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Gevrey regularity for the supercritical quasi-geostrophic equation ☆

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Dedicated to my teacher Professor Ciprian Foias on the occasion of his eightieth birthday. La Multi Ani.

Abstract

In this paper, following the techniques of Foias and Temam, we establish suitable Gevrey class regularity of solutions to the supercritical quasi-geostrophic equations in the whole space, with initial data in "critical" Sobolev spaces. Moreover, the Gevrey class that we obtain is "near optimal" and as a corollary, we obtain temporal decay rates of higher order Sobolev norms of the solutions. Unlike the Navier–Stokes or the subcritical quasi-geostrophic equations, the low dissipation poses a difficulty in establishing Gevrey regularity. A new commutator estimate in Gevrey classes, involving the dyadic Littlewood–Paley operators, is established that allow us to exploit the cancellation properties of the equation and circumvent this difficulty.

MSC: primary 35Q35; secondary 35Q30, 76D05

Keywords: Quasigeostrophic equations; Gevrey regularity; Time decay

1. Introduction

We consider the dissipative, two-dimensional (surface) quasi-geostrophic equation (henceforth, QG) on $\mathbb{R}^2 \times (0, \infty)$ given by

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$$\left.\begin{array}{l}\partial_t\theta + \Lambda^\kappa\theta - u\cdot\nabla\theta = 0, \quad \theta(0)=\theta_0,\\ u = (-R_2\theta, R_1\theta), \quad R_i = \partial_i\Lambda^{-1},\ i=1,2.\end{array}\right\} \tag{1}$$

Here u is the velocity field, θ is the temperature, the operator $\Lambda = (-\Delta)^{1/2}$ with Δ denoting the Laplacian and the operators R_i are the usual Riesz transforms. The cases $\kappa > 1$, $\kappa = 1$ and $0 < \kappa < 1$ are known as the subcritical, critical and supercritical cases respectively. The QG arises in geophysics and meteorology (see, for instance [13,15,43]). Moreover, the critical QG is the dimensional analogue of the three dimensional Navier–Stokes equations. This equation has received considerable attention recently; see [44,17,19,12], and the references therein.

In this paper, we establish higher order (Gevrey class) regularity of solutions to the supercritical QG in the whole space $\mathbb{R}^2$, with initial data in the critical Sobolev space $\mathbb{H}^{2-\kappa}$. Although such results are known for the subcritical and critical QG [22,23], to the best of our knowledge this is the first such result for the supercritical case. We follow the Gevrey class technique introduced in the seminal work of Foias and Temam [26] for the Navier–Stokes equations, where they established analyticity, and provided explicit estimates of the analyticity radius, of solutions to the Navier–Stokes equations. In their approach, one avoids cumbersome recursive estimation of higher order derivatives and intricate combinatorial arguments. Since its introduction, Gevrey class technique has become a standard tool for studying analyticity properties of solutions for a wide class of dissipative equations and in various functional spaces (see [29,8,25,4,5,2] and the references therein). In [42], and subsequently in [6], it was shown how Gevrey norm estimates can be used to derive sharp bounds for the (time) decay of higher order derivatives of solutions to a wide class of dissipative equations including the Navier–Stokes equations. Other approaches to analyticity and higher order regularity can be found in [28,41,30] for the 3D NSE and [23] for the subcritical surface quasi-geostrophic equation.

The subcritical QG is fairly well understood; it possesses a globally regular (even, analytic) solution for adequate initial data (see [17,9,23]), as well as a compact global attractor [33]. However, relatively less is known concerning the critical ($\kappa = 1$) and the supercritical ($\kappa < 1$) QG. Concerning the critical case, the authors in [14] proved the existence of unique, globally regular solution for small initial data in L^∞. The global well-posedness for the critical QG, for initial data of arbitrary size, was solved independently in recent works [7,36]; subsequently they were generalized to include initial data in larger functional spaces [1,46]. In [16], an alternative proof of global well-posedness was found using a "nonlocal maximum principle". The supercritical case also received considerable attention of late, although less is known about it. It has been shown that it is *locally* well posed for initial data of arbitrary size in appropriate functional spaces while being globally well-posed for sufficiently *small initial data* in adequate functional spaces (see [10,40,32,34,12,31,24] and the references therein). Although the global well posedness for arbitrary initial data is still open (as of this writing) for the supercritical QG, a regularity criterion for solutions has been established [18] and [24] and eventual regularity has been addressed recently in [20].

Recall that f is said to belong to L^2-based Gevrey class G_α if

$$\|f\|_{\mathbb{H}^n} \le C_f\left(\frac{n!}{\rho^n}\right)^{\frac{1}{\alpha}}, \tag{2}$$

where $\mathbb{H}^n$ denotes the usual Sobolev space of order n. This can be characterized by the finiteness of the exponential norm $\|e^{\rho'\Lambda^\alpha} f\|_{L^2}$ for all $\rho' < \rho$. If $\alpha = 1$, then f is analytic with (uniform) analyticity radius ρ (see [39,42]) while $\alpha < 1$ corresponds to sub-analytic Gevrey classes. We show that for the supercritical QG in the whole space $\mathbb{R}^2$, and for sufficiently small initial data in $\mathbb{H}^{2-\kappa}$, a solution to (1) exists which moreover satisfies satisfying $\sup_{t>0}\|e^{\rho(t)\Lambda^\alpha}\theta(t)\|_{\dot{\mathbb{H}}^{2-\kappa}}<\infty$. Here $\alpha<\kappa\le 1$ and $\rho(\cdot)$ is an adequate function. This immediately implies a higher order decay estimate similar to (2). The result also holds locally for arbitrary initial data. As noted in Remark 3, the Gevrey class we obtain is "near optimal" and moreover, our result includes as corollary the higher regularity and decay results established in [22,24] with sharper constants (see Remark 5).

The idea of the proof follows that of [47] and [27] for the Navier–Stokes equations, suitably modified for Gevrey classes. A crucial step in establishing Gevrey regularity for the Navier–Stokes or the subcritical QG (see [5,6]) is to obtain an estimate in Gevrey classes of the form (in 2D)

$$\left\|e^{\lambda\Lambda^\alpha}(fg)\right\|_{\dot{\mathbb{H}}^\zeta}\le C\left\|e^{\lambda\Lambda^\alpha}(f)\right\|_{\dot{\mathbb{H}}^{\zeta_1}}\left\|e^{\lambda\Lambda^\alpha}(g)\right\|_{\dot{\mathbb{H}}^{\zeta_2}},\quad \zeta=\zeta_1+\zeta_2-1,\ \zeta_1,\zeta_2<1,\ \zeta_1+\zeta_2>0,$$

where $\dot{\mathbb{H}}^\zeta$ denotes homogeneous Sobolev spaces. This can be derived from the corresponding inequality in Sobolev spaces, namely,

$$\|fg\|_{\dot{\mathbb{H}}^\zeta}\le C\|f\|_{\dot{\mathbb{H}}^{\zeta_1}}\|g\|_{\dot{\mathbb{H}}^{\zeta_2}},\quad \zeta=\zeta_1+\zeta_2-1,\ \zeta_1,\zeta_2<1,\ \zeta_1+\zeta_2>0.$$

However, unlike the Navier–Stokes equations or the subcritical QG, due to the low dissipation in the supercritical case, one has to work in higher regularity spaces, namely $\mathbb{H}^\delta$, $\delta>1$, for well-posedness. An inequality of the above type does not hold in general in such spaces (as ζ_1,ζ_2 will have to be taken larger than 1). This poses a hurdle in establishing Gevrey regularity. To get around this difficulty, we establish in Theorem 2.1 a new (to the best of our knowledge) commutator estimate in Gevrey classes involving the dyadic Littlewood–Paley operators, which may be of independent interest as well. This commutator estimate allows us to exploit the cancellation properties of the equation to get around the challenges posed by low dissipation. Usually, if one works in Gevrey classes, one looses the cancellation properties of the equation which is available in L^2 spaces; see however the work in [39,38] where a certain cancellation property in the analytic Gevrey class was also used to establish analyticity estimate for the space periodic Euler equation. Our technique can be generalized to initial data in critical (and noncritical) Besov spaces. Due to the lack of Hilbert space structure as well as the Plancherel theorem, the corresponding estimates are more involved and will be presented in a future work.

The organization of the paper is as follows. In Section 2, we state our main results; in Section 3, we develop the requisite notation and background material while Sections 4 and 5 are devoted to the proof of the main results.

2. Main results

Denote by Δ the Laplacian and let $\Lambda=(-\Delta)^{1/2}$. For notational parsimony, we will denote $\|f\|_{L^2}=\|f\|$. The Sobolev and the homogeneous Sobolev spaces on $\mathbb{R}^2$ are respectively denoted by $\mathbb{H}^m$ and $\dot{\mathbb{H}}^m$, $m\in\mathbb{R}$. Recall that the corresponding norms are given by

$$\|f\|_{\mathbb{H}^m}=\left\|(I+\Lambda)^m f\right\|\quad\text{and}\quad\|f\|_{\dot{\mathbb{H}}^m}=\left\|\Lambda^m f\right\|,\quad m\in\mathbb{R}.$$

Recall that by the Plancherel theorem,

$$\|f\|_{\mathbb{H}^m} = \left(\int (1+|\xi|)^{2m} \left|\mathcal{F}f(\xi)\right|^2 d\xi\right)^{1/2} \quad \text{and} \quad \|f\|_{\dot{\mathbb{H}}^m} = \left(\int |\xi|^{2m} \left|\mathcal{F}f(\xi)\right|^2 d\xi\right)^{1/2};$$

here, and henceforth, $\mathcal{F}$ denotes the Fourier transform. As is well known, the Sobolev spaces are Hilbert spaces for all m. The homogeneous Sobolev spaces on the other hand are Hilbert spaces for $m < 1$, while they are normed inner product spaces (but not complete) for all $m \geq 1$ (see [21,3]).

Gevrey Norms. Let $0 \leq \alpha \leq 1$. We denote the Gevrey norms by

$$\|f\|_{G(s)} = \left\|e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\| \quad \text{and} \quad \|f\|_{G(s),\dot{\mathbb{H}}^m} = \left\|e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\|_{\dot{\mathbb{H}}^m} \quad (\lambda > 0 \text{ fixed}).$$

The Gevrey norms are characterized by the decay rates of higher order derivatives, namely, if $\|f\|_{G(s),\dot{\mathbb{H}}^m} < \infty$ for some $m \in \mathbb{R}$, then we have the higher derivative estimates

$$\|f\|_{\dot{\mathbb{H}}^{m+n}} \leq \left(\frac{n!}{\rho^n}\right)^{\frac{1}{\alpha}} \|f\|_{G(s),\dot{\mathbb{H}}^m} \quad \text{where } \rho = \lambda \alpha s^{\frac{\alpha}{\kappa}} \text{ and } n \in \mathbb{N}. \tag{3}$$

In particular, when $\alpha = 1$, f in (3) is analytic with (uniform) analyticity radius ρ, while for $\alpha < 1$ the corresponding functional classes are referred to as subanalytic Gevrey classes. For the above mentioned facts including (3), see Theorem 4 in [39] and Theorem 5 in [42].

Remark 1. The indices of s appearing in the definition of the Gevrey norms and in inequality (5) below, allow us to establish global Gevrey regularity result for small data in the whole space. They are dictated by the scaling properties of the equation. It is not possible (at least, in our work) to establish such global results unless they are precisely of that form. The global Gevrey regularity result in turn enables us to establish decay result for higher derivatives as given in Corollary 2.3 below.

We now describe our main results. The first one is a commutator estimate involving Gevrey norms which may be of independent interest. The second, which employs the first in its proof, concerns Gevrey regularity of solutions of the critical and subcritical quasi-geostrophic equations.

2.1. A commutator estimate in Gevrey classes

The commutator of two operators is defined as

$$[A, B] = AB - BA.$$

The estimate for the commutator $[f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j]$, where Δ_j denotes the (homogeneous) dyadic Littlewood–Paley operator, is crucial for our work.

Theorem 2.1. *Let* $f, g \in L^2$ *with* $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \in \dot{\mathbb{H}}^{1+\delta_1}$, $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \in \dot{\mathbb{H}}^{\delta_2}$ *and*

$$\min\{\zeta, \delta_1, \delta_2\} > 0, \quad \delta_1 + \zeta < 1, \ \delta_2 < 1 \text{ and } \zeta < \alpha. \tag{4}$$

Denote by Δ_j, $j \in \mathbb{Z}$ *the dyadic (homogeneous) Littlewood–Paley operators. There exists a constant* C*, independent of* j, s, f *and* g*, and a sequence of constants* $\{c_j\}_{j\in\mathbb{Z}}$ *(which may depend, in addition, on* s*,* f *and* g*) satisfying* $c_j \geq 0$ *and* $\sum_j c_j^2 \leq 1$*, such that*

$$\begin{aligned} &\left\| \left[f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j \right] g \right\| \\ &\quad \leq C c_j \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \right\|_{\dot{\mathbb{H}}^{1+\delta_1}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \right\|_{\dot{\mathbb{H}}^{\delta_2}} \left\{ s^{\frac{(\alpha-\zeta)}{\kappa}} 2^{-(\delta_1+\delta_2+\zeta-\alpha)j} + 2^{-(\delta_1+\delta_2)j} \right\}. \end{aligned} \tag{5}$$

For definition of Δ_j, S_j in Theorem 2.1, see Section 3 below.

2.2. Gevrey regularity for the quasi-geostrophic equations

Here we will consider only the critical and super-critical cases, i.e., $0 < \kappa \leq 1$. For $\beta > 0$ and a measurable function $\Theta : (0, T) \to \dot{\mathbb{H}}^{2-\kappa+\beta}$, we denote

$$\left\| \Theta(\cdot) \right\|_{E_T} :== \operatorname*{ess\,sup}_{0<s<T} s^{\frac{\beta}{\kappa}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Theta(s) \right\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}, \tag{6}$$

provided the right hand side is finite.

Theorem 2.2. *Let* $\kappa \leq 1, \alpha < \kappa$ *and* $\theta_0 \in \mathbb{H}^{2-\kappa}$*. There exist* $\beta > 0$ *and* $T > 0$ *and a solution* $\theta(\cdot)$ *on* $[0, T]$ *of (1) such that*

$$\left\| \theta(\cdot) \right\|_{E_T} \leq C \|\theta_0\|_{\mathbb{H}^{2-\kappa}},$$

where the constant C *is independent of* θ_0 *and* T*. Furthermore, in case* $\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}}$ *is adequately small, we can take* $T = \infty$*.*

Remark 2. In case $\theta_0 \in \mathbb{H}^{2-\kappa+\epsilon}$ with $\epsilon > 0$, following the method presented here, we can provide an explicit estimate of T in Theorem 2.2 above in terms of $\|\theta_0\|_{\mathbb{H}^{2-\kappa+\epsilon}}$. However, in the critical space $\mathbb{H}^{2-\kappa}$ considered here, T depends in a more complicated way on θ_0, not just on its norm.

Remark 3. The Gevrey regularity result presented in Theorem 2.2 is "near optimal" since the solution in the linear case (i.e., when the nonlinearity in the quasi-geostrophic equation is not present) belongs to the same Gevrey class with $\alpha = \kappa$, and no better. Though our result for the supercritical case is new, for the critical case $\kappa = 1$, Theorem 2.2 shows that for arbitrary initial data $\theta_0 \in \mathbb{H}^1$, the solutions are in all subanalytic Gevrey classes. In [6] we showed that the solution to the critical quasi-geostrophic equation is analytic if $\|\mathcal{F}\theta_0\|_{L^1}$ is sufficiently small; see also [37] for a similar result in fractal burgers equation. Thus, it would be interesting to see if one can obtain the optimal Gevrey class regularity (i.e., $\alpha = \kappa$ in Theorem 2.2) for initial data in the critical Sobolev space $\mathbb{H}^{2-\kappa}$.

Remark 4. Following our proof, it is not difficult to show that in Theorem 2.2, the function $s \to e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\theta(s)$ in fact belongs to $C([0,T];\mathbb{H}^{2-\kappa})$. Moreover, the definition of the norm $\|\theta(\cdot)\|_{E_T}$ can be modified to

$$\|\theta(\cdot)\|_{\widetilde{E_T}} := \sup_{0<t<T} \max\left\{t^{\frac{\beta}{\kappa}}\|\theta(t)\|_{G(t),\dot{\mathbb{H}}^{2-\kappa+\beta}}, \|\theta(t)\|_{G(t),\dot{\mathbb{H}}^{2-\kappa}}\right\}.$$

The conclusions of Theorem 2.2 still hold with $\|\theta(\cdot)\|_{\widetilde{E_T}}$ in place of $\|\theta(\cdot)\|_{E_T}$. This method is inspired by the work of [47] and [27] in case of the Navier–Stokes equations, where a higher order regularity gain due to dissipation is used to control the critical norm.

Corollary 2.3. *For any $n \in \mathbb{N}$ satisfying $n > 2-\kappa$ and $\alpha < \kappa$, for some constant C independent of n and s, the solution $\theta(\cdot)$ in Theorem 2.2 above satisfies the higher order decay estimates*

$$\|\Lambda^n\theta(s)\|_{\dot{\mathbb{H}}^{2-\kappa}} \le C\|\theta_0\|\frac{(n!)^{\frac{1}{\alpha}}}{\rho^{\frac{n}{\alpha}}} \quad \text{where } \rho = \lambda\alpha s^{\frac{\alpha}{\kappa}} \text{ and } s \in (0,T). \tag{7}$$

The proof of the corollary follows immediately from Theorem 2.2 and (3).

Remark 5. In [22], it was proven that in case $\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}}$ is sufficiently small, there exists constants C_n such that the decay estimate

$$\|\Lambda^n\theta(s)\|_{\dot{\mathbb{H}}^{2-\kappa}} \le \frac{C_n}{s^{\frac{n}{\kappa}}}$$

holds. This is the same rate as in (7) except that the constants C_n were not identified there, which in our case follows as a consequence of Gevrey regularity. Moreover, the constants in (7) are "near optimal" since the (optimal) rate for the linear case is the same as in (7) with $\alpha = \kappa$ (see Remark 3).

3. Notation and preliminaries

We will need the following notions and some standard results from harmonic analysis to proceed. For more details, see for instance [11,21,3], or [45].

3.1. Littlewood–Paley decomposition

Let $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$, with ranges contained in the interval [0, 1], and such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{2}, \\ 0, & |\xi \ge 1 \end{cases} \quad \text{and} \quad \varphi(\xi) = \psi(\xi/2) - \psi(\xi).$$

Let Δ_j and S_j be the (homogeneous) dyadic Littlewood–Paley projections given by

$$\mathcal{F}(\Delta_j f) = \varphi(\cdot/2^j)\mathcal{F}f \quad \text{and} \quad \mathcal{F}(S_j f) = \psi(\cdot/2^{j-3})\mathcal{F}f.$$

We denote the (open) ball $B(r)$ and the (open) annulus $\mathcal{A}(r_1, r_2)$, $0 < r_1 < r_2$ by

$$B(r) = \{\xi : |\xi| < r\} \quad \text{and} \quad \mathcal{A}(r_1, r_2) = \{\xi : r_1 < |\xi| < r_2\}.$$

For each $j \in \mathbb{Z}$, the Fourier spectrum of $\Delta_j f$ (respectively, $S_j f$) is "localized" in $\mathcal{A}(2^{j-1}, 2^{j+1})$ (respectively, $B(2^{j-3})$), i.e.,

$$\mathcal{F}(\Delta_j f) = 0 \quad \text{for } \xi \in \mathcal{A}\big(2^{j-1}, 2^{j+1}\big)^c \quad \text{and} \quad \mathcal{F}(S_j f)(\xi) = 0 \quad \text{for } \xi \in B\big(2^{j-3}\big)^c.$$

Moreover,

$$S_j = \sum_{k \le j-4} \Delta_k,$$

where the equality holds in the space of distributions "modulo polynomials" [3,21]. The (homogeneous) paraproduct formula is given by

$$fg = T_f g + T_g f + R(f, g), \tag{8}$$

where, denoting $\widetilde{\Delta}_j = \sum_{k=j-3}^{j+3} \Delta_k g$, $T_f g$ and $R(f, g)$ are given by

$$T_f g = \sum_{j \in \mathbb{Z}} S_j f \Delta_j g, \qquad R(f, g) = \sum_{j,k:|j-k| \le 3} \Delta_j f \Delta_k g = \sum_j \Delta_j f \widetilde{\Delta}_j g.$$

The following facts concerning the paraproduct decomposition will be used throughout:

$$\Delta_i \Delta_k = 0 \quad \text{if } |i - k| \ge 2, \qquad \Delta_i (S_k f \Delta_k g) = 0 \quad \text{if } |i - k| \ge 3 \tag{9}$$

and

$$\Delta_i (\Delta_k f \widetilde{\Delta}_k g) = 0 \quad \text{if } i \ge k + 6. \tag{10}$$

The above two facts, namely (9) and (10), follow readily from the spectral localization of the operators Δ_j and S_j.

3.2. *Bernstein and other related inequalities*

Let f and g be two Schwartz class functions with Fourier spectrum localized in the ball $B(r\mu)$ and annulus $\mathcal{A}(r_1\mu, r_2\mu)$ respectively, with $\min\{r, r_1, r_2, \mu\} > 0$. Then, for some constants C, C_1, C_2, depending only on r, r_1, r_2, we have

$$\|f\|_{\dot{\mathbb{H}}^m} \le C\mu^m \|f\| \quad (m > 0) \quad \text{and} \quad C_1 \mu^{m'} \|g\| \le \|g\|_{\dot{\mathbb{H}}^{m'}} \le C_2 \mu^{m'} \|g\| \quad \big(m' \in \mathbb{R}\big). \tag{11}$$

Moreover,

$$\|S_j f\|_{\dot{\mathbb{H}}^m} \le \|f\|_{\dot{\mathbb{H}}^m} \quad \text{and} \quad C_1 \|f\| \le \left(\sum_j \|\Delta_j f\|^2\right)^{1/2} \le C_2 \|f\|. \tag{12}$$

As a consequence of the Young's convolution inequality and the Parseval equality, we have

$$\|fg\|_{L^2} \le C\|\mathcal{F}f\|_{L^p}\|\mathcal{F}g\|_{L^q}, \qquad \frac{3}{2} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p, q \le 2. \tag{13}$$

We will also need versions of Bernstein's inequalities in the Fourier space that are easy to prove, namely,

$$\begin{aligned} &C_1 2^{\alpha j}\|\mathcal{F}\Delta_j f\|_{L^p} \le \left\|\mathcal{F}\Delta_j \Lambda^\alpha f\right\|_{L^p} \le C_2 2^{\alpha j}\|\mathcal{F}\Delta_j f\|_{L^p}, \\ &\|\mathcal{F}\Delta_j f\|_{L^p} \le C 2^{2j(\frac{1}{p}-\frac{1}{q})}\|\mathcal{F}\Delta_j f\|_{L^q}, \quad j \in \mathbb{Z},\ 1 \le p \le q \le \infty, \end{aligned} \tag{14}$$

and where the constants C, C_1, C_2 are independent of f and j and the space dimension is two.

Recall that for $0 \le \alpha \le 1$, the Gevrey norms that we will use were defined in Section 2 by

$$\|f\|_{G(s)} = \left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\right\| \quad \text{and} \quad \|f\|_{G(s),\dot{\mathbb{H}}^m} = \left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\right\|_{\dot{\mathbb{H}}^m} \quad (\lambda > 0 \text{ fixed}).$$

It is clear from the definition of the Gevrey norms that

$$\|f\|_{G(s)} \ge \|f\| \quad \text{and} \quad \|f\|_{G(s),\dot{\mathbb{H}}^m} \ge \|f\|_{\dot{\mathbb{H}}^m}.$$

Moreover, since $e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}$, Δ_j, S_j are all Fourier multipliers, they commute with each other, i.e.,

$$e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j f = \Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f \quad \text{and} \quad e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} S_j f = S_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f,$$

for f in appropriate functional classes. We will use these facts throughout without any further mention. The following inequalities will also be crucial.

Let $s > 0$ and $\zeta_1, \zeta_2 \in \mathbb{R}$ satisfy

$$\zeta_1 + \zeta_2 > 0, \qquad \max\{\zeta_1, \zeta_2\} < 1.$$

Then, for functions f and g belonging to $\dot{\mathbb{H}}^{\zeta_1}$ and $\dot{\mathbb{H}}^{\zeta_2}$ respectively, there exists a constant $C = C(\zeta_1, \zeta_2)$, which is independent of s, such that

$$\|fg\|_{\dot{\mathbb{H}}^{\zeta_1+\zeta_2-1}} \le C(\zeta_1, \zeta_2)\|f\|_{\dot{\mathbb{H}}^{\zeta_1}}\|g\|_{\dot{\mathbb{H}}^{\zeta_2}} \tag{15}$$

$$\|fg\|_{G(s),\dot{\mathbb{H}}^{\zeta_1+\zeta_2-1}} \le C(\zeta_1, \zeta_2)\|f\|_{G(s),\dot{\mathbb{H}}^{\zeta_1}}\|g\|_{G(s),\dot{\mathbb{H}}^{\zeta_2}}. \tag{16}$$

The first one is well known and is a consequence of a more general convolution inequality of Kerman [35] or can be found in [45]. The second can easily be derived from the first as follows. Note first that for $\xi, \eta \in \mathbb{R}^2$ and $s \ge 0$, we have $|\xi| \le |\eta| + |\xi - \eta|$; consequently from the elementary inequality

$$(x + y)^\alpha \le x^\alpha + y^\alpha, \quad x, y \ge 0,\ 0 < \alpha \le 1, \tag{17}$$

we have

$$e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha} \le e^{\lambda s^{\frac{\alpha}{\kappa}}|\eta|^\alpha} e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}. \tag{18}$$

For notational simplicity, denote $\delta = \zeta_1 + \zeta_2 - 1$. Using the Plancherel theorem and (18),

$$\begin{aligned}
\left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha}(fg)\right\|^2_{\dot{\mathbb{H}}^\delta} &= \int \left(e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha} |\xi|^\delta \int f(\xi-\eta) g(\eta)\, d\eta \right)^2 d\xi \\
&\le \int \left(|\xi|^\delta \int e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha} \left| f(\xi-\eta)\right| e^{\lambda s^{\frac{\alpha}{\kappa}}|\eta|^\alpha} \left| g(\eta)\right| d\eta \right)^2 d\xi \\
&\le \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\|^2_{\dot{\mathbb{H}}^{\zeta_1}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g\right\|^2_{\dot{\mathbb{H}}^{\zeta_2}}.
\end{aligned}$$

The last inequality follows by applying (15) to the "auxilliary" functions $\tilde{f}$ and $\tilde{g}$, where

$$\mathcal{F}\tilde{f}(\xi) = e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha} \left| f(\xi)\right| \quad \text{and} \quad \mathcal{F}\tilde{g}(\xi) = e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha} \left| g(\xi)\right|.$$

We have also used the elementary (yet, crucial) fact that

$$\left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\|_{\dot{\mathbb{H}}^{\zeta_1}} = \|\tilde{f}\|_{\dot{\mathbb{H}}^{\zeta_1}} \quad \text{and} \quad \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g\right\|_{\dot{\mathbb{H}}^{\zeta_2}} = \|\tilde{g}\|_{\dot{\mathbb{H}}^{\zeta_2}}.$$

This finishes the proof of (15) and (16).

Let $1 \le p, q, r \le \infty$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Proceeding in an analogous manner and using Young's convolution inequality, one readily obtains also the inequality

$$\left\| \mathcal{F}\big(e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha}(fg)\big)\right\|_{L^r} \le \left\| \mathcal{F}\big(e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\big)\right\|_{L^p} \left\| \mathcal{F}\big(e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g\big)\right\|_{L^q}. \tag{19}$$

4. Proof of Theorem 2.1

In order to prove this result, we will need the following lemma, which may be regarded as central to the proof of Theorem 2.1.

Lemma 4.1. *In the notation and setting of Theorem 2.1, we have the estimate*

$$\begin{aligned}
\left\| \big[T_f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] g\right\| \le C c_j \Big\{ & s^{\frac{(\alpha-\zeta)}{\kappa}} 2^{-(\delta_1+\delta_2+\zeta-\alpha)j} \left\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\|_{\dot{\mathbb{H}}^{1+\delta_1}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g\right\|_{\dot{\mathbb{H}}^{\delta_2}} \\
& + 2^{-(\delta_1+\delta_2)j} \left\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f\right\|_{\dot{\mathbb{H}}^{1+\delta_1}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g\right\|_{\dot{\mathbb{H}}^{\delta_2}} \Big\}.
\end{aligned} \tag{20}$$

Proof. Due to (9), we have

$$\big[T_f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] g = \sum_{k:|k-j|\le 2} \big[S_k f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] \Delta_k g.$$

Note that

$$\begin{aligned}
&-\mathcal{F}\big[S_k f, e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j\big]\Delta_k g(\xi) \\
&= \int \mathcal{F}S_k f(\eta)\mathcal{F}\Delta_k g(\xi-\eta)e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha}\left[\varphi\left(\frac{\xi}{2^j}\right)-\varphi\left(\frac{\xi-\eta}{2^j}\right)\right]d\eta \\
&\quad + \int \mathcal{F}S_k f(\eta)\mathcal{F}\Delta_k g(\xi-\eta)\varphi\left(\frac{\xi-\eta}{2^j}\right)\big[e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha}-e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}\big]d\eta \\
&= I + II.
\end{aligned}$$

Recall that from (9), we have

$$I = II = 0 \quad \text{if } \xi \in \big[\mathcal{A}\big(2^{k-2}, 2^{k+2}\big)\big]^c \quad \text{and} \quad \big[S_k f, e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j\big]\Delta_k g = 0 \quad \text{if } |j-k| \geq 3.$$

Due to this, for any $\delta \in \mathbb{R}$ and $k \in [j-2, j+2] \cap \mathbb{Z}$, we have

$$\|I\| \leq C2^{-\delta k}\big\|\Lambda^\delta I\big\| \leq C2^{-\delta j}\big\|\Lambda^\delta I\big\| \quad \text{and} \quad \|II\| \leq C2^{-\delta k}\big\|\Lambda^\delta II\big\| \leq C2^{-\delta j}\big\|\Lambda^\delta II\big\|, \tag{21}$$

where the constant C is independent of j, f and g. Now note that since φ and all its derivatives are uniformly bounded, applying the mean value theorem (to φ), we obtain

$$\left|\varphi\left(\frac{\xi}{2^j}\right)-\varphi\left(\frac{\xi-\eta}{2^j}\right)\right| \leq C_\varphi 2^{-j}|\eta|, \quad C_\varphi = \big\|\varphi'\big\|_{L^\infty}.$$

Inserting the above estimate and (18) in I and using (21), we obtain

$$\begin{aligned}
&\left|\int \mathcal{F}S_k f(\eta)\mathcal{F}\Delta_k g(\xi-\eta)e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha}\left[\varphi\left(\frac{\xi}{2^j}\right)-\varphi\left(\frac{\xi-\eta}{2^j}\right)\right]d\eta\right| \\
&\quad \leq C2^{-(\delta_1+\delta_2)j}\left\|\Lambda^{(\delta_1+\delta_2-1)}\int |\eta|\big|e^{\lambda s^{\frac{\alpha}{\kappa}}|\eta|^\alpha}\mathcal{F}S_k f(\eta)\big|\big|e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}\mathcal{F}\Delta_k g(\xi-\eta)\big|d\eta\right\|.
\end{aligned}$$

Now using (15) with $\zeta_1 = \delta_1$, $\zeta_2 = \delta_2$, we finally obtain the estimate

$$\begin{aligned}
\|I\| &\leq C2^{-(\delta_1+\delta_2)j}\big\|S_k e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\big\|_{\dot{\mathbb{H}}^{1+\delta_1}}\big\|\Delta_k e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}} \\
&\leq C2^{-(\delta_1+\delta_2)j}c_j\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\big\|_{\dot{\mathbb{H}}^{1+\delta_1}}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}}, \\
&\text{where } 0 < \delta_1, \delta_2 < 1 \text{ and } c_j = \frac{(\sum_{k=j-3}^{j+3}\|\Delta_k e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\|_{\dot{\mathbb{H}}^{\delta_2}}^2)^{1/2}}{C'\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\|_{\dot{\mathbb{H}}^{\delta_2}}},
\end{aligned} \tag{22}$$

where we have also used the first inequality in (12) above. Furthermore, the constant C' in (22) may depend only on δ_1, δ_2 and $\sum c_j^2 \leq 1$. These facts follow from the second inequality in (12).

We will now estimate II. Note that

$$\left|e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi|^\alpha} - e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}\right| = \left|\int_0^1 \frac{d}{d\tau}\left(e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-(1-\tau)\eta|^\alpha}\right)d\tau\right|$$
$$\leq C\lambda s^{\frac{\alpha}{\kappa}} \int_0^1 \frac{|\eta|}{|\xi-(1-\tau)\eta|^{1-\alpha}} e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-(1-\tau)\eta|^\alpha}\, d\tau. \tag{23}$$

Since $0 < \alpha \leq 1$, from (17) it follows that

$$\left|\xi-(1-\tau)\eta\right|^\alpha = \left|(\xi-\eta)+\tau\eta\right|^\alpha \leq |\xi-\eta|^\alpha + \tau^\alpha|\eta|^\alpha.$$

Consequently,

$$e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-(1-\tau)\eta|^\alpha} \leq e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha} e^{\lambda s^{\frac{\alpha}{\kappa}}\tau^\alpha|\eta|^\alpha}. \tag{24}$$

Recall that the support of $\mathcal{F}S_k f$ is in $B(2^{k-3})$ and the support of $\mathcal{F}\Delta_k g$ is in $\mathcal{A}(2^{k-1}, 2^{k+1})$. Thus, for the integrand in II to be nonzero, we must have

$$|\xi| \geq |\xi-\eta| - |\eta| \geq 2^{k-1} - 2^{k-3} = 3\left(2^{k-3}\right) \geq 3|\eta|. \tag{25}$$

Since $0 \leq (1-\tau) \leq 1$, this immediately implies that

$$\left|\xi-(1-\tau)\eta\right| \geq |\xi| - (1-\tau)|\eta| \geq |\xi| - |\eta| \geq \frac{2}{3}|\xi|. \tag{26}$$

From (26), we obtain

$$|II| \leq C\frac{s^{\frac{\alpha}{\kappa}}}{|\xi|^{1-\alpha}} \int_0^1 \int |\eta| \left|e^{\lambda s^{\frac{\alpha}{\kappa}}\tau^\alpha|\eta|^\alpha}\mathcal{F}S_k f(\eta)\right| \left|e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}\mathcal{F}\Delta_k g(\xi-\eta)\right| d\eta\, d\tau. \tag{27}$$

Since II is non-zero only for $\xi \in \mathcal{A}(2^{k-2}, 2^{k+2})$ and $|j-k| \leq 2$ (see (9)), we have

$$\frac{1}{|\xi|^{1-\alpha}} \leq C2^{-(1-\alpha)j}.$$

Thus, from (27), we obtain

$$|II| \leq C\frac{s^{\frac{\alpha}{\kappa}}}{2^{(1-\alpha)j}} \int_0^1 \int |\eta| \left|e^{\lambda s^{\frac{\alpha}{\kappa}}\tau^\alpha|\eta|^\alpha}\mathcal{F}S_k f(\eta)\right| \left|e^{\lambda s^{\frac{\alpha}{\kappa}}|\xi-\eta|^\alpha}\mathcal{F}\Delta_k g(\xi-\eta)\right| d\eta\, d\tau. \tag{28}$$

To the inequality in (28), we apply (15) with $\zeta_1 = \delta_1 + \zeta$, $\zeta_2 = \delta_2$, followed by the Minkowski inequality (in order to switch $d\tau$ and $d\xi$ integrals while computing relevant norms). Consequently, consulting also the second inequality in (21), we obtain

$$\|II\| \le C s^{\frac{\alpha}{\kappa}} 2^{-(\delta_1+\delta_2+\zeta-\alpha)j} \int_0^1 \big\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \tau^\alpha \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1+\zeta}} \big\| \Delta_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \big\|_{\dot{\mathbb{H}}^{\delta_2}} \, d\tau. \tag{29}$$

Now note that

$$\begin{aligned} \big\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \tau^\alpha \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1+\zeta}} &= \big\| \Lambda^\zeta e^{-\lambda s^{\frac{\alpha}{\kappa}} (1-\tau^\alpha) \Lambda^\alpha} S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1}} \\ &\le \frac{C}{(s^{\frac{\alpha}{\kappa}}(1-\tau^\alpha))^{\zeta/\alpha}} \big\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1}}, \end{aligned}$$

where the last inequality follows from the Plancherel equality and the elementary estimate

$$\sup_{x>0} x^m e^{-tx^\kappa} \le \frac{C(m,\kappa)}{t^{m/\kappa}} \quad (m \ge 0). \tag{30}$$

Therefore, from (29), we obtain

$$\begin{aligned} \|II\| &\le C s^{\frac{\alpha}{\kappa}} 2^{-(\delta_1+\delta_2+\zeta-\alpha)j} \big\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1}} \big\| \Delta_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \big\|_{\dot{\mathbb{H}}^{\delta_2}} \int_0^1 \frac{1}{s^{\frac{\zeta}{\kappa}}(1-\tau^\alpha)^{\zeta/\alpha}} \, d\tau \\ &\le C s^{\frac{(\alpha-\zeta)}{\kappa}} 2^{-(\delta_1+\delta_2+\zeta-\alpha)j} \big\| S_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1}} \big\| \Delta_k e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \big\|_{\dot{\mathbb{H}}^{\delta_2}} \\ &\le C s^{\frac{(\alpha-\zeta)}{\kappa}} 2^{-(\delta_1+\delta_2+\eta-\alpha)j} c_j \big\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \big\|_{\dot{\mathbb{H}}^{1+\delta_1}} \big\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} g \big\|_{\dot{\mathbb{H}}^{\delta_2}}, \end{aligned} \tag{31}$$

where c_j is as defined in (22). To obtain (31), we also used the fact that

$$\int_0^1 \frac{1}{(1-\tau^\alpha)^{\zeta/\alpha}} \, d\tau < \infty \quad (\text{since } \alpha \le 1 \text{ and } \zeta < \alpha).$$

Putting together (22) and (31), we obtain (20). □

Proof of Theorem 2.1. Note that by (8),

$$\begin{aligned} \big[f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] g &= \big[T_f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] g + T_{e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j g} f - e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j (T_g f) \\ &\quad + R\big(f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j g\big) - e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j R(f,g). \end{aligned} \tag{32}$$

Concerning the first term $[T_f, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j] g$, the inequality stated in (5) follows immediately from (20). We will now estimate the remaining terms on the right hand side of (32). Observing that $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j = \Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha}$ and using (9), we obtain

$$\|T_{e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j g} f\| = \Big\| \sum_{k:k\ge j+2} \big(S_k \Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big)(\Delta_k f)\Big\| \le \big\|\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\| \sum_{k:k\ge j+2} \|\mathcal{F}\Delta_k f\|_{L^1}$$

$$\le C\big\|\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\| \sum_{k:k\ge j+2} \|\Delta_k f\|_{\dot{\mathbb{H}}^1} \tag{33}$$

$$\le C2^{-j\delta_2}\big\|\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}} \sum_{k:k\ge j+2} 2^{-k\delta_1}\|\Delta_k f\|_{\dot{\mathbb{H}}^{1+\delta_1}} \tag{34}$$

$$\le C2^{-j\delta_2}\big\|\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}} \Big(\sum_{k:k\ge j+2} 2^{-2k\delta_1}\Big)^{1/2}\Big(\sum_{k:k\ge j+2}\|\Delta_k f\|^2_{\dot{\mathbb{H}}^{1+\delta_1}}\Big)^{1/2}$$

$$\le Cc_j 2^{-(\delta_1+\delta_2)j}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}}\|f\|_{\dot{\mathbb{H}}^{1+\delta_1}}, \tag{35}$$

where c_j is as in (22) and in order to obtain (33) and (34), we successively used Young's convolution inequality, (14) and (11). Proceeding in a similar manner, with c_j as in (22), we obtain

$$\big\|R\big(f, e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j g f\big)\big\| \le C2^{-(\delta_1+\delta_2)j}c_j\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}}\|f\|_{\dot{\mathbb{H}}^{1+\delta_1}}. \tag{36}$$

We will now estimate $\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j(T_g f)\|$. Due to (9), we have

$$e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j(T_g f) = \sum_{k:|k-j|\le 2} e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j S_k g\Delta_k f = \sum_{k:|k-j|\le 2}\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} S_k g\Delta_k f.$$

We have

$$\begin{aligned}\big\|\Delta_j e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}(S_k g\Delta_k f)\big\| &\le C2^{-(\delta_1+\delta_2-1)j}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}(S_k g\Delta_k f)\big\|_{\dot{\mathbb{H}}^{\delta_1+\delta_2-1}}\\ &\le C2^{-(\delta_1+\delta_2-1)j}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k f\big\|_{\dot{\mathbb{H}}^{\delta_1}}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} S_k g\big\|_{\dot{\mathbb{H}}^{\delta_2}}\\ &\le C2^{-(\delta_1+\delta_2)j}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k f\big\|_{\dot{\mathbb{H}}^{1+\delta_1}}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} S_k g\big\|_{\dot{\mathbb{H}}^{\delta_2}}\\ &\le Cc_j 2^{-(\delta_1+\delta_2)j}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\big\|_{\dot{\mathbb{H}}^{1+\delta_1}}\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}},\\ c_j = \frac{(\sum_{k=j-2}^{j+2}\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k f\|^2_{\dot{\mathbb{H}}^{1+\delta_1}})^{1/2}}{\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\|_{\dot{\mathbb{H}}^{1+\delta_1}}},\end{aligned}$$

where the first inequality in the above line is obtained using (11), the second using (16) and the third again by (11). Additionally, we have also used the fact that $k \in [j-2, j+2] \cap \mathbb{N}$.

Finally, we will prove that there exists $\{c_j\}_{-\infty}^{\infty}$, $c_j \ge 0$ with $\sum c_j^2 \le 1$ such that

$$\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j R(f,g)\big\| \le C2^{-(\delta_1+\delta_2)j}c_j\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} f\big\|_{\dot{\mathbb{H}}^{1+\delta_1}}\|\big\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha} g\big\|_{\dot{\mathbb{H}}^{\delta_2}}, \delta_1+\delta_2 > 0. \tag{37}$$

From (10) we have

$$\begin{aligned}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_j R(f,g)\right\| &\le 2^{-(\delta_1+\delta_2)j}\sum_{k\ge j-6}2^{(\delta_1+\delta_2)j}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}(\widetilde{\Delta}_k f\Delta_k g)\right\| \\ &\le 2^{-(\delta_1+\delta_2)j}\sum_{k\ge j-6}2^{(\delta_1+\delta_2)j}\left\|\mathcal{F}e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\widetilde{\Delta}_k f\right\|_{L^1}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k g\right\| \qquad (38)\\ &\le 2^{-(\delta_1+\delta_2)j}\sum_{k\ge j-6}2^{(\delta_1+\delta_2)j}2^k\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\widetilde{\Delta}_k f\right\|\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k g\right\| \qquad (39)\\ &\le 2^{-(\delta_1+\delta_2)j}\sum_{k\ge j-6}2^{(\delta_1+\delta_2)(j-k)}2^{(\delta_1+1)k}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\widetilde{\Delta}_k f\right\|2^{\delta_2 k}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k g\right\|, \qquad (40)\end{aligned}$$

where to obtain (38), we used (19), while to obtain (39), we used (14). Let $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ be sequences defined by

$$a_k = 2^{(\delta_1+1)k}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\widetilde{\Delta}_k f\right\|2^{\delta_2 k}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k g\right\|, \qquad b_k = \chi_{[-6,\infty)}(k)2^{-(\delta_1+\delta_2)k}.$$

Applying Cauchy–Schwartz and the second inequality in (12), we have

$$\left\|(a_k)_{k\in\mathbb{Z}}\right\|_{\ell_1} \le C\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f\right\|_{\dot{\mathbb{H}}^{1+\delta_1}}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}g\right\|_{\dot{\mathbb{H}}^{\delta_2}}. \qquad (41)$$

Define

$$\begin{aligned}c_j &= \frac{1}{C\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f\|_{\dot{\mathbb{H}}^{1+\delta_1}}\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}g\|_{\dot{\mathbb{H}}^{\delta_2}}}\sum_{k\in\mathbb{Z}}b_{j-k}a_k \\ &= \frac{1}{C\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f\|_{\dot{\mathbb{H}}^{1+\delta_1}}\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}g\|_{\dot{\mathbb{H}}^{\delta_2}}}\sum_{k\ge j-6}2^{(\delta_1+\delta_2)(j-k)}2^{(\delta_1+1)k} \\ &\quad\times\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\widetilde{\Delta}_k f\right\|2^{\delta_2 k}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}\Delta_k g\right\|,\end{aligned}$$

where C is as in (41). Now using (41), the fact that $\|(b_k)\|_{\ell_2} < \infty$ (since $\delta_1+\delta_2 > 0$) and Young's convolution inequality for sequences, we get that $\sum c_j^2 \le 1$. Using this fact, we immediately obtain (37) from (40). □

5. Proof of main result

We will need the following lemma, the proof of which follows that of Lemma 8 in [42].

Lemma 5.1. *Let $\alpha < \kappa$ and f is such that $f \in \dot{\mathbb{H}}^{\alpha/2}$ and $e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f \in \dot{\mathbb{H}}^{\kappa/2}$. Then $e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f \in \dot{\mathbb{H}}^{\alpha/2}$ and we have the estimate*

$$\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f\right\|^2_{\dot{\mathbb{H}}^{\alpha/2}} \le e\|f\|^2_{\dot{\mathbb{H}}^{\alpha/2}} + (2\lambda)^{\frac{\kappa}{\alpha}-1}s^{1-\frac{\alpha}{\kappa}}\left\|e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^\alpha}f\right\|^2_{\dot{\mathbb{H}}^{\kappa/2}}. \qquad (42)$$

Proof. From the Plancherel equality, we have

$$\left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} f \right\|^2_{\dot{\mathbb{H}}^{\alpha/2}} = \int |\xi|^\alpha e^{2\lambda s^{\frac{\alpha}{\kappa}} |\xi|^\alpha} \left|(\mathcal{F} f)(\xi)\right|^2 d\xi. \tag{43}$$

Moreover, for all $x \geq 0$ and $m > 0$, we have the inequality $e^x \leq e + x^m e^x$. This is due to the fact that $e^x \leq e$ for $x \in [0, 1]$ and $e^x \leq x^m e^x$ for $x \geq 1$. Applying this to (43) with $x = 2\lambda s^{\frac{\alpha}{\kappa}} |\xi|^\alpha$ and $m = \frac{\kappa}{\alpha} - 1$, we obtain the desired conclusion. □

We will also need an estimate for the linear term given in the lemma below.

Lemma 5.2. *Let $\theta_0 \in \dot{\mathbb{H}}^{2-\kappa}$ and $\beta > 0$. Denote*

$$\|\theta_0\|_{E_T} = \sup_{0<s\leq T} s^{\frac{\beta}{\kappa}} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} e^{-s\Lambda^\kappa} \theta_0 \right\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}. \tag{44}$$

In this setting, with a constant C independent of T and θ_0, we have

$$\|\theta_0\|_{E_T} \leq C\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}} \quad \text{and} \quad \lim_{T\to 0} \|\theta_0\|_{E_T} = 0. \tag{45}$$

Proof. Observe that

$$\begin{aligned} \left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} e^{-s\Lambda^\kappa} \theta_0 \right\|^2_{\dot{\mathbb{H}}^{2-\kappa+\beta}} &= \int \left(|\xi|^{2-\kappa+\beta} e^{\lambda s^{\frac{\alpha}{\kappa}} |\xi|^\alpha} e^{-s|\xi|^\kappa} \left|(\mathcal{F}\theta_0)(\xi)\right|\right)^2 d\xi \\ &= \int \left(|\xi|^{2-\kappa+\beta} e^{\lambda s^{\frac{\alpha}{\kappa}} |\xi|^\alpha - \frac{s}{2}|\xi|^\kappa} e^{-\frac{s}{2}|\xi|^\kappa} \left|(\mathcal{F}\theta_0)(\xi)\right|\right)^2 d\xi. \end{aligned} \tag{46}$$

Now observe that

$$\sup_{s\geq 0, \xi\in\mathbb{R}^2} e^{\lambda s^{\frac{\alpha}{\kappa}} |\xi|^\alpha - \frac{s}{2}|\xi|^\kappa} = \sup_{s\geq 0, \xi\in\mathbb{R}^2} e^{\lambda (s^{\frac{1}{\kappa}} |\xi|)^\alpha - \frac{1}{2}(s^{\frac{1}{\kappa}} |\xi|)^\kappa} \leq C(\lambda, \kappa, \alpha), \tag{47}$$

since, for $\alpha < \kappa$, the function $f(x) = \lambda x^\alpha - \frac{1}{2} x^\kappa \leq C(\lambda, \alpha, \kappa)$ for all $x > 0$. Applying (47) and (30) to (46), we obtain the first inequality in (45). In case $\theta_0' \in \dot{\mathbb{H}}^{2-\kappa+\beta}$, a similar calculation using (47) yields

$$\left\| e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} e^{-s\Lambda^\kappa} \theta_0' \right\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}} \leq C \left\| \theta_0' \right\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}. \tag{48}$$

Given $\epsilon > 0$, we can choose θ_0' such that

$$\left\| \theta_0' - \theta_0 \right\|_{\dot{\mathbb{H}}^{2-\kappa}} \leq \epsilon \quad \text{and} \quad \theta_0' \in \dot{\mathbb{H}}^{2-\kappa+\beta}. \tag{49}$$

From the first inequality in (45), (48) and (49), the second assertion in (45) immediately follows. □

Before embarking on the proof of Theorem 2.2, we note that

$$\|u\|_{\dot{\mathbb{H}}^m} \simeq \|\theta\|_{\dot{\mathbb{H}}^m}, \quad m \in \mathbb{R},$$

since they are related by the Riesz transform as given in (1).

Proof of Theorem 2.2. As is customary, we consider the following approximate sequence of solutions:

$$\left.\begin{aligned} &\partial_t \theta^{(n+1)} + \Lambda^\kappa \theta^{n+1} + u^{(n)} \cdot \nabla \theta^{(n+1)} = 0, \quad \theta^{(n+1)}\big|_{t=0} = \theta_0, \\ &u^{(n)} = \big(-R_2\theta^{(n)}, R_1\theta^{(n)}\big), \quad n = 0, 1, \cdots, \end{aligned}\right\} \tag{50}$$

with the convention that $\theta^{(-1)} \equiv 0$ and $u^{(-1)} = 0$. Denote

$$\widetilde{\theta}^{(n)}(s) = e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \theta^{(n)}(s), \qquad \widetilde{u}^{(n)}(s) = e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} u^{(n)}(s) = \big(-R_2 \widetilde{\theta}^{(n)}, R_1 \widetilde{\theta}^{(n)}\big); \tag{51}$$

the very last equality above is due to the fact that R_i, $i = 1, 2$, commute with $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha}$.

Due to Theorem 1 and Corollary 1 in [40], provided either $\|\theta_0\|_{\mathbb{H}^{2-\kappa}}$ or T is sufficiently small, the sequence $\{\theta^{(n)}\}$ converges (in $\mathbb{H}^{2-\kappa}$) to a solution θ of (1) which additionally belongs to $C([0, T]; \mathbb{H}^{2-\kappa})$. Moreover, for all n, $\{\theta^{(n)}\}$ satisfies

$$\big\|\theta^{(n)}\big\|_{E_T} = \sup_{0<s\le T} s^{\frac{\beta}{\kappa}} \big\|\theta^{(n)}(s)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}} \le C\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}} \quad \text{and} \quad \lim_{T\to 0} \sup_n \big\|\theta^{(n)}\big\|_{E_T} = 0; \tag{52}$$

the constant C above is independent of T and θ_0. Thus, in order to prove the Theorem 2.2, it will be sufficient to demonstrate *a priori* estimates, i.e., to obtain bounds on $\|\widetilde{\theta}^{(n)}(\cdot)\|_{E_T}$, independent of n.

For the remainder of the proof, we choose, and fix, the parameters β, ζ by

$$0 < \beta < \min\left\{\frac{\kappa}{2}, 2(\kappa - \alpha), \alpha\right\} \quad \text{and} \quad \zeta = \alpha - \frac{\beta}{2}. \tag{53}$$

Using respectively the facts that $u^{(n)}$ is divergence free and the operators $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j$ and $e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} S_j$ are Fourier multipliers (and hence commute with ∇), we have

$$\begin{aligned} &e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j \nabla \theta^{(n)}(s) = \nabla \Delta_j \widetilde{\theta}^{(n)}(s), \qquad e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} S_j \nabla \theta^{(n)}(s) = \nabla S_j \widetilde{\theta}^{(n)}(s) \quad \text{and} \\ &\big\langle u^{(n)} \cdot \nabla \Delta_j \widetilde{\theta}^{(n+1)}, \Delta_j \widetilde{\theta}^{(n+1)} \big\rangle = 0, \end{aligned} \tag{54}$$

where $\widetilde{\theta}^{(n)}, \widetilde{u}^{(n)}$ are as in (51). From (1) and (54), taking L^2-inner product, we readily obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\big\|\Delta_j \widetilde{\theta}^{(n+1)}\big\|^2 + \big\|\Lambda^{\kappa/2} \Delta_j \widetilde{\theta}^{(n+1)}\big\|^2 \\ &\quad = \lambda \frac{\alpha}{\kappa} s^{\frac{\alpha}{\kappa}-1} \big\|\Lambda^{\alpha/2} \Delta_j \widetilde{\theta}^{(n+1)}\big\|^2 + \big\langle \big[u^{(n)}, e^{\lambda s^{\frac{\alpha}{\kappa}} \Lambda^\alpha} \Delta_j\big] \nabla \theta^{(n+1)}, \Delta_j \widetilde{\theta}^{(n+1)} \big\rangle \\ &\quad \le C(\alpha, \kappa) \lambda^{\frac{\kappa}{\alpha}} \big\|\Lambda^{\kappa/2} \Delta_j \widetilde{\theta}^{(n+1)}\big\|^2 + \lambda s^{\frac{\alpha}{\kappa}-1} C(\alpha, \kappa) \big\|\Lambda^{\alpha/2} \Delta_j \theta^{(n+1)}\big\|^2 \end{aligned}$$

$$+\langle [u^{(n)}, e^{\lambda s^{\frac{\alpha}{\kappa}}\Lambda^{\alpha}}\Delta_j]\nabla\theta^{(n+1)}, \Delta_j\widetilde{\theta}^{(n+1)}\rangle, \tag{55}$$

where to obtain the inequality (55), we applied (42). Since $\alpha < \kappa$ and $C(\alpha,\kappa)$ is independent of λ, we can choose (and henceforth, fix) λ small enough so that $C(\alpha,\kappa)\lambda^{\frac{\kappa}{\alpha}} < \frac{1}{2}$. Note that due to (53), the parameters β and ζ satisfy the conditions

$$\min\{\beta,\zeta\} > 0, \qquad \beta+\zeta < \kappa \quad \text{and} \quad \zeta < \alpha. \tag{56}$$

We can now apply Theorem 2.1 to the commutator term on the right hand side of inequality (55) with

$$\delta_1 = 1-\kappa+\beta, \qquad \delta_2 = 1-\kappa+\beta, \qquad f = u^{(n)}, \qquad g = \nabla\theta^{(n+1)}, \tag{57}$$

and Bernstein's inequality (11) to the term $\|\Lambda^{\kappa/2}\Delta_j\widetilde{\theta}^{(n+1)}\|^2$ on the left hand side of (55), to obtain

$$\begin{aligned}\frac{d}{ds}\big\|\Delta_j\widetilde{\theta}^{(n+1)}\big\|^2 + C_1 2^{\kappa j}\big\|\Delta_j\widetilde{\theta}^{(n+1)}\big\|^2 \le C\big\{&2^{\alpha j}s^{\frac{\alpha}{\kappa}-1}\big\|\Delta_j\theta^{(n+1)}\big\|^2 \\ &+ c_j\big(2^{-(2-2\kappa+2\beta)j} + s^{\frac{(\alpha-\zeta)}{\kappa}}2^{-(2-2\kappa+2\beta+\zeta-\alpha)j}\big) \\ &\times \big\|\widetilde{\theta}^{(n)}\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\|\widetilde{\theta}^{(n+1)}\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\|\Delta_j\widetilde{\theta}^{(n+1)}\big\|\big\}.\end{aligned}$$

Now divide both sides by $\|\Delta_j\widetilde{\theta}^{(n+1)}\|$ and recall that $\frac{\|\Delta_j\theta^{(n+1)}\|}{\|\Delta_j\widetilde{\theta}^{(n+1)}\|} \le 1$. This yields

$$\begin{aligned}\frac{d}{ds}\big\|\Delta_j\widetilde{\theta}^{(n+1)}\big\| + C_1 2^{\kappa j}\big\|\Delta_j\widetilde{\theta}^{(n+1)}\big\| \le C\big\{&s^{\frac{\alpha}{\kappa}-1}2^{\alpha j}\big\|\Delta_j\theta^{(n+1)}\big\| \\ &+ c_j\big(2^{-(2-2\kappa+2\beta)j} + s^{\frac{(\alpha-\zeta)}{\kappa}}2^{-(2-2\kappa+2\beta+\zeta-\alpha)j}\big) \\ &\times \big\|\widetilde{\theta}^{(n)}\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\|\widetilde{\theta}^{(n+1)}\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\}.\end{aligned}$$

The variation of parameters formula, and the fact that $\widetilde{\theta}^{(n+1)}(0) = \theta^{(n+1)}(0) = \theta_0$, now yield

$$\begin{aligned}\big\|\Delta_j\widetilde{\theta}^{(n+1)}(t)\big\| \le e^{-C_1 2^{\kappa j}t}\|\Delta_j\theta_0\| + C\int_0^t s^{\frac{\alpha}{\kappa}-1}2^{\alpha j}e^{-C_1 2^{\kappa j}(t-s)}\big\|\Delta_j\theta^{(n+1)}(s)\big\|\,ds \\ + C\int_0^t c_j 2^{-(2-2\kappa+2\beta)j}e^{-C_1 2^{\kappa j}(t-s)}\big\|\widetilde{\theta}^{(n)}(s)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\|\widetilde{\theta}^{(n+1)}(s)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\,ds \\ + C\int_0^t c_j s^{\frac{(\alpha-\zeta)}{\kappa}}2^{-(2-2\kappa+2\beta+\zeta-\alpha)j}e^{-C_1 2^{\kappa j}(t-s)} \\ \times \big\|\widetilde{\theta}^{(n)}(s)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\big\|\widetilde{\theta}^{(n+1)}(s)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}}\,ds.\end{aligned} \tag{58}$$

Multiply both sides of the inequality (58) by $2^{(2-\kappa+\beta)j}$ and apply (30). Subsequently, take the ℓ_2-norm of the resulting sequence and apply the Minkowski inequality (to interchange ds and $\sum_j$).

Consequently, from (58), the first relation in (52) and the fact that $\sum c_j^2 \le 1$, we obtain for all $t > 0$ the estimate

$$\begin{aligned}\big\|\widetilde{\theta}^{(n+1)}(t)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}} &\le \widetilde{C}_1\frac{\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}}}{t^{\frac{\beta}{\kappa}}} + \widetilde{C}_2\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}}\int_0^t \frac{ds}{(t-s)^{\frac{\alpha}{\kappa}}s^{1+\frac{\beta-\alpha}{\kappa}}} + \widetilde{C}_3\big\|\widetilde{\theta}^{(n+1)}\big\|_{E_T}\big\|\widetilde{\theta}^{(n)}\big\|_{E_T} \\ &\quad\times\left\{\int_0^t \frac{ds}{s^{2\frac{\beta}{\kappa}}(t-s)^{1-\frac{\beta}{\kappa}}} + \int_0^t \frac{ds}{s^{\frac{2\beta-\alpha+\zeta}{\kappa}}(t-s)^{\frac{\alpha+\kappa-\zeta-\beta}{\kappa}}}\right\}, \qquad (59)\end{aligned}$$

where the constants $\widetilde{C}_i$ above are independent of n, T, t and θ_0 as well as the sequences $\{\theta^{(n)}\}$ and $\{\widetilde{\theta}^{(n)}\}$. The integrals on the right hand side of (59) are finite because $\alpha < \kappa$, and due to (53), the parameters β, ζ satisfy

$$\beta < \min\left\{\alpha, \frac{\kappa}{2}\right\}, \qquad \beta < \frac{\kappa}{2} - \frac{\zeta-\alpha}{2} \quad \text{and} \quad \alpha < \zeta + \beta. \qquad (60)$$

From (59), we easily obtain

$$\begin{aligned}\big\|\widetilde{\theta}^{(n+1)}(\cdot)\big\|_{E_T} &= \sup_{0<t<T} t^{\frac{\beta}{\kappa}}\big\|\widetilde{\theta}^{(n+1)}(t)\big\|_{\dot{\mathbb{H}}^{2-\kappa+\beta}} \\ &\le \widetilde{C}_1\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}} + \widetilde{C}_4\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}} + \widetilde{C}_5\big\|\widetilde{\theta}^{(n+1)}\big\|_{E_T}\big\|\widetilde{\theta}^{(n)}\big\|_{E_T}, \qquad (61)\end{aligned}$$

where

$$\widetilde{C}_4 = t^{\frac{\beta}{\kappa}}\widetilde{C}_2\int_0^t \frac{ds}{(t-s)^{\frac{\alpha}{\kappa}}s^{1+\frac{\beta-\alpha}{\kappa}}} \quad \text{and}$$

$$\widetilde{C}_5 = t^{\frac{\beta}{\kappa}}\widetilde{C}_3\left\{\int_0^t \frac{ds}{s^{2\frac{\beta}{\kappa}}(t-s)^{1-\frac{\beta}{\kappa}}} + \int_0^t \frac{ds}{s^{\frac{2\beta-\alpha+\eta}{\kappa}}(t-s)^{\frac{\alpha+\kappa-\eta-\beta}{\kappa}}}\right\}.$$

The integrals in the definition of $\widetilde{C}_4$ and $\widetilde{C}_5$ above are finite due to (60).

In a similar manner, following the derivation of (61) and using (45), we can also obtain

$$\big\|\widetilde{\theta}^{(n+1)}\big\|_{E_T} \le \widetilde{C}_1\|\theta_0\|_{E_T} + \widetilde{C}_4\|\theta_0\|_{E_T} + \widetilde{C}_5\big\|\widetilde{\theta}^{(n+1)}\big\|_{E_T}\big\|\widetilde{\theta}^{(n)}\big\|_{E_T}, \qquad (62)$$

where $\|\theta_0\|_{E_T}$ is as defined in (44). Assume that $\widetilde{C}_5\|\widetilde{\theta}^{(n)}\|_{E_T} < 1/2$. From (61), we readily obtain,

$$\big\|\widetilde{\theta}^{(n+1)}\big\|_{E_T} \le 2(\widetilde{C}_1 + \widetilde{C}_4)\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}}. \qquad (63)$$

Provided

$$2\widetilde{C}_5(C_1 + C_4)\|\theta_0\|_{\dot{\mathbb{H}}^{2-\kappa}} \le \frac{1}{2},$$

we see that $\widetilde{C}_5\|\widetilde{\theta}^{(n+1)}\|_E \le \frac{1}{2}$ and by induction, (63) holds for all n. In case $\theta_0 \in \dot{\mathbb{H}}^{2-\kappa}$ is arbitrary, for sufficiently small T, we can similarly derive uniform (in n) bound on $\|\widetilde{\theta}^{(n+1)}\|_{E_T}$ from (62), by applying (45) and the second relation in (52).

In order that the parameters satisfy (56) and (60), it is sufficient that the conditions

$$\alpha < \eta + \beta < \kappa, \qquad \eta < \alpha \quad \text{and} \quad \beta < \frac{\kappa}{2}$$

hold. As long as $\alpha < \kappa$, simply take $\eta = \alpha - \frac{\beta}{2}$ with $\beta < \frac{\kappa}{2}$. All conditions are met and we finish the proof. □

References

[1] H. Abidi, T. Hmidi, On the global well-posedness of the critical quasi-geostrophic equation, SIAM J. Math. Anal. 40 (2008) 167–185.

[2] H. Bae, A. Biswas, E. Tadmor, Analyticity and decay estimates of the Navier–Stokes equations in critical Besov spaces, Arch. Ration. Mech. Anal. 205 (3) (2012) 963–991.

[3] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, vol. 343, Springer Verlag, 2011.

[4] A. Biswas, D. Swanson, Existence and generalized Gevrey regularity of solutions to the Kuramoto–Sivashinsky equation in $\mathbb{R}^n$, J. Differential Equations 240 (1) (2007) 145–163.

[5] A. Biswas, D. Swanson, Navier–Stokes equations and weighted convolution inequalities in groups, Comm. Partial Differential Equations 35 (4) (2010) 559–589.

[6] A. Biswas, Gevrey regularity for a class of dissipative equations with applications to decay, J. Differential Equations 253 (10) (2012) 2739–2764.

[7] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 171 (2010) 1903–1930.

[8] C. Cao, M.A. Rammaha, E.S. Titi, The Navier–Stokes equations on the rotating 2-D sphere: Gevrey regularity and asymptotic degrees of freedom, Z. Angew. Math. Phys. 50 (3) (1999) 341–360.

[9] J.A. Carrillo, L.C.F. Ferreira, The asymptotic behavior of subcritical dissipative quasi-geostrophic equations, Nonlinearity 21 (2008) 1001–1018.

[10] D. Chae, J. Lee, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, Comm. Math. Phys. 233 (2003) 297–311.

[11] J.Y. Chemin, Perfect Incompressible Fluids, Oxford University Press, USA, 1998.

[12] Q. Chen, C. Miao, Z. Zhang, A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation, Comm. Math. Phys. 271 (2007) 821–838.

[13] P. Constantin, Euler equations, Navier–Stokes equations and turbulence, in: Mathematical Foundation of Turbulent Viscous Flows, 2006, pp. 1–43.

[14] P. Constantin, D. Cordoba, J. Wu, On the critical dissipative quasi-geostrophic equation. Dedicated to Professors Ciprian Foias and Roger Temam, Indiana Univ. Math. J. 50 (Special Issue) (2001) 97–107.

[15] P. Constantin, A.J. Majda, E. Tabak, Formation of strong fronts in the 2-d quasigeostrophic thermal active scalar, Nonlinearity 7 (1994) 1495.

[16] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geom. Funct. Anal. 22 (5) (2012) 1289–1321.

[17] P. Constantin, J. Wu, Behavior of solutions of 2d quasi-geostrophic equations, SIAM J. Math. Anal. 30 (1999) 937.

[18] P. Constantin, J. Wu, Regularity of Hölder Continuous Solutions of the Supercritical Quasi-geostrophic Equation, Ann. Inst. Henri Poincare Anal. Non Lineaire, vol. 25, Elsevier, 2008, pp. 1103–1110.

[19] A. Cordoba, D. Cordoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (3) (2004) 511–528.

[20] M. Dabkowsky, Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation, Geom. Funct. Anal. 21 (2011) 1–13.
[21] R. Danchin, Fourier Analysis Methods for PDES, Lecture Notes, vol. 14, 2005.
[22] H. Dong, Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness, Discrete Contin. Dyn. Syst. 26 (2010) 1197–1211.
[23] H. Dong, D. Li, Spatial analyticity of the solutions to the subcritical dissipative quasi-geostrophic equations, Arch. Ration. Mech. Anal. 189 (2008) 131–158.
[24] H. Dong, D. Li, On the 2D critical and supercritical dissipative quasi-geostrophic equation in Besov spaces, J. Differential Equations 248 (11) (2010) 2684–2702.
[25] A.B. Ferrari, E.S. Titi, Gevrey regularity for nonlinear analytic parabolic equations, Comm. Partial Differential Equations 23 (1) (1998) 424–448.
[26] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, J. Funct. Anal. 87 (1989) 359–369.
[27] H. Fujita, T. Kato, On the Navier–Stokes initial value problem. I, Arch. Ration. Mech. Anal. 16 (4) (1964) 269–315.
[28] P. Germain, N. Pavlović, G. Staffilani, Regularity of solutions to the Navier–Stokes equations evolving from small data in BMO^{-1}, Int. Math. Res. Not. IMRN 21 (2007).
[29] Z. Grujic, I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L_p, J. Funct. Anal. 152 (2) (1998) 447–466.
[30] R. Guberović, Smoothness of the Koch–Tataru solutions to the Navier–Stokes equations revisited, Discrete Contin. Dyn. Syst. 27 (1) (2010) 231–236.
[31] T. Hmidi, S. Keraani, Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces, Adv. Math. 214 (2) (2007) 618–638.
[32] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys. 251 (2) (2004) 365–376.
[33] N. Ju, The maximum principle and the global attractor for the dissipative 2d quasi-geostrophic equations, Comm. Math. Phys. 255 (1) (2005) 161–181.
[34] N. Ju, Global solutions to the two dimensional quasi-geostrophic equation with critical or super-critical dissipation, Math. Ann. 334 (3) (2006) 627–642.
[35] R. Kerman, Convolution theorems with weights, Trans. Amer. Math. Soc. 280 (1) (1983) 207–219.
[36] A. Kiselev, F. Nazarov, A. Volberg, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Invent. Math. 167 (3) (2007) 445–453.
[37] A. Kiselev, F. Nazarov, R. Shterenberg, Blow up and regularity for fractal Burgers' equation, Dyn. Partial Differ. Equ. 5 (3) (2008) 211–240.
[38] I. Kukavica, V. Vicol, On the radius of analyticity of solutions to the three-dimensional Euler equations, in: Proc. Amer. Math. Soc., vol. 137, 2009, pp. 669–677.
[39] C.D. Levermore, M. Oliver, Analyticity of solutions for a generalized Euler equation, J. Differential Equations 133 (2) (1997) 321–339.
[40] H. Miura, Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space, Comm. Math. Phys. 267 (1) (2006) 141–157.
[41] H. Miura, O. Sawada, On the regularizing rate estimates of Koch–Tataru's solution to the Navier–Stokes equations, Asymptot. Anal. 49 (1) (2006) 1–15.
[42] M. Oliver, E.S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in $\mathbb{R}^n$, J. Funct. Anal. 172 (1) (2000) 1–18.
[43] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, 1986.
[44] S. Resnick, Dynamical problems in nonlinear advective partial differential equations, Ph.D. Thesis, 1995.
[45] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, vol. 3, de Gruyter, 1996.
[46] H. Wang, Z. Zhang, A frequency localized maximum principle applied to the 2D quasi-geostrophic equation, Comm. Math. Phys. 301 (1) (2011) 105–129.
[47] F.B. Weissler, The Navier–Stokes initial value problem in L^p, Arch. Ration. Mech. Anal. 74 (3) (1980) 219–230.