1+|g|)^{\alpha p}|\vec{\mathbf{u}}(g)|^p d\mu(g)\\ &= \int_G e^{-\lambda p(t-s)|g|^2 + \lambda p\sqrt{t}|g| - \lambda p\sqrt{s}|g|} e^{-(1-\lambda)(t-s)p|g|^2} e^{\lambda p\sqrt{s}|g|}(1+|g|)^{\alpha p}|\vec{\mathbf{u}}(g)|^p d\mu(g)\\ &\le e^{\lambda p}\int_G e^{-(1-\lambda)(t-s)p|g|^2} e^{\lambda p\sqrt{s}|g|}(1+|g|)^{\alpha p}|\vec{\mathbf{u}}(g)|^p d\mu(g),\end{aligned} \tag{29}$$

where (29) follows by noting that for $0 \le s \le t$,

$$\big(\sqrt{t} - \sqrt{s}\big)|g| - (t-s)|g|^2 = \big(\sqrt{t} - \sqrt{s}\big)|g|\big(1 - (\sqrt{t} + \sqrt{s})|g|\big) \le 1.$$

This can be easily seen by considering the cases $\sqrt{t}|g| \geq 1$ and $\sqrt{t}|g| \leq 1$ separately. This finishes the proof of the proposition. $\square$

For each $T > 0$ and $\beta \geq 0$, we let $\Sigma = \Sigma_T$ denote the space of all $\vec{\mathbf{u}}(\cdot) \in C([0, T]; \mathbb{V}_{\alpha,p})$ with the property that

$$\|\vec{\mathbf{u}}(\cdot)\|_\Sigma := \max\left\{\sup_{0\leq t\leq T} \|\vec{\mathbf{u}}(t)\|_{\lambda\sqrt{t},\alpha,p}, \quad \sup_{0<t\leq T} t^{\frac{\beta}{2}}\|\vec{\mathbf{u}}(t)\|_{\lambda\sqrt{t},\alpha+\beta,p}\right\} < \infty.$$

For convenience we denote by $\|\cdot\|_{\Sigma'}$ the norm

$$\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'} = \sup_{0<t\leq T} t^{\frac{\beta}{2}}\|\vec{\mathbf{u}}(t)\|_{\lambda\sqrt{t},\alpha+\beta,p}.$$

Let $\vec{\mathbf{u}}_0 \in \mathbb{V}_{\alpha,p}$ and define

$$\vec{\mathbf{G}}(t) = e^{-tA}\vec{\mathbf{u}}_0. \tag{30}$$

Propositions 4.4 and 4.5 (with $s = 0$) imply that

$$\|\vec{\mathbf{G}}(t)\|_{\lambda\sqrt{t},\alpha,p} \leq e^{\lambda}\|e^{-(1-\lambda)tA}\vec{\mathbf{u}}_0\|_{\alpha,p} \leq C_\lambda\|\vec{\mathbf{u}}_0\|_{\alpha,p}$$

and

$$\|\vec{\mathbf{G}}(t)\|_{\lambda\sqrt{t},\alpha+\beta,p} \leq e^{\lambda}\|e^{-(1-\lambda)tA}\vec{\mathbf{u}}_0\|_{\alpha+\beta,p} \leq C_\beta((1-\lambda)t)^{-\frac{\beta}{2}}\|\vec{\mathbf{u}}_0\|_{\alpha,p}.$$

Consequently

$$\|\vec{\mathbf{G}}(\cdot)\|_\Sigma \leq C_{\beta,\lambda}\|\vec{\mathbf{u}}_0\|_{\alpha,p}. \tag{31}$$

Next define

$$N(T) = \|\vec{\mathbf{G}}(\cdot)\|_{\Sigma'} \tag{32}$$

so that $N(T) \leq C_{\beta,\lambda}\|\vec{\mathbf{u}}_0\|_{\alpha,p}$. Moreover, for $1 \leq p < \infty$ and $\beta > 0$, by Proposition 4.4 we also have

$$\lim_{T\to 0^+} N(T) = 0. \tag{33}$$

Finally define

$$b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot)) = -\int_0^t e^{-(t-s)A}B[\vec{\mathbf{u}}(s), \vec{\mathbf{v}}(s)]ds.$$

We will show that $b : \Sigma \times \Sigma \to \Sigma$. Recall from the preceding section that if $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{V}_{\lambda,\gamma,p}$, then

$$\|\tau\vec{\mathbf{u}} * \tau\vec{\mathbf{v}}\|_{\lambda,2\gamma-\frac{n}{p'},p} \leq C_{n,\gamma,p}\|\vec{\mathbf{u}}\|_{\lambda,\gamma,p}\|\vec{\mathbf{v}}\|_{\lambda,\gamma,p} \tag{34}$$

if $\frac{n}{2p'} < \gamma < \frac{n}{p'}$, and

$$\|\tau\vec{\mathbf{u}} * \tau\vec{\mathbf{v}}\|_{\lambda,\gamma,p} \le C_{n,\gamma,p}\|\vec{\mathbf{u}}\|_{\lambda,\gamma,p}\|\vec{\mathbf{v}}\|_{\lambda,\gamma,p} \tag{35}$$

if $\gamma > \frac{n}{p'}$.

Proposition 4.6. *Let* $1 < p \le \infty$ *and* $0 \le \lambda < 1$ *and choose* $\beta \ge 0$ *satisfying*

$$\begin{aligned} &\max\left\{0, \frac{n}{2p'} - \alpha\right\} < \beta < \frac{n}{p'} - \alpha && \textit{for } \frac{n}{p'} - 1 \le \alpha < \frac{n}{p'},\\ &0 < \beta < 1 && \textit{for } \alpha = \frac{n}{p'},\\ &\beta = 0 && \textit{for } \alpha > \frac{n}{p'}. \end{aligned} \tag{36}$$

If $\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot) \in \Sigma$ *then*

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \le C_{\alpha,\beta,\lambda,p}e^{T}\begin{cases} T^{\frac{1}{2}(1-\frac{n}{p'}+\alpha)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} & \textit{if } \frac{n}{p'} - 1 \le \alpha < \frac{n}{p'},\\ T^{\frac{1}{2}(1-\beta)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} & \textit{if } \alpha = \frac{n}{p'},\\ T^{\frac{1}{2}}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} & \textit{if } \alpha > \frac{n}{p'}. \end{cases}$$

When $p = 1$ *and* $\alpha > -1$, *with* $\beta = \max\{-\alpha, 0\}$, *we have*

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \le C_{\alpha,\beta,\lambda,p}e^{T}T^{\frac{1}{2}(1-\beta)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'}.$$

Proof. Let $\delta \in \mathbf{R}$ satisfy $\delta > 2\alpha + 2\beta - \frac{n}{p'} - 1$. The triangle inequality and Proposition 4.5 imply

$$\begin{aligned} \|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\delta,p} &\le \int_0^t \|e^{-(t-s)A}B[\vec{\mathbf{u}}(s), \vec{\mathbf{v}}(s)]\|_{\lambda\sqrt{t},\delta,p}ds\\ &\le e^{\lambda}\int_0^t \|e^{-(1-\lambda)(t-s)A}B[\vec{\mathbf{u}}(s), \vec{\mathbf{v}}(s)]\|_{\lambda\sqrt{s},\delta,p}ds \end{aligned} \tag{37}$$

for all $0 < t \le T$. Using (27), we obtain from the previous inequalities that

$$\|e^{-(1-\lambda)(t-s)A}B[\vec{\mathbf{u}}(s), \vec{\mathbf{v}}(s)]\|_{\lambda\sqrt{s},\delta,p} \le \|e^{-(1-\lambda)(t-s)A}((\tau\vec{\mathbf{u}})(s) * (\tau\vec{\mathbf{v}})(s))\|_{\lambda\sqrt{s},\delta+1,p} \tag{38}$$

for all $0 \le s \le t$. Due to the choice of β as in the proposition above, we now have two cases:

Case I. In case $\frac{n}{2p'} < \alpha + \beta < \frac{n}{p'}$, Proposition 4.4 implies

$$\begin{aligned} &\|e^{-(1-\lambda)(t-s)A}((\tau\vec{\mathbf{u}})(s) * (\tau\vec{\mathbf{v}})(s))\|_{\lambda\sqrt{s},\delta+1,p}\\ &\quad \le C_{\lambda,\alpha,\beta,\delta,p}e^{(t-s)}\frac{\|(\tau\vec{\mathbf{u}})(s) * (\tau\vec{\mathbf{v}})(s)\|_{\lambda\sqrt{s},2\alpha+2\beta-\frac{n}{p'},p}}{(t-s)^{\frac{1}{2}(1+\frac{n}{p'}+\delta-2\alpha-2\beta)}}, \end{aligned} \tag{39}$$

and (34) implies in turn that

$$\begin{aligned}\|(\tau\vec{\mathbf{u}})(s)*(\tau\vec{\mathbf{v}})(s)\|_{\lambda\sqrt{s},2\alpha+2\beta-\frac{n}{p'},p} &\leq C_{\alpha,\beta,p}\|(\tau\vec{\mathbf{u}})(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}\|(\tau\vec{\mathbf{v}})(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}\\ &= C_{\alpha,\beta,p}\|\vec{\mathbf{u}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}\|\vec{\mathbf{v}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}. \qquad (40)\end{aligned}$$

The definition of the Σ' norm implies that

$$\|\vec{\mathbf{u}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}\|\vec{\mathbf{v}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p} \leq s^{-\beta}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'}, \qquad (41)$$

so we obtain from (37)–(41) that

$$\|b(\vec{\mathbf{u}}(\cdot),\vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\delta,p} \leq C_{\alpha,\beta,\delta,\lambda,p}e^T\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}(1+\frac{n}{p'}+\delta-2\alpha-2\beta)}s^{\beta}}ds$$

for all $0 < t \leq T$. A simple calculation shows that if $c, d < 1$ then

$$\int_0^t \frac{1}{(t-s)^c s^d}ds = C_{c,d}t^{1-c-d}$$

for all $t > 0$, and therefore

$$\|b(\vec{\mathbf{u}}(\cdot),\vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\delta,p} \leq C_{\alpha,\beta,\delta,\lambda,p}e^T t^{\frac{1}{2}(1-\frac{n}{p'}+2\alpha-\delta)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma} \qquad (42)$$

provided that

$$\beta < 1 \quad \text{and} \quad \frac{1}{2}\left(1+\frac{n}{p'}+\delta-2\alpha-2\beta\right) < 1.$$

The choice of β in (36) implies that this condition is satisfied for $\delta = \alpha$ and $\delta = \alpha+\beta$. With $\delta = \alpha$ in (42) we obtain

$$\|b(\vec{\mathbf{u}}(\cdot),\vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\alpha,p} \leq C_{\alpha,\beta,\lambda,p}e^T T^{\frac{1}{2}(1-\frac{n}{p'}+\alpha)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} \qquad (43)$$

for all $0 \leq t \leq T$, and with $\delta = \alpha+\beta$ in (42) we obtain

$$t^{\frac{\beta}{2}}\|b(\vec{\mathbf{u}}(\cdot),\vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\alpha+\beta,p} \leq C_{\alpha,\beta,\lambda,p}e^T T^{\frac{1}{2}(1-\frac{n}{p'}+\alpha)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} \qquad (44)$$

for all $0 < t \leq T$.

Case II. In case $\frac{n}{p'} < \alpha+\beta$, we use Proposition 4.4 and (35) to obtain

$$\begin{aligned}&\|e^{-(1-\lambda)(t-s)A}((\tau\vec{\mathbf{u}})(s)*(\tau\vec{\mathbf{v}})(s))\|_{\lambda\sqrt{s},\delta+1,p} \qquad (45)\\ &\leq C_{\lambda,\alpha,\beta,\delta,p}e^{(t-s)}\frac{\|(\tau\vec{\mathbf{u}})(s)*(\tau\vec{\mathbf{v}})(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}}{(t-s)^{\frac{1}{2}(1+\delta-\alpha-\beta)}}\\ &\leq C_{\lambda,\alpha,\beta,\delta,p}e^{(t-s)}\frac{\|\vec{\mathbf{u}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}\|\vec{\mathbf{v}}(s)\|_{\lambda\sqrt{s},\alpha+\beta,p}}{(t-s)^{\frac{1}{2}(1+\delta-\alpha-\beta)}}\\ &\leq C_{\lambda,\alpha,\beta,\delta,p}e^{(t-s)}\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}\|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'}\frac{1}{s^{\beta}(t-s)^{\frac{1}{2}(1+\delta-\alpha-\beta)}}. \qquad (46)\end{aligned}$$

Proceeding as in the previous case, provided that

$$\beta < 1 \quad \text{and} \quad \frac{1}{2}(1+\delta-\alpha-\beta) < 1,$$

we obtain

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\lambda\sqrt{t},\delta,p} \leq C_{\alpha,\beta,\delta,\lambda,p} e^T T^{\frac{1}{2}(1+\alpha-\beta-\delta)} \|\vec{\mathbf{u}}(\cdot)\|_{\Sigma} \|\vec{\mathbf{v}}(\cdot)\|_{\Sigma}$$

Using (4) with $\delta = \alpha$ and $\delta = \alpha + \beta$ and in view of (43) and (44), we get

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \leq C_{\alpha,\beta,\lambda,p} e^T \begin{cases} T^{\frac{1}{2}(1-\frac{n}{p'}+\alpha)} \|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'} \|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} & \text{if } \dfrac{n}{2p'} < \alpha+\beta < \dfrac{n}{p'}, \\ T^{\frac{1}{2}(1-\beta)} \|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'} \|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'} & \text{if } \dfrac{n}{p'} < \alpha+\beta. \end{cases}$$

The proof for the case $p = 1$ is similar and is omitted. □

The following proposition is a general existence theorem for solutions to a nonlinear equation in a Banach space. The presence of the functional q serves to make the proposition directly applicable to the proof of our main results which follow. The proof is an easy application of the Banach fixed point theorem and is not given here. The details can be found e.g., in [3].

Proposition 4.7. *Let Σ be a Banach space with norm $\|\cdot\|_{\Sigma}$ and let $q : \Sigma \to [0,\infty)$ be a subadditive functional on Σ satisfying $q \leq \|\cdot\|_{\Sigma}$. Suppose that $G \in \Sigma$, and that $b : \Sigma \times \Sigma \to \Sigma$ is bilinear. Let*

$$E = \{u \in \Sigma : q(u-G) \leq q(G)\}.$$

If there exists $0 < \theta < 1/2$ with the property that

$$\|b(u,v)\|_{\Sigma} \leq \theta q(v)$$

whenever $u \in E$ and $v \in \Sigma$, and

$$\|b(u,v)\|_{\Sigma} \leq \theta q(u)$$

whenever $u \in \Sigma$ and $v \in E$, then there exists a unique $u \in E$ satisfying

$$u = G + b(u,u).$$

Now we are ready to prove Theorem 4.1. In order to find a solution to (26) it is enough to find a time T so that the equation

$$\vec{\mathbf{u}}(\cdot) = \vec{\mathbf{G}}(\cdot) + b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{u}}(\cdot)) \tag{47}$$

has a solution in Σ. This is accomplished by letting

$$E = \left\{\vec{\mathbf{u}}(\cdot) \in \Sigma : \|\vec{\mathbf{u}}(\cdot) - \vec{\mathbf{G}}(\cdot)\|_{\Sigma'} \leq \|\vec{\mathbf{G}}(\cdot)\|_{\Sigma'}\right\} \tag{48}$$

and verifying the hypotheses of Proposition 4.7 with $q = \|\cdot\|_{\Sigma'}$. If $\vec{\mathbf{u}}(\cdot) \in E$, then (32), (48) and the triangle inequality imply that $\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'} \le 2N(T)$. If in addition $\vec{\mathbf{v}}(\cdot) \in \Sigma$, then Proposition 4.6 implies

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \le 2C_{\alpha,\beta,\lambda,p} e^{T} T^{\gamma} N(T) \|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'},$$

for an appropriate $\gamma \ge 0$ as given by Proposition 4.6. We only note here that $\gamma = 0$ in case $1 < p \le \infty$ and $\alpha = \frac{n}{p'} - 1$ and $\gamma > 0$ otherwise. Likewise, if $\vec{\mathbf{u}}(\cdot) \in \Sigma$ and $\vec{\mathbf{v}}(\cdot) \in E$, then

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \le 2C_{\alpha,\beta,\lambda,p} e^{T} T^{\gamma} N(T) \|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}.$$

In light of (31) and (32) when $\gamma > 0$ and (33) when $\gamma = 0$, we may choose T sufficiently small to ensure that

$$C_{\alpha,\beta,\lambda,p} e^{T} T^{\gamma} N(T) < \frac{1}{3}. \tag{49}$$

With this choice of T, Proposition 4.7 implies the existence of a function $\vec{\mathbf{u}}(\cdot) \in E$ satisfying (47). This concludes the proof Theorem 4.1.

When $G = \mathbf{R}^n$ or $G = \mathbf{Z}^n$, this argument may be modified to give a global Gevrey regular solution for small initial data. The case $G = \mathbf{Z}^n$ has been done in [1]. We will sketch the proof for $G = \mathbf{R}^n$. In order to prove the existence of global solutions to (26), we introduce the space $\mathbb{V}'_{\lambda,\alpha,p}$ given by

$$\mathbb{V}'_{\lambda,\alpha,p} = \left\{ \vec{\mathbf{u}} \in \mathbb{V} : \|\vec{\mathbf{u}}\|'_{\lambda,\alpha,p} := \left(\int_G e^{\lambda p|x|} |x|^{\alpha p} |\vec{\mathbf{u}}(x)| d\mu(x) \right)^{1/p} < \infty \right\}.$$

If $\vec{\mathbf{u}} \in V'_{\lambda,\alpha,p}$, then the conclusion of Proposition 4.4 may be improved to read

$$\|e^{-\eta A}\vec{\mathbf{u}}\|'_{\lambda,\alpha+\beta,p} \le C_{\beta} \eta^{-\frac{\beta}{2}} \|\vec{\mathbf{u}}\|_{\lambda,\alpha,p}.$$

Provided that the definitions of the spaces Σ and Σ' are suitably modified to accommodate the $\|\cdot\|'_{\lambda,\alpha,p}$-norm, the global existence result may be obtained from the following proposition which establishes the boundedness of the bilinear operator b. The proof of this proposition closely resembles the proof of Proposition 4.6.

Proposition 4.8. *Let* $1 < p \le \infty$, $\alpha = \frac{n}{p'} - 1$ *and* $0 \le \lambda < 1$ *and let* β *satisfy* $\max\{0, 1 - \frac{n}{2p'}\} < \beta < 1$. *If* $\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot) \in \Sigma$ *then*

$$\|b(\vec{\mathbf{u}}(\cdot), \vec{\mathbf{v}}(\cdot))\|_{\Sigma} \le C_{\alpha,\beta,\lambda,p} \|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'} \|\vec{\mathbf{v}}(\cdot)\|_{\Sigma'}.$$

The following theorem asserts the existence of solutions in $\mathbb{V}'_{\alpha,p}$.

Theorem 4.9. *Assume that* $1 < p \le \infty$, $\alpha = \frac{n}{p'} - 1$ *and* $\vec{\mathbf{u}}_0 \in \mathbb{V}'_{\alpha,p}$. *Then there exist* $T > 0$ *and a solution* $\vec{\mathbf{u}}(\cdot) \in C([0, T]; \mathbb{V}'_{\alpha,p})$ *of* (26) *which satisfies*

$$\sup_{0 \le t \le T} \|\vec{\mathbf{u}}(t)\|'_{\lambda\sqrt{t},\alpha,p} < \infty.$$

Moreover, there exists an adequate constant $C_{\lambda,\alpha,p}$ *such that we may take* $T=\infty$ *provided* $\|\vec{\mathbf{u}}_0\|'_{\alpha,p}\le C_{\lambda,\alpha,p}$.

Proof. We proceed exactly as in the proof of Theorem 4.1 but apply Proposition 4.8 instead of Proposition 4.6. Consequently, if $\vec{\mathbf{u}}(\cdot)\in\Sigma'$ and $\vec{\mathbf{v}}(\cdot)\in E$, or if $\vec{\mathbf{u}}(\cdot)\in E$ and $\vec{\mathbf{v}}(\cdot)\in\Sigma'$, then we obtain

$$\|b(\vec{\mathbf{u}}(\cdot),\vec{\mathbf{v}}(\cdot))\|_\Sigma\le 2C_{\alpha,\beta,\lambda,p}N(T)\|\vec{\mathbf{u}}(\cdot)\|_{\Sigma'}.$$

In view of (31) and (32), we have that

$$N(T)\le C_{\beta,\lambda}\|\vec{\mathbf{u}}_0\|'_{\alpha,p},$$

meaning that the inequality

$$2C_{\alpha,\beta,\lambda,p}N(T)<\frac{1}{2}$$

will be satisfied with $T=\infty$ provided that $\|\vec{\mathbf{u}}_0\|'_{\alpha,p}$ is sufficiently small. In view of Proposition 4.7, the proof of the theorem is now complete. □

5. Some Necessary Conditions

In this section we consider conditions necessary for the inequality

$$\|f*g\|_{\omega_3,r}\le C\|f\|_{\omega_1,p}\|g\|_{\omega_2,q} \tag{50}$$

to hold in a group G.

Theorem 5.1. *Let G be a discrete group. Suppose that* $1\le p,q,r<\infty$ *and that* (50) *holds for all* $f\in L^p_{\omega_1}(G)$ *and* $g\in L^q_{\omega_2}(G)$. *Then*

$$\omega_3(x)\le C\omega_1(y)\omega_2(x-y)$$

for all $x,y\in G$.

Proof. Let $x_0,y_0\in G$. Define $f(x)=\chi_{x_0}(x)$ and $g(x)=\chi_{y_0}(x)$ so that $\|f\|_{\omega_1,p}=\omega_1(x_0)$ and $\|g\|_{\omega_2,q}=\omega_2(y_0)$. A simple computation shows

$$f*g(x)=\chi_{x_0}*\chi_{y_0}(x)=\chi_{x_0+y_0}(x),$$

and therefore $\|f*g\|_{\omega_3,r}=\omega_3(x_0+y_0)$. □

Lemma 5.2. *Let* $G=\mathbf{R}^n$. *Suppose that* $1\le p,q,r<\infty$ *and that* (50) *holds for all* $f\in L^p_{\omega_1}(G)$ *and* $g\in L^q_{\omega_2}(G)$. *Then*

$$\rho^n\left(\int_{B(x_0+y_0,\rho)}\omega_3^r dx\right)^{1/r}\le C\left(\int_{B(x_0,\rho)}\omega_1^p dx\right)^{1/p}\left(\int_{B(y_0,\rho)}\omega_2^q dx\right)^{1/q} \tag{51}$$

for all $x,y\in G$ *and* $\rho>0$.

Proof. Let $x_0, y_0 \in \mathbf{R}^n$ and let $\rho > 0$. Define the functions

$$f(x) = \chi_{B(x_0,\rho)}(x) \quad \text{and} \quad g(x) = \chi_{B(y_0,\rho)}(x),$$

so that

$$\|f\|_{\omega_1,p}\|g\|_{\omega_2,q} = \left(\int_{B(x_0,\rho)} \omega_1^p dx\right)^{1/p}\left(\int_{B(y_0,\rho)} \omega_2^q dx\right)^{1/q}. \tag{52}$$

Now let $x, y \in \mathbf{R}^n$. Since $x - y \in B(x_0, \rho)$ if and only if $y \in B(x - x_0, \rho)$ we have

$$\begin{aligned} f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)dy &= \int_{B(y_0,\rho)} \chi_{B(x_0,\rho)}(x-y)dy \\ &= \int_{B(y_0,\rho)} \chi_{B(x-x_0,\rho)}(y)dy \\ &= |B(y_0,\rho) \cap B(x-x_0,\rho)|. \end{aligned}$$

This implies that

$$\begin{aligned} \|f * g\|_{\omega_3,r} &= \left(\int_{\mathbf{R}^n} |B(y_0,\rho) \cap B(x-x_0,\rho)|^r \omega_3(x)^r dx\right)^{1/r} \\ &\geq \left(\int_{B(x_0+y_0,\rho)} |B(y_0,\rho) \cap B(x-x_0,\rho)|^r \omega_3(x)^r dx\right)^{1/r}. \end{aligned}$$

Now, if $x \in B(x_0 + y_0, \rho)$ then $B(y_0, \rho) \cap B(x - x_0, \rho)$ contains the ball $B(\xi, \rho/2)$, where $\xi = \frac{1}{2}(y_0 - x_0 + x)$. This means that

$$|B(y_0,\rho) \cap B(x-x_0,\rho)| \geq C_n\rho^n \quad \text{for all } x \in B(x_0+y_0,\rho),$$

which yields

$$\int_{B(x_0+y_0,\rho)} |B(x_0,\rho) \cap B(x-x_0,\rho)|^r \omega_3(x)^r dx \geq C\rho^{rn} \int_{B(x_0+y_0,\rho)} \omega_3(x)^r dx.$$

Putting these inequalities together we have

$$\|f * g\|_{\omega_3,r} \geq C_n\rho^n \left(\int_{B(x_0+y_0,\rho)} \omega_3^r dx\right)^{1/r},$$

which, along with (50) and (52), gives (51). □

As an application of this result, we have a simple proof of the following known theorem concerning power weights on $\mathbf{R}^n$.

Theorem 5.3. *Let $G = \mathbf{R}^n$ and suppose the convolution inequality*

$$\left(\int_{\mathbf{R}^n} |f * g(x)|^r |x|^{cr} dx\right)^{1/r} \leq C\left(\int_{\mathbf{R}^n} |f(x)|^p |x|^{ap} dx\right)^{1/p}\left(\int_{\mathbf{R}^n} |g(x)|^p |x|^{bq} dx\right)^{1/q} \tag{53}$$

holds for all f and g. Then, we must have

$$\frac{n}{t'} = a + b - c.$$

Proof. If (53) holds for all f and g we may take $x_0 = y_0 = 0$ in (51) to obtain

$$\rho^n \left(\int_{B(0,\rho)} |x|^{cr} dx \right)^{1/r} \leq C \left(\int_{B(0,\rho)} |x|^{ap} dx \right)^{1/p} \left(\int_{B(0,\rho)} |x|^{bq} dx \right)^{1/q},$$

which implies that

$$\rho^{n+c+n/r} \leq C\rho^{a+n/p+b+n/q}$$

for all $\rho > 0$. This is possible only if the exponents are equal, hence

$$n + c + \frac{n}{r} = a + \frac{n}{p} + b + \frac{n}{q}.$$

Therefore a necessary condition for (53) is

$$\frac{n}{t'} = a + b - c.$$

□

Next we consider the particular case $t = 1$.

Theorem 5.4. *Suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and that* (50) *holds. Then*

$$\omega_3(x) \leq C\omega_1(y)\omega_2(x - y)$$

for almost all $x, y \in \mathbf{R}^n$.

Proof. In this case, (51) may be written

$$\left(\fint_{B(x_0+y_0,\rho)} \omega_3^r dx \right)^{1/r} \leq C \left(\fint_{B(x_0,\rho)} \omega_1^p dx \right)^{1/p} \left(\fint_{B(y_0,\rho)} \omega_2^q dx \right)^{1/q},$$

where $\fint_E$ denotes the average integral over the set E. Letting $\rho \to 0^+$ we have

$$\omega_3(x_0 + y_0) \leq Cw_1(x_0)\omega_2(y_0)$$

whenever x_0, y_0, and $x_0 + y_0$ are Lebesgue points of ω_1, ω_2, and ω_3, respectively. □

Finally we consider the convolution inequality

$$\left(\int_{\mathbf{R}^n} |(f * g)(x)|^r \varphi(x) dx \right)^{1/r} \leq C \left(\int_{\mathbf{R}^n} |f(x)|^p \varphi(x) dx \right)^{1/p} \left(\int_{\mathbf{R}^n} |g(x)|^q \varphi(x) dx \right)^{1/q} \tag{54}$$

for a weight $\varphi : \mathbf{R}^n \to [0, \infty)$ and exponents $1 \leq p, q, r < \infty$. This corresponds to the case $\omega_1 = \varphi^{1/p}$, $\omega_2 = \varphi^{1/q}$, and $\omega_3 = \varphi^{1/r}$ in (50).

Proposition 5.5. *If* (54) *holds then either* $\varphi = 0$ *almost everywhere or* $\varphi > 0$ *almost everywhere.*

Proof. Suppose $|\{x : \varphi(x) = 0\}| > 0$ and let $A \subset \{x : \varphi(x) = 0\}$ be a measurable set satisfying $0 < |A| < \infty$. Define

$$f = g = \chi_A.$$

Since $|A| > 0$ there exists $\varepsilon > 0$ and a ball $B = B(x_0, \rho)$ with the property that $f * g > \varepsilon$ on B. It follows from (54) that

$$\int_B \varphi dx \le 0,$$

and therefore φ vanishes almost everywhere on B. Let $y_0 \in \mathbf{R}^n$. According to Lemma 5.2 it follows that

$$\rho^n\bigg(\int_{B(y_0,\rho)} \varphi\, dx\bigg)^{1/r} \le C\bigg(\int_{B(x_0,\rho)} \varphi\, dx\bigg)^{1/p}\bigg(\int_{B(y_0-x_0,\rho)} \varphi\, dx\bigg)^{1/q} = 0,$$

hence φ vanishes on a neighborhood of y_0. Since y_0 is arbitrary, φ vanishes almost everywhere on $\mathbf{R}^n$. □

For the remainder of this section we assume that

$$K = \frac{1}{p} + \frac{1}{q} - \frac{1}{r}. \tag{55}$$

Proposition 5.6. *If* (54) *holds and* $K > 1$, *then* $\varphi = 0$ *almost everywhere.*

Proof. Since φ is locally integrable, the Hardy–Littlewood maximal function

$$M_1\varphi(x) = \sup_{0<r\le 1} ⨍_{B(x,r)} |\varphi(z)|dz \tag{56}$$

is finite almost everywhere. It follows from the Lebesgue differentiation theorem that the set

$$E = \bigg\{x : \lim_{\rho\to 0^+} ⨍_{B(x,\rho)} \varphi(z)dz = \varphi(x) \quad \text{and} \quad M_1\varphi\Big(\frac{x}{2}\Big) < \infty\bigg\}$$

satisfies $|\mathbf{R}^n \backslash E| = 0$. Let $x \in E$. Appealing to Lemma 5.2 with $x_0 = y_0 = x/2$ we have

$$\rho^n\bigg(\int_{B(x,\rho)} \varphi\, dz\bigg)^{1/r} \le C\bigg(\int_{B(x/2,\rho)} \varphi\, dz\bigg)^{1/p}\bigg(\int_{B(x/2,\rho)} \varphi\, dz\bigg)^{1/q}$$

for all $\rho > 0$. Thus

$$\begin{aligned}\bigg(⨍_{B(x,\rho)} \varphi\, dz\bigg)^{1/r} &\le C\rho^{n(K-1)}\bigg(⨍_{B(x/2,\rho)} \varphi\, dz\bigg)^{1/p+1/q} \\ &\le C\rho^{n(K-1)} M_1\varphi(x/2)^{1/p+1/q}\end{aligned}$$

for all $0 < \rho \leq 1$. It follows that

$$\varphi(x) = \lim_{\rho \to 0^+} \fint_{B(x,\rho)} \varphi \, dx = 0.$$

Therefore $\varphi(x) = 0$ for all $x \in E$, hence $\varphi = 0$ almost everywhere. □

Proposition 5.7. *If* (54) *holds and* $K < \max\{\frac{1}{p}, \frac{1}{q}\}$, *then* $\varphi = 0$ *almost everywhere.*

Proof. Assume to the contrary that $\varphi > 0$ almost everywhere. Appealing to Lemma 5.2 with $x_0 = y_0 = 0$ we have

$$\rho^n \left(\int_{B(0,\rho)} \varphi \, dx \right)^{1/r} \leq C \left(\int_{B(0,\rho)} \varphi \, dx \right)^{1/p} \left(\int_{B(0,\rho)} \varphi \, dx \right)^{1/q}$$

for all $\rho > 0$. Thus

$$\rho^n \leq C \left(\int_{B(0,\rho)} \varphi \, dx \right)^K \tag{57}$$

In case $K \leq 0$ we obtain a contradiction, since the right-hand side of (57) is nonincreasing as $\rho \to \infty$, whereas the left-hand side increases to $+\infty$. Thus we may assume that $K > 0$. Moreover we may assume without loss of generality that $p \leq q$, so that $0 < K < \frac{1}{p}$. In this case note that

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = K < \frac{1}{p}$$

implies $1/r - 1/q > 0$. For each $x \in \mathbf{R}^n$ define

$$\Phi(x) = \int_{B(x,1)} \varphi(z) dz.$$

Appealing to Lemma 5.2 with $x_0 = x$ and $y_0 = 0$ we have

$$\Phi(x)^{1/r} \leq C\Phi(0)^{1/p}\Phi(x)^{1/q},$$

hence

$$\Phi(x)^{1/r-1/q} \leq C\Phi(0)^{1/p}. \tag{58}$$

Since $1/r > 1/q$ it follows from (58) that Φ is bounded. Next we claim that

$$\int_{B(x,\rho)} \varphi(z) dz \leq C\rho^n \|\Phi\|_\infty \tag{59}$$

whenever $x \in \mathbf{R}^n$ and $\rho > 1$. Since

$$B(x, \rho) \subset \bigcup_{z \in B(x,\rho)} B(z, 1/5)$$

we may employ an elementary covering argument (see e.g., [21, Theorem 1.3.1]) to find a (necessarily finite) sequence $x_1, x_2, \ldots, x_N \in B(x, \rho)$ with the property that the balls $B(x_k, 1/5)$ are disjoint, yet

$$B(x, \rho) \subset \bigcup_{k=1}^{N} B(x_k, 1).$$

Thus

$$\int_{B(x,\rho)} \varphi(z)dz \leq \sum_{k=1}^{N} \int_{B(x_k,1)} \varphi(z)dz = \sum_{k=1}^{N} \Phi(x_k) \leq N\|\Phi\|_\infty. \tag{60}$$

Evidently $B(x_k, 1/5) \subset B(x, 2\rho)$ for all k, and since these balls are disjoint it follows that

$$\sum_{k=1}^{N} |B(x_k, 1/5)| \leq C_n \rho^n.$$

Therefore $N \leq C_n \rho^n$. This fact, along with (60), yields (59). It follows from (57) that

$$\rho^n \leq C\|\Phi\|_\infty^K \rho^{nK}$$

for all $\rho > 1$, which is a contradiction for large ρ since $K < 1$. □

The preceding results are summarized in the following theorem.

Theorem 5.8. *Let* $\varphi : \mathbf{R}^n \to [0, \infty)$, *let* $1 \leq p, q, r < \infty$, *and let* K *be given by* (55).

(1) *If* $K = 1$, *then* (54) *is satisfied if and only if either* $\varphi \equiv 0$ *or*

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}^n} \left(\varphi(x)^{1/r}\|\varphi^{-1/p}(\cdot)\varphi^{-1/q}(x-\cdot)\|_\infty\right) < \infty.$$

(2) *If* $\max\{p^{-1}, q^{-1}\} \leq K < 1$, *then* (54) *is satisfied if either* $\varphi \equiv 0$ *or*

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}^n} \left(\varphi(x)^{1/r}\|\varphi^{-1/p}(\cdot)\varphi^{-1/q}(x-\cdot)\|_{t'}\right) < \infty,$$

where t' *is the Hölder conjugate of* $t = 1/K$.

(3) *In all other cases,* (54) *holds if and only if* $\varphi \equiv 0$.

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