YJFAN:7322

Journal of Functional Analysis  $\bullet \bullet \bullet (\bullet \bullet \bullet \bullet) \bullet \bullet \bullet - \bullet \bullet \bullet$ 



Contents lists available at ScienceDirect

### Journal of Functional Analysis

www.elsevier.com/locate/jfa

# On Gevrey regularity of the supercritical SQG equation in critical Besov spaces $\stackrel{\bigstar}{\approx}$

### Animikh Biswas<sup>a</sup>, Vincent R. Martinez<sup>b,\*</sup>, Prabath Silva<sup>b</sup>

 <sup>a</sup> Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, United States
 <sup>b</sup> Department of Mathematics, Indiana University, Bloomington, IN 47405, United States

#### ARTICLE INFO

Article history: Received 1 August 2014 Accepted 14 August 2015 Available online xxxx Communicated by F. Otto

Keywords: Supercritical surface quasigeostrophic equation Gevrey regularity Besov spaces Bilinear multiplier operators

#### ABSTRACT

In this paper, we show that the solution of the supercritical surface quasi-geostrophic (SQG) equation, with initial data in a critical Besov space, belongs to a subanalytic Gevrey class. In order to prove this, a suitable estimate on the nonlinear term, in the form of a commutator, is required. We express the commutator as a bilinear multiplier operator and obtain single-scale estimates for its symbol. In particular, we show that the localized symbol is of Marcinkiewicz type, and show that due to the localizations inherited from working in the Besov spaces, this condition implies the requisite boundedness of the corresponding operator. This starting from initial data in critical Besov spaces are classical. As a direct consequence of our method, decay estimates of higher-order derivatives are easily deduced.

© 2015 Elsevier Inc. All rights reserved.

 $^{*}$  The research of A. Biswas was supported in part by NSF grant DMS-1425877 and the CNMS start-up fund at University of Maryland-Baltimore County. The research of V.M. was supported in part by NSF grant numbers DMS-1008661 and DMS-1109638.

\* Corresponding author.

*E-mail addresses:* abiswas@umbc.edu (A. Biswas), vinmarti@indiana.edu (V.R. Martinez), pssilve@indiana.edu (P. Silva).

 $\label{eq:http://dx.doi.org/10.1016/j.jfa.2015.08.010} 0022\text{-}1236/ \ensuremath{\textcircled{\odot}}\ 2015$  Elsevier Inc. All rights reserved.

#### $\mathbf{2}$

A. Biswas et al. / On Gevrey regularity for supercritical SQG

### 1. Introduction

We consider the two-dimensional dissipative surface quasi-geostrophic (SQG) equation, arising in geophysics, meteorology, and oceanography. It is given by

$$\begin{cases} \partial_t \theta + \Lambda^{\kappa} \theta + u \cdot \nabla \theta = 0, \\ u = (-R_2 \theta, R_1 \theta), \\ \theta(x, 0) = \theta_0(x), \end{cases}$$
(1.1)

where the scalar quantity  $\theta$  is the potential temperature, u is the velocity of the fluid,  $R_j$  is the *j*-th Riesz transform, and  $\Lambda^{\kappa} := (-\Delta)^{\kappa/2}$  for  $0 < \kappa \leq 2$ . When  $\kappa = 1$ , SQG is structurally similar to the 3D Navier–Stokes equations and contains a mechanism for vortex stretching. In this case, it is derived from the more general 3D quasigeostrophic equation and describes the evolution of temperature on the 2D boundary of a rapidly rotating half-space with small Rossby and Ekman numbers (see [55]).

The investigation of (1.1) can be divided into three cases: supercritical ( $\kappa < 1$ ), critical ( $\kappa = 1$ ), and subcritical ( $\kappa > 1$ ), while the case of no diffusion is called the *inviscid* case. The analytical and numerical study of the inviscid SQG equation was initiated by Constantin, Majda, and Tabak in [15] and global existence of weak solutions, in all cases, was first established in [56], subsequently sparking great interest within the mathematical community. It is known that the subcritical SQG is globally well-posed [18] (see [50] for related results on a generalized SQG equation) and in fact, has a global attractor [37]. In the supercritical case, only local well-posedness for large initial data, global wellposedness for small data, and eventual regularity and conditional regularity results are known (see for instance [2,10,11,19,21,22,24,25,35,36,52,56,59,61]). In the critical ( $\kappa = 1$ ) case, local existence as well as global existence for small data in  $L^{\infty}$  was first proven in [13] while the problem of global regularity for arbitrary smooth data was solved, first in [9] and [42] independently, using different techniques, and subsequently by yet another technique in [17] (see also [1,40,41,16]). Under additional regularity assumptions, the framework of [9] has been extended to include the supercritical ( $\kappa < 1$ ) case and other hydrodynamic equations in [19,20], (see [14,23,51] for cases where these difficulties are resolved and global regularity is recovered), while the techniques of [17] were applied to study long term dynamics for the critical SQG in [16].

This article focuses on the supercritical SQG. In particular, we establish that the solutions to the initial value problem (1.1), with initial value  $\theta_0$  belonging to the critical Besov space  $B_{p,q}^{1+2/p-\kappa}(\mathbb{R}^2)$ , immediately becomes Gevrey regular (see (2.3)) for at least a short time, and will remain Gevrey regular provided that the *homogeneous* Besov norm (see (4.11) and (4.12)) of the initial data is sufficiently small. The study of Gevrey regularity of solutions was inspired by the seminal work of Foias–Temam in [27], who introduced the technique of Gevrey norms to establish analyticity in both space and time of solutions to the Navier–Stokes equations in two and three dimensions. The technique of Gevrey norms has since become a standard tool in studying analyticity and higher-order

3

regularity for a large class of dissipative equations and in various functional spaces, as well as small length scales in classical turbulence theory (cf. [3,5,7,8,26,29,32,34,33,40,43-46,48,49,53,54]). Our result therefore properly extends that of [6] for supercritical SQG to  $L^p$ -based Besov spaces and moreover, strengthens that of [25], where it was shown that the solutions obtained in [11], are actually classical solutions. As a consequence of working with Gevrey norms, we obtain decay of higher-order derivatives of the corresponding solutions (Corollary 3.3) in Besov spaces.

The notations and conventions used throughout the paper are introduced in Section 4, while the statements of our main theorems are located in Section 3. We establish our commutator estimate in Section 5 and Gevrey regularity of solutions to SQG in Section 6. The proof of our multiplier theorem is classical and elementary; it can be found in Appendix A.

### 2. Overview

In this section, we provide a brief overview of the results and techniques. In order to do this, we need to first define Gevrey classes. Let  $0 < \alpha \leq 1$  and  $\gamma > 0$ . We denote the *Gevrey operator* by the linear multiplier operator  $T_{G_{\gamma}} = \mathcal{F}^{-1}G_{\gamma}\mathcal{F}$ , where  $\mathcal{F}$  denotes the Fourier transform and

$$G_{\gamma}(\xi) := \exp(\gamma \|\xi\|^{\alpha}). \tag{2.1}$$

In the definition above,  $\|\cdot\|$  denotes the two-dimensional Euclidean norm in  $\mathbb{R}^2$ . For convenience, we write the multiplier operator,  $T_{G_{\gamma}}f$ , simply as

$$G_{\gamma}f \text{ or } \tilde{f}.$$
 (2.2)

We say that a function f is *Gevrey regular* if

$$\|G_{\gamma}f\|_{\dot{B}^{s}_{p,q}} < \infty, \tag{2.3}$$

for some  $s \in \mathbb{R}$ ,  $\gamma > 0$ , and  $1 \le p, q \le \infty$ . Here  $\dot{B}_{p,q}^s$  denotes the homogeneous  $L^p$ -based Besov spaces with regularity index s and summability index q (see for instance [4] or Section 4 below for definition). Note that when p = q = 2, one recovers the usual definition of Gevrey classes (cf. [27,48]) in the space-periodic setting, for instance.

An important property of Gevrey regular functions is that estimates on higher-order derivatives follow immediately. In particular, it is elementary to show that functions in Gevrey classes for which the estimate (2.3) holds, automatically satisfy for any k > 0 the following inequality:

$$\|D^{k}f\|_{\dot{B}^{s}_{p,q}} \leq C^{k} \frac{k^{k/\alpha}}{(\gamma\alpha)^{k/\alpha}} \|G_{\gamma}f\|_{\dot{B}^{s}_{p,q}},$$
(2.4)

Please cite this article in press as: A. Biswas et al., On Gevrey regularity of the supercritical SQG equation in critical Besov spaces, J. Funct. Anal. (2015), http://dx.doi.org/10.1016/j.jfa.2015.08.010

#### A. Biswas et al. / On Gevrey regularity for supercritical SQG

for some absolute constant C > 0. Indeed, this is one of the main reasons for working with Gevrey norms. In particular, the Gevrey class corresponding to  $\alpha = 1$  is called the analytic Gevrey class, as the finiteness of the corresponding Gevrey norm implies that the functions are (real-)analytic with (uniform) analyticity radius  $\gamma$ , while the Gevrey classes corresponding to  $\alpha > 1$  are comprised of entire functions. When  $\alpha < 1$ , the corresponding functions are no longer analytic, although in view of (2.4), these functions are  $C^{\infty}$ ; the corresponding classes of such functions are the so-called sub-analytic Gevrey classes.

One of the main points in this article is how the product in the nonlinear term in (1.1) is estimated in  $L^p$  Besov space-based Gevrey norms. In the case of ( $L^2$ -based) Sobolev spaces and in analytic Gevrey classes, i.e.,  $\alpha = 1$ , this can be done via the Plancherel theorem as follows (cf. [27]):

$$\begin{split} \|G_{\gamma}(uv)\|_{L^{2}}^{2} &\leq \int \left(\int e^{\gamma \|\xi\|} |\hat{u}(\xi-\eta)| |\hat{v}(\eta)| \, d\eta\right)^{2} d\xi \\ &\leq \int \left(\int e^{\gamma \|\xi-\eta\|} |\hat{u}(\xi-\eta)| e^{\gamma \|\eta\|} |\hat{v}(\eta)| \, d\eta\right)^{2} d\xi = \|UV\|_{L^{2}}^{2}, \end{split}$$

where,  $(\mathcal{F}U)(\xi) = e^{\gamma \|\xi\|} |\hat{u}(\xi)|$ ,  $(\mathcal{F}V)(\xi) = e^{\gamma \|\xi\|} |\hat{v}(\xi)|$  and to obtain the last line, we have used triangle inequality  $\|\xi\| \leq \|\xi - \eta\| + \|\eta\|$ . One may now proceed to estimate  $\|UV\|_{L^2}$  using the usual Sobolev inequalities after one makes the crucial observation that  $\|U\|_{H^s} = \|G_{\gamma}u\|_{H^s}$ , which is again due to the Plancherel theorem. This readily yields, for instance in two space dimensions, an estimate of the form

$$\|G_{\gamma}(uv)\|_{L^{2}} \leq C \|G_{\gamma}(u)\|_{H^{1/2}} \|G_{\gamma}(v)\|_{H^{1/2}}.$$

Consequently, the product of two functions in Gevrey classes has been estimated as a product of two Gevrey norms in adequate functional spaces. This has been generalized in [47] for analytic Gevrey classes, to the  $L^p$  space setting, where it is shown that for  $1 < p, q, r < \infty$ :

$$\|G_{\gamma}(uv)\|_{L^{p}} \leq \|G_{\gamma}u\|_{L^{q}}\|G_{\gamma}v\|_{L^{r}}, \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$
(2.5)

Although the Plancherel theorem is no longer available, the approach taken in [47] is to first rewrite the product in Gevrey space as a bilinear multiplier operator (see (3.4) for a definition) and then slightly modify the definition of the Gevrey norm to an equivalent one in which the Euclidean norm,  $||\xi||$ , in the symbol is replaced by the  $\ell_1$  norm, i.e.,  $|\xi|_{\ell^1} := |\xi_1| + |\xi_2|$ . Then, taking advantage of the special form of this symbol for analytic Gevrey class (namely  $e^{\gamma|\xi|_{\ell^1}}$ ), it is shown in [47] that this operator can be decomposed as a finite sum of tensor products of the Hilbert transform and the identity operator. The inequality (2.5) then immediately follows from the Hölder inequality and the classical  $L^p$ -boundedness of the one-dimensional Hilbert transform.

In our setting, the dissipation operator is  $\Lambda^{\kappa}$ , where  $\kappa < 1$ , and consequently, the solution of the linear equation lies only in a sub-analytic Gevrey classes and is not, in general, analytic. We are therefore forced to work with sub-analytic Gevrey norms where the above technique of [47] does not seem to apply since the bilinear multiplier operator no longer appears to admit such a decomposition involving one-dimensional Calderón–Zygmund operators. Nevertheless, it can be checked that *at each scale* the symbol m in our case satisfies the inequality

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}m(\xi,\eta)\right| \lesssim_{\beta} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}$$

This condition is weaker than the Coifman–Meyer condition on the symbol which guarantees  $L^p$  boundedness. Indeed, such is not the case in general (cf. [31]). Yet, by taking advantage of additional localization and decay properties of our symbol (see Theorem 3.7), we are able to show that a product estimate of the type (2.5) holds in  $L^p$ -based Besov spaces (see Theorem 3.4).

Another difficulty that arises in the supercritical regime, which is not present in the subcritical case or in the case of the Navier–Stokes equations, is the fact that the dissipation operator is of a lower order than the derivative present in the nonlinear term. For instance, in case of the Navier–Stokes equations one has  $\|\nabla e^{t\Delta}\|_{L^p \to L^p} = O(t^{-1/2}),$ which is therefore integrable over the interval (0,T). This is no longer the case for the supercritical SQG. This obliges one to exploit cancellation properties of the equation via commutator estimates in the functional class that one is working in as was done in [10,52], for instance. This was in particular done for Sobolev-based Gevrey classes in [6] by employing the Littlewood–Paley decomposition, Bony paraproduct formula, and the Plancherel theorem. When one tries to generalize that technique to the  $L^p$ -based setting, the setting of bilinear multiplier operators becomes natural. We show through elementary harmonic analysis techniques that one can establish the commutator estimate given in Theorem 3.5 in Besov space-based Gevrey classes. Finally, with such estimates in hand, we prove the main result, Theorem 3.1, following the semigroup approach pioneered for the Navier–Stokes equations in [28,30,39,58]. In particular, we show that solutions,  $\theta$ , of (1.1), whose initial values satisfy  $\theta_0 \in B_{p,q}^{\sigma}$  with  $\sigma \geq 1 + 2/p - \kappa$ , are Gevrey regular up to some time T > 0. Additionally, the time T can be taken to be  $T = \infty$  if the initial data is adequately small in the critical *homogeneous* Besov space,  $\dot{B}_{p,q}^{1+2/p-\kappa}$ .

### 3. Main results

We will now state our main results more precisely.

**Theorem 3.1.** Let  $2 \le p < \infty$  and  $1 \le q \le \infty$  and define

$$\sigma_c := 1 + 2/p - \kappa.$$

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Assume that  $\theta_0 \in B^{\sigma}_{p,q}(\mathbb{R}^2)$  for some  $\sigma \geq \sigma_c$  and let  $\alpha < \kappa$ . Then there exists T > 0 and a unique solution  $\theta$  of (1.1) such that  $\theta \in C([0,T); B^{\sigma}_{p,q}(\mathbb{R}^2))$  and additionally,  $\theta$  satisfies the estimate

$$\sup_{0 \le t \le T} \|G_{\gamma}\theta(\cdot, t)\|_{\dot{B}^{\sigma}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}, \text{ where } \gamma := \lambda t^{\alpha/\kappa}.$$

Moreover, if  $\sigma = \sigma_c$  and  $\|\theta_0\|_{\dot{B}^{\sigma}_{n,d}} \leq C$  for an adequate constant C > 0, then  $T = \infty$ .

**Remark 3.2.** It will be clear from the proof that in fact, the Gevrey norm of  $\theta$  satisfies an additional regularity property, namely,

$$\sup_{0 < t < T} t^{\beta/\kappa} \|G_{\gamma}\theta(\cdot, t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \lesssim \|\theta_0\|_{B^{\sigma}_{p,q}}, \text{ for some } \beta > 0.$$

It immediately follows from Theorem 3.1, (2.4), and Stirling's approximation that the solutions of (1.1) with initial data belonging to  $B_{p,q}^{1+2/p-\kappa}(\mathbb{R}^2)$  automatically satisfy certain higher-order decay estimates.

**Corollary 3.3.** Let  $k > 1 + 2/p - \kappa$ . Then the solution  $\theta$  in Theorem 3.1 satisfies

$$\|D^{k}\theta(t)\|_{\dot{B}^{1+2/p-\kappa}_{p,q}} \lesssim C^{k} \frac{(k!)^{1/\alpha}}{t^{k/\kappa}} \|\theta_{0}\|_{\dot{B}^{1+2/p-\kappa}_{p,q}},$$
(3.1)

for all 0 < t < T, where  $C := C(q, \alpha, \beta, \kappa)$ .

The crucial estimate for the product of two functions in Gevrey spaces, which may also have independent interest, is stated in the following theorem. This is a suitable generalization to non-analytic Gevrey classes of Lemma 24.8 in [47], which applies to analytic Gevrey classes.

**Theorem 3.4.** Let  $1 and <math>1 \le q \le \infty$ . Suppose  $s, t \in \mathbb{R}$  satisfy the following

- (i) s, t < 2/p,
- (ii) s + t > 0.

Then there exists C > 0 such that

$$\|G_{\gamma}(fg)\|_{\dot{B}^{s+t-2/p}_{p,q}(\mathbb{R}^{2})} \le C \|G_{\gamma}f\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^{2})} \|G_{\gamma}g\|_{\dot{B}^{t}_{p,q}(\mathbb{R}^{2})}.$$
(3.2)

It is well-documented (cf. [38,52] for instance) that in the presence of supercritical dissipation, product estimates are insufficient to control the nonlinearity in (1.1), and that commutators must be used instead to ensure that one remains in a perturbative regime. The proof of Theorem 3.1 will make use of the following commutator estimate for Gevrey regular functions, which is an extension of that found by Biswas (cf. [6]) for

7

A. Biswas et al. / On Gevrey regularity for supercritical SQG

homogeneous Sobolev spaces, to homogeneous Besov spaces. Recall that the commutator bracket [A, B] is defined as

$$[A,B] := AB - BA. \tag{3.3}$$

**Theorem 3.5.** Let  $1 and <math>1 \le q \le \infty$ . Let  $\gamma, \delta > 0$  such that  $\delta < 1$ . Suppose  $s, t \in \mathbb{R}$  satisfy the following

(i) 2/p < s < 1 + 2/p, (ii) t < 2/p, (iii) s + t > 2/p.

Then, denoting by  $\triangle_j$  the dyadic Littlewood-Paley operator, we have

$$\|[G_{\gamma} \triangle_{j}, f]g\|_{L^{p}(\mathbb{R}^{2})} \lesssim 2^{-(s+t-2/p)j} C_{j} \|G_{\gamma}f\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^{2})} \|G_{\gamma}g\|_{\dot{B}^{t}_{p,q}(\mathbb{R}^{2})},$$

where

$$C_j = C_j(\alpha, \delta, \gamma) := c_j \left( \gamma^{(\alpha-\delta)/\alpha} 2^{(\alpha-\delta)j} + 1 \right),$$

for some  $0 \leq \delta < \alpha$  such that  $s < 1 + 2/p - \delta$  and with  $(c_j)_{j \in \mathbb{Z}}$  such that  $||(c_j)||_{\ell^q(\mathbb{Z})} \leq C$ for some absolute constant C > 0.

When one formally sets  $\gamma = 0$ , p = 2, and  $\delta < \alpha$ , Theorem 3.5 extends the commutator estimate of Miura (cf. [52]) to homogeneous Besov spaces.

**Corollary 3.6.** Suppose that p, q satisfy the conditions of Theorem 3.5 with  $\delta = 0$ . Then there exists  $(c_j)_{j \in \mathbb{Z}} \in \ell^q$  such that

$$\|[\Delta_j, f]g\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-(s+t-2/p)j} c_j \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} \|g\|_{\dot{B}^t_{p,q}(\mathbb{R}^2)}.$$

This corollary can be proved by closely following the proof of Theorem 3.5 and so we omit the details.

In order to prove Theorems 3.5 and 3.4, we apply the Bony paraproduct decomposition and view the resulting terms of both the commutator,  $[G_{\gamma} \Delta_j, f]g$ , and the product,  $G_{\gamma}(fg)$ , as bilinear multiplier operators,  $T_m(f,g)$ , which are written as

$$T_m(f,g) := \int \int e^{ix \cdot (\xi+\eta)} m(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) \ d\xi d\eta, \qquad (3.4)$$

and show that for their corresponding symbols, m, the following estimate is satisfied for each multi-index  $\beta$ :

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}m(\xi,\eta)\right| \lesssim_{\beta} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}.$$
(3.5)

In other words, we show that m is of Marcinkiewicz type. Note that condition (3.5) is weaker than that of Coifman–Meyer (cf. [12]). In general, such multipliers need not map  $L^p \times L^q$  into  $L^r$  for any  $1 < p, q < \infty$  and 1/r = 1/p + 1/q (cf. [31]). In our case however, the special structure of our symbol and the fact that we work with Besov spaces, provides additional localizations which greatly simplify the situation.

**Theorem 3.7.** Suppose  $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$  satisfies (3.5) for finitely many, but sufficiently large number of multi-indices  $|\beta| \ge 0$  with  $\beta = \beta_1 + \beta_2$ . Assume moreover that for each fixed  $\xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}, m(\xi, \eta)$  is a smooth function of  $\eta$ , with support contained in  $[2^{j_0-1} \le ||\eta|| \le 2^{j_0+1}]$  for some  $j_0 \in \mathbb{N}$ . Then for all 1 such that<math>1/r = 1/p + 1/q, the associated bilinear multiplier operator  $T_m : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to$  $L^r(\mathbb{R}^d)$  satisfies

$$||T_m(f,g)||_{L^r} \lesssim ||f||_{L^p} ||g||_{L^q}$$

**Remark 3.8.** Note that the same conclusion holds with the roles of  $\xi$ ,  $\eta$  and p, q reversed together in the above hypotheses. Also, the required number of derivatives,  $|\beta|$ , can be determined from the proof of the theorem.

A prototypical example of a bilinear operator satisfying (3.5) is  $T(f,g) = Hf \cdot Hg$ , where H is the Hilbert transform. Indeed, boundedness would then follow from Hölder's inequality. The role then of the smooth localization in  $\eta$  in Theorem 3.7 is that it essentially allows us to treat the bilinear multiplier as a product of linear ones, effectively reducing the situation to the simpler case of  $Hf \cdot Hg$ . Thus, Besov spaces provide an appropriate setting with which to work with bilinear Marcinkiewicz multipliers.

The proof of Theorem 3.7 relies on classical harmonic analysis techniques and we provide its proof in Appendix A. The theorem itself may be of independent interest, particularly in PDE applications. The proofs of Theorems 3.1 and 3.5 will be given in Sections 6 and 5, respectively.

**Remark 3.9.** The notation  $T_m$  will be used to denote either a linear multiplier operator,  $T_m f = \mathcal{F}^{-1}(m\mathcal{F}f)$ , where  $\mathcal{F}$  denotes the Fourier transform, or a bilinear multiplier operator  $T_m(f,g)$ , defined as in (3.4). However, it will be quite clear from the context which type of operator  $T_m$  is denoting.

### 4. Notation and preliminaries

#### 4.1. Littlewood–Paley decomposition and related inequalities

Let  $\psi_0$  be a radial bump function such that  $\psi_0(\xi) = 1$  when  $[||\xi|| \le 1/2] \subset \mathbb{R}^d$ , and

$$0 \le \psi_0 \le 1$$
 and spt  $\psi_0 = [\|\xi\| \le 1]$ .

9

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Define  $\phi_0(\xi) := \psi_0(\xi/2) - \psi_0(\xi)$ . Observe that

$$0 \le \varphi_0 \le 1$$
 and spt  $\phi_0 = [2^{-1} \le ||\xi|| \le 2]$ 

Now for each  $j \in \mathbb{Z}$ , define  $\psi_j := (\psi_0)_{2^{-j}}$  and  $\varphi_j := (\varphi_0)_{2^{-j}}$ , where we use the notation

$$f_{\lambda}(x) := f(\lambda x), \tag{4.1}$$

for any  $\lambda \geq 0$  (the notation  $f_{\lambda}$  should not be confused with the Gevrey operator  $G_{\gamma}$ in (2.1); the usage will be clear from context). In view of the above definitions, clearly  $\varphi_0 := \psi_1 - \psi_0$  and  $\psi_{j+1} = \psi_j + \varphi_j$ , so that

spt 
$$\psi_j = [\|\xi\| \le 2^{j-1}]$$
 and spt  $\varphi_j = [2^{j-1} \le \|\xi\| \le 2^{j+1}].$  (4.2)

Moreover, we have

$$\sum_{j\in\mathbb{Z}}\varphi_j(\xi)=1, \text{ for } \xi\in\mathbb{R}^d\setminus\{\mathbf{0}\}.$$

One can then define

We call the operators  $\triangle_k$  Littlewood–Paley blocks. For convenience, we will sometimes use the shorthand  $f_k := \triangle_k f$ .

For functions which are spectrally supported in a compact set, one has the Bernstein inequalities (cf. [4]), which we will invoke copiously throughout the article. We state it here in terms of Littlewood–Paley blocks. Note that we will use the following convention throughout the paper.

**Notation.**  $A \leq B$  to denote the relation  $A \leq cB$  for some absolute constant c > 0. In our estimates, the constant c may change line to line, but will nevertheless remain an absolute constant.

**Lemma 4.1** (Bernstein inequalities). Let  $1 \le p \le q \le \infty$  and  $f \in \mathcal{S}'(\mathbb{R}^d)$ . Then

$$2^{js} \| \triangle_j f \|_{L^q} \lesssim \| \Lambda^s \triangle_j f \|_{L^q} \lesssim 2^{js+d(1/p-1/q)} \| \triangle_j f \|_{L^p}, \tag{4.3}$$

for each  $j \in \mathbb{Z}$  and  $s \in \mathbb{R}$ .

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Since we will be working with  $L^p$  norms, we will also require the generalized Bernstein inequalities, which was proved in [11] and [60].

**Lemma 4.2** (Generalized Bernstein inequalities). Let  $2 \leq p < \infty$  and  $f \in \mathcal{S}'(\mathbb{R}^d)$ . Then

$$2^{\frac{2s_j}{p}} \|\Delta_j f\|_{L^p} \lesssim \|\Lambda^s |\Delta_j f|^{p/2} \|_{L^2}^{\frac{2}{p}} \lesssim 2^{\frac{2s_j}{p}} \|\Delta_j f\|_{L^p},$$
(4.4)

for each  $j \in \mathbb{Z}$  and  $s \in [0, 1]$ .

In order to apply these inequalities, we will first need the following positivity lemma, which was initially proved in [21], and generalized by Ju in [37] (see also [17,16]).

**Lemma 4.3** (Positivity lemma). Let  $2 \le p < \infty$ ,  $f, \Lambda^s f \in L^p(\mathbb{R}^2)$ . Then

$$\int \Lambda^{s} f |f|^{p-2} f \, dx \ge \frac{2}{p} \int (\Lambda^{\frac{s}{2}} |f|^{\frac{p}{2}})^{2} \, dx.$$
(4.5)

We will also make use of the following heat kernel estimate, which was proved in [52] for  $L^2$ . We extend it to  $L^p$ .

**Lemma 4.4.** Let  $2 \leq p < \infty$ . Then there exist constants  $c_1, c_2 > 0$  such that

$$e^{-c_1 t 2^{\kappa_j}} \| \triangle_j u \|_{L^p} \le \| e^{-t\Lambda^{\kappa}} \triangle_j u \|_{L^p} \le e^{-c_2 t 2^{\kappa_j}} \| \triangle_j u \|_{L^p},$$
(4.6)

holds for all t > 0.

**Proof.** Let  $u_j := e^{-t\Lambda^{\kappa}} \triangle_j u$ . Then  $u_j$  satisfies the initial value problem

$$\begin{cases} \partial_t u_j + \Lambda^{\kappa} u_j = 0\\ u_j(x,0) = \Delta_j u(x). \end{cases}$$
(4.7)

Multiplying (4.7) by  $u_j |u_j|^{p-2}$  and integrating gives

$$\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p + \int (\Lambda^{\kappa} u_j)u_j|u_j|^{p-2} dx = 0.$$

By applying Lemmas 4.2 and 4.3, then dividing by  $||u_j||_{L^p}^{p-1}$  we obtain

$$\frac{d}{dt} \|u_j\|_{L^p} + c_1 2^{\kappa j} \|u_j\|_{L^p} \le 0.$$

Similarly, by Hölder's inequality we obtain

$$\frac{d}{dt} \|u_j\|_{L^p} + c_2 2^{\kappa j} \|u_j\|_{L^p} \ge 0.$$

11

A. Biswas et al. / On Gevrey regularity for supercritical SQG

An application of Gronwall's inequality gives

$$e^{-c_2 2^{\kappa_j t}} \|u_j(0)\|_{L^p} \le \|u_j(t)\|_{L^p} \le e^{-c_1 2^{\kappa_j t}} \|u_j(0)\|_{L^p},$$
(4.8)

which completes the proof.  $\hfill\square$ 

#### 4.2. Besov spaces

Let  $s \in \mathbb{R}$  and  $1 \le p, q \le \infty$ . The *inhomogeneous Besov space*  $B^s_{p,q}$  is the space defined by

$$B_{p,q}^{s} := \{ f \in \mathcal{S}'(\mathbb{R}^{d}) : \| f \|_{B_{p,q}^{s}} < \infty \},$$
(4.9)

where,  $\mathcal{S}'$  denotes the space of tempered distributions, and one can define the norm by

$$\|f\|_{B^{s}_{p,q}} := \|\check{\psi}_{0} * f\|_{L^{p}} + \left(\sum_{j \ge 0} 2^{jsq} \|\triangle_{j}f\|_{L^{p}}^{q}\right)^{1/q},$$
(4.10)

provided that  $q < \infty$ . The homogeneous Besov space  $\dot{B}_{p,q}^s$  is the space defined by

$$\dot{B}_{p,q}^{s} := \{ f \in \mathcal{Z}'(\mathbb{R}^{d}) : \|f\|_{\dot{B}_{p,q}^{s}} < \infty \},$$
(4.11)

where  $\mathcal{Z}'(\mathbb{R}^d)$  denotes the dual space of  $\mathcal{Z}(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) : \partial^{\beta} \hat{f}(0) = 0, \forall \beta \in \mathbb{N}^d\},$ and for  $q < \infty$ , the (semi)norm is given by

$$\|f\|_{\dot{B}^{s}_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_{j}f\|_{L^{p}}^{q}\right)^{1/q}.$$
(4.12)

One then makes the usual modification for  $q = \infty$ . For more details, see [4] or [57].

### 5. Product and commutator estimates

In this section, we establish estimates for the product

$$G_{\gamma} \triangle_j (fg),$$
 (5.1)

and for the commutator

$$[G_{\gamma} \triangle_j, f]g := G_{\gamma} \triangle_j (fg) - fG_{\gamma} \triangle_j g, \qquad (5.2)$$

#### A. Biswas et al. / On Gevrey regularity for supercritical SQG

where  $G_{\gamma} := e^{\gamma \Lambda^{\alpha}}$  and  $0 < \alpha < \kappa \leq 1$ , where  $\kappa$  is the order of dissipation in (1.1). For convenience, we will use the notation

$$\widetilde{f} := G_{\gamma} f. \tag{5.3}$$

To prove Theorems 3.5 and 3.4, we will require the Faà di Bruno formula, whose statement we recall from [4] for convenience. Note that by  $\mathbb{N}$  and  $\mathbb{N}^*$  we mean the set of positive integers with zero and the set  $\mathbb{N} \setminus \{0\}$ , respectively.

**Lemma 5.1** (Faà di Bruno formula). Let  $u : \mathbb{R}^d \to \mathbb{R}^m$  and  $F : \mathbb{R}^m \to \mathbb{R}$  be smooth functions. For each multi-index  $\alpha \in \mathbb{N}^d$  with  $|\alpha| > 0$  we have

$$\partial^{\alpha}(F \circ u) = \sum_{\mu,\nu} C_{\mu,\nu} \partial^{\mu} F \prod_{\substack{1 \le |\beta| \le |\alpha| \\ 1 \le j \le m}} (\partial^{\beta} u^{j})^{\nu_{\beta_{j}}},$$
(5.4)

where the coefficients  $C_{\mu,\nu}$  are nonnegative integers, and the sum is taken over those  $\mu$ and  $\nu$  such that  $1 \leq |\mu|, |\nu| \leq |\alpha|, \nu_{\beta_i} \in \mathbb{N}^*$ ,

$$\sum_{1 \le |\beta| \le |\alpha|} \nu_{\beta_j} = \mu_j, \text{ for } 1 \le j \le m, \text{ and } \sum_{\substack{1 \le |\beta| \le |\alpha| \\ 1 \le j \le m}} \beta \nu_{\beta_j} = \alpha.$$
(5.5)

We will repeatedly apply this formula to functions of the form

$$(F \circ u)(\xi, \eta) = e^{\gamma R_{\alpha,\sigma}(\xi, \eta)},$$

where

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi + \sigma\eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha}$$

or

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi\sigma + \eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha},$$

where  $\sigma \in [0, 1]$ . For convenience, we provide that application here. By Lemma 5.1 we have

$$\partial^{\beta}(F \circ u)(\xi, \eta) = \sum_{\mu, \nu} C_{\mu, \nu} \gamma^{|\mu|} e^{\gamma R_{\alpha, \sigma}(\xi, \eta)} \prod_{1 \le |b| \le |\beta|} (\partial^{b} R_{\alpha, \sigma}(\xi, \eta))^{\nu_{b}}$$
(5.6)

for all  $\beta \in \mathbb{N}^2$ , where  $\nu = (\nu_1, \nu_2), 1 \le |\mu| \le |\beta|$  and

$$\sum_{1 \le |b| \le |\beta|} \nu_b = \mu \quad \text{and} \quad \sum_{1 \le |b| \le |\beta|} b\nu_b = \beta.$$
(5.7)

13

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Thus, in order to apply Theorem 3.7, we will require  $R_{\alpha,\sigma}$  to satisfy certain derivative estimates.

**Proposition 5.2.** Let  $0 < \alpha \leq 1$ ,  $\sigma \in [0,1]$ , and define  $R_{\alpha,\sigma} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$  by

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi + \eta\sigma\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha}$$

$$(5.8)$$

Suppose that  $\ell + 3 \le k$  and  $2^{k-1} \le \|\xi\| \le 2^{k+1}$  and  $2^{\ell-1} \le \|\eta\| \le 2^{\ell+1}$ . Then

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim_{\beta,\alpha} 2^{\ell\alpha} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|},\tag{5.9}$$

for all multi-indices  $\beta_1, \beta_2 \in \mathbb{N}^2$ .

If  $j+3 \le k$  with  $2^{j-1} \le \|\eta\| \le 2^{j+1}$  and  $2^{k-1} \le \|\xi\|, \|\xi+\eta\| \le 2^{k+1}$ , then

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}R_{\alpha,1}(\xi,-\xi-\eta)\right| \lesssim_{\beta,\alpha} 2^{k\alpha} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|},\tag{5.10}$$

for all  $\beta_1, \beta_2 \in \mathbb{N}^d$ .

**Remark 5.1.** If  $R_{\alpha,\sigma}$  is given instead by

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi\sigma + \eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha},$$
(5.11)

then (5.9) and (5.10) all hold with the roles of k and  $\ell$  reversed.

**Proof.** We prove (5.9). The inequality (5.10) can be obtained by a more straightforward estimation of derivatives.

Let  $\beta \in \mathbb{N}^4 \times \mathbb{N}^4$ , where  $\beta = (\beta_1, \beta_2) = (\beta_{\xi}, \beta_{\eta}), \beta_j = (\beta_j^{\xi}, \beta_j^{\eta}), \beta_j^{\xi}, \beta_j^{\eta} \in \mathbb{N}^2$  for j = 1, 2, and  $\beta_{\xi} = \beta_1^{\xi} + \beta_2^{\xi}$  and  $\beta_{\eta} = \beta_1^{\eta} + \beta_2^{\eta}$ . Firstly, from the triangle inequality

$$|R_{\alpha,\sigma}(\xi,\eta)| \lesssim (1-\sigma) \|\eta\|^{lpha} \lesssim 2^{\ell lpha}$$

This proves (5.9) for  $|\beta| = 0$ . For  $|\beta| \neq 0$ , we apply the mean value theorem to write

$$R_{\alpha,\sigma}(\xi,\eta) = \int_0^1 \|\xi + \eta\tau\sigma\|^{\alpha-2} ((\xi\cdot\eta)\sigma + \|\eta\|^2 \sigma^2 \tau) \ d\tau - \|\eta\|^{\alpha}.$$

Then observe that

$$\begin{split} \left|\partial^{\beta}R_{\alpha}(\xi,\eta)\right| \\ \lesssim \sum_{\beta=\beta_{1}+\beta_{2}}c_{\beta}\int_{0}^{1}\left(\|\xi+\eta\sigma\tau\|^{\alpha-2-|\beta_{1}|}\partial^{\beta_{2}}((\xi\cdot\eta)\sigma+\|\eta\|^{2}\sigma^{2}\tau)\right) d\tau + N_{\alpha}(\beta,\eta), \end{split}$$

14

A. Biswas et al. / On Gevrey regularity for supercritical SQG

where  $N_{\alpha}(\beta,\eta) = 0$  if  $|\beta_j^{\xi}| \neq 0$  for some j, and  $N_{\alpha}(\beta,\eta) = ||\eta||^{\alpha-|\beta|}$  otherwise. Next observe that since  $k \geq \ell + 3$ ,  $\sigma, \tau \in [0, 1]$ , and  $\xi \sim 2^k$ ,  $\eta \sim 2^{\ell}$ , we have

$$\|\xi + \eta \sigma \tau\| \gtrsim 2^k \gtrsim 2^\ell. \tag{5.12}$$

We also have

$$\left|\partial^{\beta_{2}}((\xi \cdot \eta)\sigma + \|\eta\|^{2}\sigma^{2}\tau)\right| \lesssim \begin{cases} 2^{k+\ell}, & |\beta_{2}| = 0\\ 2^{k}, & |\beta_{2}| = |\beta_{2}^{\eta}| = 1\\ 2^{\ell}, & |\beta_{2}| = |\beta_{2}^{\xi}| = 1\\ 1, & |\beta_{2}| = 2 \text{ and } |\beta_{2}^{\xi}| < 2\\ 0, & |\beta_{2}| \geq 3 \text{ or } |\beta_{2}^{\xi}| = 2. \end{cases}$$
(5.13)

Now we consider three cases. First suppose that  $|\beta_1| = 0, |\beta_2| \neq 0$ . Using (5.12) and the fact that  $\alpha < 1$ , observe that

$$\|\xi + \eta \sigma \tau\|^{\alpha - 2} \lesssim \begin{cases} 2^{\ell(\alpha - 1)} 2^{-k}, & |\beta_2| = 1 \text{ or } |\beta_2^{\eta}| = |\beta_2^{\xi}| = 1\\ 2^{\ell(\alpha - 2)}, & |\beta_2| = |\beta_2^{\eta}| = 2. \end{cases}$$
(5.14)

Thus, combining (5.13) and (5.14) gives

$$\left|\partial^{\beta} R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim 2^{\ell\alpha} 2^{-k|\beta_{2}^{\xi}|} 2^{-\ell|\beta_{2}^{\eta}|},$$

which implies (5.9) since  $\beta = (0, 0, \beta_2^{\xi}, \beta_2^{\eta})$ .

Now suppose  $|\beta_1| \neq 0$  and  $|\beta_2| = 0$ . Applying (5.12) then gives

$$\|\xi + \eta \sigma \tau\|^{\alpha - 2 - |\beta_1|} \lesssim \begin{cases} 2^{\ell(\alpha - 1)} 2^{-k - |\beta_1^{\xi}|}, & |\beta_1^{\eta}| = 0 \neq |\beta_1^{\xi}| \\ 2^{\ell(\alpha - 1 - |\beta_1^{\eta}|)} 2^{-k}, & |\beta_1^{\xi}| = 0 \neq |\beta_1^{\eta}| \\ 2^{\ell(\alpha - 1 - |\beta_1^{\eta}|)} 2^{k(-1 - |\beta_1^{\xi}|)}, & |\beta_1^{\xi}|, |\beta_1^{\eta}| \neq 0. \end{cases}$$
(5.15)

Thus, by combining (5.13) and (5.15) we get

 $\left|\partial^{\beta} R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim 2^{\ell\alpha} 2^{-k|\beta_{1}^{\xi}|} 2^{-\ell|\beta_{1}^{\eta}|},$ 

which again implies (5.9) since  $\beta = (\beta_1^{\xi}, \beta_1^{\eta}, 0, 0)$ .

Finally, if  $\beta_1 \neq 0, \beta_2 \neq 0$ , we may combine the argumentation of the previous two cases to obtain

$$\left|\partial^{\beta} R_{\sigma,\alpha}(\xi,\eta)\right| \lesssim 2^{\ell\alpha} 2^{-k|\beta_{\xi}|} 2^{-\ell|\beta_{\eta}|}.$$
(5.16)

This establishes (5.9) for all  $\beta \in \mathbb{N}^4 \times \mathbb{N}^4$ .  $\Box$ 

We will also need the following "rotation" lemma.

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

15

A. Biswas et al. / On Gevrey regularity for supercritical SQG

**Lemma 5.3.** Let  $T_m$  be a bilinear multiplier operator with multiplier  $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ . Then for  $\widetilde{m}(\xi, \eta) := m(\xi, -\xi - \eta)$ , we have

$$\langle T_m(f,g),h\rangle = \langle T_{\widetilde{m}}(h,g),f\rangle, \tag{5.17}$$

for all  $f, g, h \in \mathcal{S}(\mathbb{R}^d)$ . Moreover, if  $T_m : L^p \times L^q \to L^r$  is bounded for some 1/r = 1/p + 1/q, then  $T_{\widetilde{m}} : L^{r'} \times L^q \to L^{p'}$  is bounded, where p', r' are the Hölder conjugates of p, r, respectively.

**Proof.** By change of variables we have

$$\begin{split} \int T_m(f,g)(x)h(x) \, dx &= \int \int \int e^{ix \cdot (\xi+\eta)} m(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta)h(x) \, d\xi \, d\eta \, dx \\ &= \int \int \int e^{-ix \cdot \nu} m(\xi,\xi-\nu) \hat{f}(\xi) \hat{g}(-\nu-\xi)h(x) \, dx \, d\xi \, d\nu \\ &= \int \int \int m(\xi,-\nu-\xi) \hat{g}(-\xi-\nu) \hat{h}(\nu) \hat{f}(\xi) \, d\nu \, d\xi \\ &= \int \int \int \int e^{-ix \cdot \xi} m(\xi,-\xi-\nu) \hat{g}(-\xi-\nu) \hat{h}(\nu) f(x) \, d\nu \, d\xi \, dx \\ &= \langle T_{\widetilde{m}}(h,g),f \rangle, \end{split}$$

as desired. Boundedness of  $T_{\widetilde{m}}$  then follows from duality.  $\Box$ 

**Remark 5.2.** Observe that if  $1 < p, r < \infty$ , then  $1 < p', r' < \infty$  as well. Therefore, if  $T_m$  is bounded in the range 1/r = 1/p + 1/q for  $1 < p, r < \infty$ , then  $T_{\widetilde{m}}$  is also bounded in the same range.

We will first prove Theorem 3.4 since the estimates there will be used to prove Theorem 3.5. As a preliminary, we recall the paraproduct decomposition:

$$fg = \sum_{k} S_k f \triangle_k g + \sum_{k} \triangle_k f S_k g + \sum_{k} \breve{\triangle}_k f \triangle_k g.$$
(5.18)

This implies that

$$[G_{\gamma} \triangle_{j}, f]g = \sum_{k} G_{\gamma} \triangle_{j} (S_{k} f \triangle_{k} g) + G_{\gamma} \triangle_{j} (\triangle_{k} f S_{k} g) + G_{\gamma} \triangle_{j} (\check{\Delta}_{k} f \triangle_{k} g) - \left( \sum_{k} (S_{k} f) (\triangle_{j} \triangle_{k} \widetilde{g}) + (\triangle_{k} f) (\triangle_{j} S_{k} \widetilde{g}) + (\check{\Delta}_{k} f) (\triangle_{j} \triangle_{k} \widetilde{g}) \right). (5.19)$$

Then by the localization properties in (4.2), we can reduce (5.19) to

A. Biswas et al. / On Gevrey regularity for supercritical SQG

$$[G_{\gamma} \triangle_{j}, f]g = \sum_{|k-j| \le 4} \left\{ [G_{\gamma} \triangle_{j}, S_{k}f] \triangle_{k}g + G_{\gamma} \triangle_{j}(\triangle_{k}fS_{k}g) + G_{\gamma} \triangle_{j}(\check{\Delta}_{k}f\triangle_{k}g) \right\}$$
$$+ \sum_{k \ge j+5} G_{\gamma} \triangle_{j}(\check{\Delta}_{k}f\triangle_{k}g)$$
$$- \sum_{k \ge j+1} \triangle_{k}f \triangle_{j}S_{k}\widetilde{g} - \sum_{|k-j| \le 2} \check{\Delta}_{k}f\triangle_{j}\Delta_{k}\widetilde{g}.$$
(5.20)

### 5.1. Proof of Theorem 3.4

Observe that  $G_{\gamma} \triangle_j(fg)$  is precisely the first line of (5.19). By symmetry and localization, it suffices to consider only

$$\sum_{|k-j| \leq 4} \left[ G_{\gamma} \triangle_j (\triangle_k f S_k g) + G_{\gamma} \triangle_j (\check{\bigtriangleup}_k f \triangle_k g) \right] \text{ and } \sum_{k \geq j+5} G_{\gamma} \triangle_j (\check{\bigtriangleup}_k f \triangle_k g)$$

Case:  $k \ge j+5$  First, we rewrite  $G_{\gamma} \bigtriangleup_j (\breve{\bigtriangleup}_k f \bigtriangleup_k g)$  as

$$G_{\gamma} \triangle_j (G_{\gamma}^{-1} \breve{\triangle}_k \widetilde{f} G_{\gamma}^{-1} \triangle_k \widetilde{g}) \tag{5.21}$$

The multiplier associated to (5.21) is

$$m_{k,j}(\xi,\eta) := e^{\gamma(\|\xi+\eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha})} \varphi_j(\xi+\eta) \breve{\varphi}_k(\xi) \varphi_k(\eta),$$
(5.22)

where  $\breve{\varphi}_k = \sum_{|k-\ell| \leq 2} \varphi_{\ell}$ . By Lemma 5.3, in order to apply Theorem 3.7, it suffices to prove

$$|\partial_{\xi}^{\beta_1}\partial_{\eta}^{\beta_2}\widetilde{m}_{k,j}(\xi,\eta)| \lesssim \|\xi\|^{-|\beta_1|} \|\eta\|^{-|\beta_2|},\tag{5.23}$$

where  $\widetilde{m}_{k,j}(\xi,\eta) = m_{k,j}(\xi,-\xi-\eta)$ . Once the required  $L^p$  bounds are deduced, we then show that the obtained estimate is summable in  $\ell^q$  with respect to j.

So first observe that for  $\beta = (\beta_1, \beta_2)$ , by (5.6), (5.7), and (5.10) we have

$$\begin{aligned} |\partial^{\beta} m_{k,j}(\xi, -\xi - \eta)| \\ &\lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{\gamma(||\eta||^{\alpha} - ||\xi||^{\alpha} - ||\xi + \eta||^{\alpha})} \prod_{1 \le |b| \le |\beta|} (2^{k(\alpha - |b|)})^{\nu_{b}} \\ &\lesssim 2^{-k|\beta|} \sum_{\mu,\nu} C_{\mu,\nu} (\gamma 2^{k\alpha})^{|\mu|} e^{\gamma(||\eta||^{\alpha} - ||\xi||^{\alpha} - ||\xi + \eta||^{\alpha})} \end{aligned}$$

Also, by the triangle inequality

$$\|\eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\xi + \eta\|^{\alpha} \lesssim -c_{\alpha}2^{k\alpha},$$

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

for some absolute constant  $c_{\alpha} > 0$ . Thus

$$|\partial^{\beta} m_{k,j}(\xi, -\xi - \eta)| \lesssim 2^{-k|\beta|} \sum_{\mu,\nu} C_{\mu,\nu}(\gamma 2^{k\alpha})^{|\mu|} e^{-c_{\alpha}\gamma 2^{k\alpha}} \lesssim 2^{-k|\beta_1|} 2^{-k|\beta_2|}$$
(5.24)

holds for all  $\xi \in \mathbb{R}^2$ , which implies (5.23).

Now, let  $\sigma := s + t - 2/p$ . Observe that by the Bernstein and Theorem 3.7 we have

$$\begin{split} \|G_{\gamma} \triangle_{j}(\breve{\Delta}_{k} f \triangle_{k} g)\|_{L^{p}} &\lesssim 2^{j(2/p)} \|G_{\gamma} \triangle_{j}(\breve{\Delta}_{k} f \triangle_{k} g)\|_{L^{p/2}} \\ &\lesssim 2^{j(2/p)} \|\breve{\Delta}_{k} \widetilde{f}\|_{L^{p}} \|\triangle_{k} \widetilde{g}\|_{L^{p}} \\ &\lesssim 2^{-\sigma j} \underbrace{2^{-(s+t)(k-j)}}_{a_{k-j}} \underbrace{(2^{sk} \|\breve{\Delta}_{k} \widetilde{f}\|_{L^{p}})}_{b_{k}} \underbrace{(2^{tk} \|\triangle_{k} \widetilde{g}\|_{L^{p}})}_{c_{k}} \end{split}$$

It follows that

$$\begin{split} \sum_{k\geq j+5} 2^{\sigma j} \|G_{\gamma} \triangle_{j}(\breve{\Delta}_{k} f \triangle_{k} g)\|_{L^{p}} \lesssim \sum_{k} \chi_{[n\geq 5]}(k-j) a_{k-j} b_{k} c_{k} \\ \lesssim \left(\sum_{k} \chi_{[n\geq 5]}(k-j) a_{k-j} b_{k}\right) \left(\sup_{k} c_{k}\right). \end{split}$$

Observe that by Young's convolution inequality we have

$$\left(\sum_{k\geq j+5} (a_{k-j}b_k)^q\right)^{1/q} \le \left(\sum_{k\geq 5} a_k\right) \left(\sum_k b_k^q\right)^{1/q},$$

which is finite provided that s + t > 0.

Therefore

$$2^{(s+t-2/p)j} \sum_{k \ge j+5} \|G_{\gamma} \triangle_j(\breve{\Delta}_k f \triangle_k g)\|_{L^r} \lesssim c_j \|\widetilde{f}\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}},$$
(5.25)

where

$$c_j := \|\widetilde{f}\|_{\dot{B}^s_{p,q}}^{-1} \sum_{k \ge j+5} a_{k-j} b_k$$

and satisfies  $(c_j)_{j \in \mathbb{Z}} \in \ell^q$ . This finishes the case  $k \ge j + 5$ .

Case:  $|k-j| \leq 4$  It suffices to consider  $G_{\gamma} \triangle_j (\triangle_k f S_k g)$  since the term  $G_{\gamma} \triangle_j (\check{\triangle}_k f \triangle_k g)$  is easier.

A. Biswas et al. / On Gevrey regularity for supercritical SQG

First, let us we rewrite  $G_{\gamma} \triangle_j (\triangle_k f S_k g)$  as

$$\sum_{\ell \le k-3} G_{\gamma} \triangle_j (G_{\gamma}^{-1} \triangle_k \tilde{f} G_{\gamma}^{-1} \triangle_\ell \tilde{g}).$$
(5.26)

We claim that the associated multiplier satisfies the following bounds

$$|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}m_{j,k,\ell}(\xi,\eta)| \lesssim \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|},$$
(5.27)

where

$$m_{j,k,\ell}(\xi,\eta) = e^{\gamma(\|\xi+\eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha})} \varphi_j(\xi+\eta) \varphi_k(\xi) \varphi_\ell(\eta).$$
(5.28)

To this end, let  $\beta = (\beta_{\xi}, \beta_{\eta})$  and observe that by (5.6) and Proposition 5.2 we have

$$|\partial^{\beta} e^{\gamma R_{\alpha}(\xi,\eta)}| \lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{\gamma R_{\alpha}(\xi,\eta)} \prod_{1 \le |b| \le |\beta|} (2^{\ell(\alpha-|b_{\eta}|)} 2^{-k|b_{\xi}|})^{\nu_{b}}.$$

Since  $\|\eta\| \sim 2^{\ell}$  and  $k - \ell \geq 3$ , it follows by Lemma A.1 that

$$\|\xi+\eta\|^{\alpha}-\|\xi\|^{\alpha}-\|\eta\|^{\alpha}\lesssim -c_{\alpha}2^{\ell\alpha}.$$

Thus, by (5.7) we get

$$\begin{aligned} |\partial^{\beta} e^{\gamma R_{\alpha}(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{-c_{\alpha} \gamma 2^{\ell \alpha}} 2^{\ell(\alpha|\mu|-|\beta_{\eta}|)} 2^{-k|\beta_{\xi}|} \\ &\lesssim 2^{-k|\beta_{\xi}|} 2^{-\ell|\beta_{\eta}|} \sum_{\mu,\nu} C_{\mu,\nu} (\gamma 2^{\ell \alpha})^{|\mu|} e^{-c_{\alpha} \gamma 2^{\ell \alpha}} \\ &\lesssim 2^{-k|\beta_{\xi}|} 2^{-\ell|\beta_{\eta}|} \end{aligned}$$
(5.29)

holds for all  $\xi \in \mathbb{R}^2$ 

Hence, by the product rule and the fact that  $2^k \sim 2^j$ , we can conclude that

$$|\partial_{\xi}^{\beta_1}\partial_{\eta}^{\beta_2}m_{j,k}(\xi,\eta)| \lesssim 2^{-k|\beta_1|}2^{-\ell|\beta_2|},$$

for all  $\xi \in \mathbb{R}^2$ , which implies (5.27).

Therefore by Theorem 3.7, we have

$$\|G_{\gamma} \triangle_j (\triangle_k f S_k g)\|_{L^r} \lesssim \sum_{\ell \le k-1} \|\triangle_k \widetilde{f}\|_{L^p} \|\triangle_\ell \widetilde{g}\|_{L^q},$$
(5.30)

where 1/r = 1/p + 1/q and  $1 \le r < \infty$ ,  $1 , <math>1 < q \le \infty$ .

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

19

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Now let  $\sigma = s + t - 2/p$  and N > 1. Let  $p^* := (pN)/(N-1)$ . Then by (5.30), the Bernstein inequalities, and the fact that  $|k - j| \le 4$ , we have

$$2^{\sigma j} \|G_{\gamma} \triangle_{j}(\triangle_{k} f S_{k} g)\|_{L^{p}}$$
  

$$\lesssim \sum_{\ell \leq k-1} 2^{(\sigma-s-t+2/p^{*})k} 2^{sk} \|\triangle_{k} \widetilde{f}\|_{L^{p^{*}}} 2^{t\ell} \|\triangle_{\ell} \widetilde{g}\|_{L^{p}} 2^{-(2/p^{*}-t)(k-\ell)}$$
  

$$\lesssim 2^{sj} \|\triangle_{j} \widetilde{f}\|_{L^{p}} \sum_{\ell \leq k-1} 2^{t\ell} \|\triangle_{\ell} \widetilde{g}\|_{L^{p}} 2^{-(2/p^{*}-t)(k-\ell)}$$

Let t < 2/p. Observe that for N large enough, we have  $t < 2/p^*$ . Then

$$2^{(s+t-2/p)j} \|G_{\gamma} \triangle_j(\triangle_k f S_k g)\|_{L^p} \lesssim C_j \|\widetilde{f}\|_{\dot{B}^s_{p,\infty}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}},$$
(5.31)

where

$$C_j := \|\widetilde{g}\|_{\dot{B}_{p,q}^t}^{-1} \sum_{\ell \le j+2} 2^{t\ell} \|\triangle_{\ell} \widetilde{g}\|_{L^p} 2^{(t-2/p^*)(j-\ell)},$$

which satisfies  $(C_j)_{j \in \mathbb{Z}} \in \ell^q$ . Combining the estimates (5.25) and (5.31) completes the proof of Theorem 3.4.  $\Box$ 

### 5.2. Proof of Theorem 3.5

Cases:  $k \ge j + 1$  and  $|k - j| \le 1$  The corresponding terms are  $\triangle_k f \triangle_j S_k \tilde{g}$  and  $\triangle_k f \triangle_j \triangle_k \tilde{g}$ , respectively. By Hölder's inequality and Bernstein we have

$$2^{\sigma j} \| \triangle_k f \triangle_j S_k \widetilde{g} \|_{L^p} \lesssim c_j 2^{-(s-2/p)(k-j)} 2^{sk} \| \triangle_k f \|_{L^p} \| \widetilde{g} \|_{\dot{B}^t_{p,q}},$$
(5.32)

where

$$c_j := \|\widetilde{g}\|_{\dot{B}_{p,q}^t}^{-1} 2^{tj} \| \triangle_j \widetilde{g} \|_{L^p}.$$

Observe that by Hölder's inequality

$$\sum_{k \ge j+1} 2^{-(s-2/p)(k-j)} \chi_{[n\ge 1]}(k-j) 2^{sk} \| \triangle_k f \|_{L^p} \lesssim \left( \sum_{k\ge 1} 2^{-(s-2/p)kq'} \right)^{1/q'} \| f \|_{\dot{B}^s_{p,q}},$$

which is finite provided that s - 2/p > 0. Therefore

$$2^{(s+t-2/p)j} \sum_{k \ge j+1} \|\Delta_k f \Delta_j S_k \widetilde{g}\|_{L^p} \lesssim c_j \|f\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}}$$
(5.33)

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Similarly, we have for any  $s \in \mathbb{R}$ 

$$2^{(s+t-2/p)j} \sum_{|k-j| \le 1} \|\breve{\Delta}_k f \Delta_k \widetilde{g}\|_{L^p} \lesssim c_j \|f\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}},$$
(5.34)

where

$$c_j := \|\widetilde{g}\|_{\dot{B}_{p,q}^t}^{-1} 2^{tj} \| \triangle_j \widetilde{g} \|_{L^p}.$$

Case:  $k \ge j+5$  The derivative estimates for the corresponding multiplier remain the same as those from Theorem 3.4, except that we sum over k differently since now it is assumed that s + t - 2/p > 0.

Since (5.23) holds, we know that Theorem 3.7 implies

$$\|G_{\gamma} \triangle_{j}(\check{\bigtriangleup}_{k} f \triangle_{k} g)\|_{L^{p}} \lesssim \|\check{\bigtriangleup}_{k} \widetilde{f}\|_{L^{p}} \|\triangle_{k} \widetilde{g}\|_{L^{\infty}} \lesssim 2^{(2/p)k} \|\check{\bigtriangleup}_{k} \widetilde{f}\|_{L^{p}} \|\triangle_{k} \widetilde{g}\|_{L^{p}}$$

Thus, for  $\sigma := s + t - 2/p$ , by the Bernstein inequalities we have that

$$\sum_{k\geq j+5} 2^{\sigma j} \|G_{\gamma} \triangle_{j}(\triangle_{k} f \triangle_{k} g)\|_{L^{p}}$$

$$\lesssim \sum_{k} \underbrace{\chi_{[n\geq 5]}(k-j)}_{\mu_{k-j}} \underbrace{2^{-(s+t-2/p)(k-j)}}_{a_{k-j}} \underbrace{2^{sk} \|\check{\Delta}_{k} \widetilde{f}\|_{L^{p}}}_{b_{k}} \underbrace{2^{tk} \|\triangle_{k} \widetilde{g}\|_{L^{p}}}_{c_{k}}$$

As before Young's convolution inequality implies

$$\left(\sum_{k} (\mu_{k-j}a_{k-j}b_k)^q\right)^{1/q} \le \left(\sum_{k\ge 5} a_k\right) \left(\sum_{k} b_k^q\right)^{1/q},$$

which will be finite provided that

$$s + t - 2/p > 0.$$

Thus

$$2^{(s+t-2/p)j} \sum_{k \ge j+5} \|G_{\gamma} \bigtriangleup_j(\breve{\bigtriangleup}_k f \bigtriangleup_k g)\|_{L^r} \lesssim c_j \|\widetilde{f}\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}},$$
(5.35)

with  $c_j$  given by

$$c_j := \|\widetilde{f}\|_{\dot{B}^s_{p,q}}^{-1} \sum_k \mu_{k-j} a_{k-j} b_k.$$

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

21

#### A. Biswas et al. / On Gevrey regularity for supercritical SQG

Case:  $|k-j| \leq 4$  From the proof of Theorem 3.4, it suffices to consider the commutator term,  $[G_{\gamma} \triangle_j, S_k f] \triangle_k g$ , which we view as  $T_{m_{j,k}}(S_k f, \triangle_k g)$ . Indeed, observe that

$$T_{m_{j,k}}(S_k f, \triangle_k g)(x)$$
  
=  $\int \int e^{ix \cdot (\xi+\eta)} \left[ G_{\gamma}(\xi+\eta)\varphi_j(\xi+\eta) - G_{\gamma}(\eta)\varphi_j(\eta) \right] \psi_k(\xi)\varphi_k(\eta)\hat{f}(\xi)\hat{g}(\eta) \ d\xi d\eta.$ 

Then by the mean value theorem

$$T_{m_{j,k}}(S_k f, \triangle_k g)(x) = \sum_{i=1,2} \sum_{\ell \le k-3} \int_0^1 T_{m_{i,j,k,\ell,\sigma}}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})(x) \ d\sigma,$$

where

$$m_{i,j,k,\ell,\sigma}(\xi,\eta) = m_A(\xi,\eta) + m_B(\xi,\eta),$$

and

$$m_A(\xi,\eta) := \alpha \gamma e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \|\xi \sigma + \eta\|^{\alpha-2} (\xi_i \sigma + \eta_i) \varphi_j(\xi \sigma + \eta) \varphi_\ell(\xi) \varphi_k(\eta)$$
$$m_B(\xi,\eta) := e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} (\partial_i \varphi_0) (2^{-j} (\xi \sigma + \eta)) 2^{-j} \varphi_\ell(\xi) \varphi_k(\eta).$$

Now observe that since  $\|\xi\| \sim 2^{\ell}$ ,  $\|\eta\| \sim 2^k$ , and  $k - \ell \ge 3$ , by Lemma A.1 there exists a constant  $c_{\alpha} > 0$  such that

$$\|\xi\sigma + \eta\|^{\alpha} - \|\xi\sigma\|^{\alpha} - \|\eta\|^{\alpha} \le -c_{\alpha}\|\xi\|^{\alpha}, \text{ for } \sigma \ge 1/2,$$
(5.36)

and by the triangle inequality

$$\|\xi\sigma + \eta\|^{\alpha} - \|\xi\|^{\alpha} - \|\eta\|^{\alpha} \le -c'_{\alpha}\|\xi\|^{\alpha}, \text{ for } \sigma \le 1/2.$$
(5.37)

This implies that

$$e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \lesssim \begin{cases} e^{-c'_{\alpha}\gamma \|\xi\|^{\alpha}}, & \sigma \le 1/2\\ e^{-c_{\alpha}\gamma \|\xi\|^{\alpha}} e^{-(1-\sigma^{\alpha})\|\xi\|^{\alpha}}, & \sigma > 1/2. \end{cases}$$
(5.38)

Suppose that  $\sigma \leq 1/2$  and observe that by Proposition 5.2, Faà di Bruno, and (5.38), we have

$$\begin{aligned} |\partial^{\beta} e^{\gamma R_{\alpha,\sigma}(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \prod_{1 \le |b| \le |\beta|} (2^{\ell(\alpha-|b_{\xi}|)} 2^{-k|b_{\eta}|})^{\nu_{b}} \\ &\lesssim 2^{-\ell|\beta_{1}|} 2^{-k|\beta_{2}|} \sum_{\mu,\nu} C_{\mu,\nu} (\gamma 2^{\ell\alpha})^{|\mu|} e^{-c_{\alpha} \gamma 2^{\ell\alpha}} \\ &\lesssim e^{-(c_{\alpha}'/2) \gamma 2^{\ell\alpha}} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}. \end{aligned}$$
(5.39)

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Similarly, for  $\sigma \geq 1/2$ , using (5.38) we obtain

$$\begin{aligned} |\partial^{\beta} e^{\gamma R_{\alpha,\sigma}(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{-c_{\alpha} \gamma 2^{\ell \alpha}} e^{-(1-\sigma^{\alpha})\gamma \|\xi\|^{\alpha}} \prod_{1 \le |b| \le |\beta|} (\partial^{b} R_{\alpha,\sigma}(\xi,\eta))^{\nu_{b}} \\ &\lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{-c_{\alpha} \gamma 2^{\ell \alpha}} \prod_{1 \le |b| \le |\beta|} (2^{\ell(\alpha-|b_{\xi}|)} 2^{-k|b_{\eta}|})^{\nu_{b}} \\ &\lesssim e^{-(c_{\alpha}/2)\gamma 2^{\ell \alpha}} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}. \end{aligned}$$
(5.40)

For the other factors, observe that since  $\|\xi\sigma + \eta\| \sim 2^j$  we have

$$\left|\partial^{\beta}\|\xi\sigma + \eta\|^{\alpha-2}\right| \lesssim \|\xi\sigma + \eta\|^{\alpha-2-|\beta|} \lesssim 2^{j(\alpha-2)}\|\xi\|^{-|\beta_{\xi}|}\|\eta\|^{-|\beta_{\eta}|} \tag{5.41}$$

$$\left|\partial^{\beta}(\xi_{i}\sigma + \eta_{i})\right| \lesssim \begin{cases} 2 + 2 , & |\beta\xi| = 0\\ 1, & |\beta| = 1 \text{ and } |\beta_{\xi}^{i}| \text{ or } |\beta_{\eta}^{i}| = 1\\ 0, & |\beta| \ge 2 \text{ or } |\beta^{i'}| \neq 0 \text{ } i' \neq i \end{cases}$$
(5.42)

It follows from (5.41) and (5.42) that

$$\left|\partial^{\beta} \left( \|\xi\sigma + \eta\|^{\alpha - 2} (\xi_i \sigma + \eta_i) \right) \right| \lesssim 2^{j(\alpha - 1)} 2^{-\ell |\beta_{\xi}|} 2^{-k|\beta_{\eta}|}.$$
(5.43)

We also have

$$\left|\partial_{\xi}^{\beta}\varphi_{\ell}(\xi)\right| \lesssim 2^{-\ell|\beta|} \lesssim \|\xi\|^{-|\beta|},\tag{5.44}$$

$$\left|\partial_{\eta}^{\beta}\varphi_{k}(\eta)\right| \lesssim 2^{-k|\beta|} \lesssim \|\eta\|^{-|\beta|} \tag{5.45}$$

for all  $\eta \in \mathbb{R}^2$ .

Therefore, combining (5.39), (5.40) and (5.43)–(5.45), we can deduce that

$$\begin{aligned} \left| \partial_{\xi}^{\beta_{1}} \partial_{\eta}^{\beta_{2}} m_{A}(\xi, \eta) \right| &\lesssim \gamma 2^{-j(1-\alpha)} e^{-(c_{\alpha}/2)\gamma 2^{\ell\alpha}} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|} \\ &\lesssim \gamma^{1-\delta/\alpha} 2^{-j(1-\alpha)} 2^{-\ell\delta} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}, \end{aligned}$$
(5.46)

for any  $\delta \geq 0$ .

On the other hand, we can estimate  $m_B$  using (5.39) and (5.40) by

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}m_{B}(\xi,\eta)\right| \lesssim 2^{-j} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|}.$$
(5.47)

Fix N > 1 and let  $p^* = (pN)/(N-1)$  with  $(p^*)' = pN$  so that  $1/p = 1/(p^*)' + 1/p^*$ . Then by Theorem 3.7 and the Bernstein inequalities

$$\|T_{m_A}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})\|_{L^p} \lesssim \gamma^{1-\delta/\alpha} 2^{-j(1-\alpha)} 2^{(1-\delta)\ell} \|\triangle_\ell \widetilde{f}\|_{L^{(p^*)'}} \|\triangle_k \widetilde{g}\|_{L^{p^*}}, \quad (5.48)$$

$$\|T_{m_B}(\Delta_\ell \partial_i \widetilde{f}, \Delta_k \widetilde{g})\|_{L^p} \lesssim 2^{-j} 2^\ell \|\Delta_\ell \widetilde{f}\|_{L^{(p^*)'}} \|\Delta_k \widetilde{g}\|_{L^{p^*}}.$$
(5.49)

23

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Suppose that s < 1 + 2/p and consider  $0 \le \delta < \alpha$  such that  $s < 1 + 2/p - \delta$ . Moreover, choose N > 0 large enough so that  $s < 1 + 2/p^* - \delta$ . From (5.49), we apply the Bernstein inequalities again and the fact that  $|k - j| \le 4$  to get

$$\begin{aligned} \|T_{m_{B}}(\triangle_{\ell}\partial_{i}\widetilde{f}, \triangle_{k}\widetilde{g})\|_{L^{p}} \\ &\lesssim 2^{-(s+t)k} 2^{(2/p)k} \|\widetilde{g}\|_{\dot{B}^{t}_{p,\infty}} \sum_{\ell \leq k-3} 2^{-(1+2/p^{*}-s)(k-\ell)} 2^{s\ell} \|\triangle_{\ell}\widetilde{f}\|_{L^{p}} \\ &\lesssim 2^{-(s+t-2/p)j} C_{j} \|\widetilde{f}\|_{\dot{B}^{s}_{p,q}} \|\widetilde{g}\|_{\dot{B}^{t}_{p,\infty}}, \end{aligned}$$
(5.50)

where

$$C_j := \|\widetilde{f}\|_{\dot{B}^s_{p,q}}^{-1} \sum_{j \ge \ell-2} 2^{-(1+2/p^*-s)(j-\ell)} 2^{s\ell} \|\triangle_\ell \widetilde{f}\|_{L^p},$$

which satisfies  $(C_i)_{i \in \mathbb{Z}} \in \ell^q$  since  $s < 1 + 2/p^*$ .

Similarly, since  $s < 1 + 2/p^* - \delta$ , from (5.48) we can estimate

$$\begin{aligned} \|T_{m_{A}}(\triangle_{\ell}\partial_{i}\widetilde{f},\triangle_{k}\widetilde{g})\|_{L^{p}} \\ &\lesssim \gamma^{1-\delta/\alpha}2^{-(\delta-\alpha+s+t-2/p)j}\|\widetilde{g}\|_{\dot{B}^{t}_{p,\infty}}\sum_{\ell\leq k-3}2^{-(1+2/p^{*}-\delta-s)(k-\ell)}2^{s\ell}\|\triangle_{\ell}\widetilde{f}\|_{L^{p}} \\ &\lesssim \gamma^{(\alpha-\delta)/\alpha}2^{(\alpha-\delta)j}2^{-(s+t-2/p)j}C_{j}\|\widetilde{f}\|_{\dot{B}^{s}_{p,q}}\|\widetilde{g}\|_{\dot{B}^{t}_{p,\infty}}, \end{aligned}$$
(5.51)

where

$$C_j := \|\widetilde{f}\|_{\dot{B}^{s}_{p,q}}^{-1} \sum_{j \ge \ell-2} 2^{-(1+2/p^*-\delta-s)(j-\ell)} 2^{s\ell} \|\triangle_{\ell} \widetilde{f}\|_{L^p},$$

which satisfies  $(C_j)_{j \in \mathbb{Z}} \in \ell^q$  since  $s < 1 + 2/p^* - \delta$ .

Combining the estimates (5.33), (5.34), (5.35), (5.50), and (5.51) completes the proof of Theorem 3.5.  $\Box$ 

### 6. Proof of Theorem 3.1

We will provide the proof only for the critical regularity index  $\sigma_c := 1 + 2/p - \kappa$ , the proof for  $\sigma > \sigma_c$  being similar. The proof will proceed in three steps. In the first step (Part I) we will make two preliminary estimates that arises from the linear part of the equation. Next, in Part II, we will construct and establish properties for the approximating solution sequence in the usual Besov spaces that will be necessary for our purposes. Finally, in Part III, we conclude the proof by making the relevant *a priori* estimates in Gevrey classes.

A. Biswas et al. / On Gevrey regularity for supercritical SQG

#### 6.1. Part I: Preliminary estimates

We will need to control the linear term that appears from differentiating the Gevrey norm with respect to t in the a priori estimates. To do so, we adapt the approach in [54] where the  $L^2$  case is dealt with, and modify the proof to accommodate the general case of  $p \neq 2$ .

**Lemma 6.1.** Let  $0 < \alpha < \kappa$  and  $1 \le p \le \infty$ . If  $\Lambda^{\alpha} f, G_{\gamma} \Lambda^{\kappa} f \in L^p$ , then

$$\|G_{\gamma}\Lambda^{\alpha}\triangle_{j}f\|_{L^{p}} \lesssim \|\Lambda^{\alpha}\triangle_{j}f\|_{L^{p}} + \gamma^{-(1-\kappa/\alpha)}\|G_{\gamma}\Lambda^{\kappa}\triangle_{j}f\|_{L^{p}}, \tag{6.1}$$

for all  $j \in \mathbb{Z}$ .

**Proof.** Fix an integer k, to be chosen later, such that  $N := 2^{k-3}$ . Denote by  $\check{\Delta}_j$  the augmented operator  $\Delta_{j-1} + \Delta_j + \Delta_{j+1}$ . Observe that

$$G_{\gamma}\Lambda^{\alpha}\triangle_{j}f = G_{\gamma}S_{k}(\Lambda^{\alpha}\triangle_{j}f) + \Lambda^{-(\kappa-\alpha)}(I-S_{k})\triangle_{j}(G_{\gamma}\Lambda^{\kappa}\mathring{\triangle}_{j}f).$$

Observe that  $G_{\gamma}S_k \in L^1$ . Indeed, by Lemma 4.1 we have

$$\|G_{\gamma}S_k\|_{L^1} \le \sum_{n=0}^{\infty} \frac{\lambda^n \gamma^n}{n!} \|\Lambda^{\alpha n}S_k\|_{L^1} \le e^{c\gamma 2^{k\alpha}},\tag{6.2}$$

for some absolute constant c > 0. On the other hand, observe that  $\check{m} := \Lambda^{-(\kappa-\alpha)}(I - S_k) \Delta_j$  is smooth with compact support. Let  $g := G_{\gamma} \Lambda^{\kappa} \check{\Delta}_j f$ . We consider three cases.

If  $2^{j+2} \leq N$ , then  $g \equiv 0$ . If  $N \leq 2^{j-2}$ , then Lemma 4.1 and Young's convolution inequality implies that

$$||T_m g||_{L^1} \lesssim 2^{-(\kappa - \alpha)j} \lesssim N^{-(\kappa - \alpha)},$$

where  $T_m$  is convolution with  $\check{m}$ . Similarly, if  $2^{j-1} \leq N \leq 2^{j+1}$ , then

$$||T_m g||_{L^1} \lesssim N^{-(\kappa - \alpha)}. \tag{6.3}$$

Therefore, for any N > 0

$$\|G_{\gamma}\Lambda^{\alpha}\triangle_{j}f\|_{L^{p}} \lesssim e^{\gamma N^{\alpha}}\|\Lambda^{\alpha}\triangle_{j}f\|_{L^{p}} + N^{-(\kappa-\alpha)}\|G_{\gamma}\Lambda^{\kappa}\check{\bigtriangleup}_{j}f\|_{L^{p}}.$$

Finally, choose  $k := [\alpha^{-1} \log_2(1/\gamma)]$ , where [x] denotes the greatest integer  $\leq x$ . Then  $N \sim \gamma^{-1/\alpha}$ , which gives (6.1).  $\Box$ 

We will also require the following properties for the solution to the linear heat equation (4.7). Let us consider the space

$$X_T := \{ v \in C((0,T); \dot{B}_{p,q}^{\sigma+\beta}(\mathbb{R}^2)) : \|v\|_{X_T} < \infty \},$$
(6.4)

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

such that

$$\|v\|_{X_T} := \sup_{0 < t < T} t^{\beta/\kappa} \|G_{\gamma}v(\cdot, t)\|_{\dot{B}^{\sigma+\beta}_{p,q}},$$
(6.5)

where  $\gamma = \lambda t^{\alpha/\kappa}, \, \lambda > 0, \, \sigma \in \mathbb{R}$  and  $\beta \ge 0$ .

**Lemma 6.2.** Let  $\alpha < \kappa$ ,  $\sigma > 0$ , and  $\beta \ge 0$  and suppose that  $\theta_0 \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^2)$ . Then

- (i)  $\|e^{-(\cdot)\Lambda^{\kappa}}\theta_0\|_{X_T} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}$ , for any  $T \ge 0$ , and
- (ii)  $\lim_{T \to 0} \|e^{-(\cdot)\Lambda^{\kappa}}\theta_0\|_{X_T} = 0.$

**Proof.** Observe that for b < 1, we have  $e^{ax^b - cx} \le 1$  for x > 1 and  $e^{ax^b - cx} \le e^{-cx}$  for  $0 \le x \le 1$ . If  $t2^{j\kappa} \le 1$ , then arguing as in Lemma 6.1, e.g., (6.2),

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}}e^{-t\Lambda^{\kappa}}\triangle_{j}\theta_{0}\|_{L^{p}} \lesssim e^{c_{1}\lambda(t2^{j\kappa})^{\alpha/\kappa}}\|e^{-t\Lambda^{\kappa}}\triangle_{j}\theta_{0}\|_{L^{p}} \lesssim e^{\lambda}\|e^{-t\Lambda^{\kappa}}\triangle_{j}\theta_{0}\|_{L^{p}}$$

for some  $c_1 > 0$ . If  $t2^{j\kappa} > 1$ , then arguing as in Lemma 6.1 and applying Lemma 4.4

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}}e^{-t\Lambda^{\kappa}} \triangle_{j}\theta_{0}\|_{L^{p}} \lesssim e^{c_{1}\lambda t^{\alpha/\kappa}2^{j\alpha}-c_{2}t2^{j\kappa}}\|e^{-(t/2)\Lambda^{\kappa}} \triangle_{j}\theta_{0}\|_{L^{p}} \lesssim \|e^{-c_{3}t\Lambda^{\kappa}} \triangle_{j}\theta_{0}\|_{L^{p}},$$

for some  $c_1, c_2, c_3 > 0$ . Therefore, a final application of Lemma 4.4 proves

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}}e^{-t\Lambda^{\kappa}}\Delta_{j}\theta_{0}\|_{L^{p}} \lesssim \|e^{-c_{3}t\Lambda^{\kappa}}\Delta_{j}\theta_{0}\|_{L^{p}} \lesssim e^{-c_{4}t2^{j\kappa}}\|\Delta_{j}\theta_{0}\|_{L^{p}}, \tag{6.6}$$

for some  $c_4 > 0$ . Now by (6.6) we have

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}}e^{-t\Lambda^{\kappa}}\theta_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}}^{q} = \sum_{j} 2^{(\sigma+\beta)jq} \|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}-t\Lambda^{\kappa}} \triangle_{j}\theta_{0}\|_{L^{p}}^{q}$$
$$\lesssim \sum_{j} 2^{\beta jq}e^{-qc_{4}t2^{j\kappa}} \left(2^{\sigma j}\|\triangle_{j}\theta_{0}\|_{L^{p}}\right)^{q}$$
$$\lesssim t^{-(\beta q)/\kappa} \|\theta_{0}\|_{\dot{B}^{\sigma}_{p,q}}^{q}.$$
(6.7)

This proves (i). Now we prove (ii). Then for let  $\epsilon > 0$ , there exists  $\theta_0^{\epsilon} \in S$  such that  $\mathcal{F}\theta_0^{\epsilon}$  is supported away from the origin and  $\|\theta_0 - \theta_0^{\epsilon}\|_{\dot{B}^{\sigma}_{p,q}} < \epsilon$ . In particular,  $\theta_0^{\epsilon} \in \dot{B}^{\sigma+\beta}_{p,q}$ . Observe that for  $0 < t \leq T$ 

$$\begin{split} \|e^{-t\Lambda^{\kappa}}\widetilde{\theta}_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}} &\lesssim \|e^{-t\Lambda^{\kappa}}\widetilde{\theta}^{\epsilon}_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}} + \|e^{-t\Lambda^{\kappa}}\widetilde{\theta}_{0} - e^{-t\Lambda^{\kappa}}\widetilde{\theta}^{\epsilon}_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ &\lesssim \|\theta^{\epsilon}_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}} + \|e^{\lambda t^{\alpha/\kappa}\Lambda^{\alpha}}e^{-t\Lambda^{\kappa}}(\theta_{0} - \theta^{\epsilon}_{0})\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ &\lesssim t^{-\beta/\kappa}\left(T^{\beta/\kappa}\|\theta^{\epsilon}_{0}\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right) + t^{-\beta/\kappa}\|\theta_{0} - \theta^{\epsilon}_{0}\|_{\dot{B}^{\sigma}_{p,q}}, \end{split}$$

where we have applied (6.7) to  $\theta_0 - \theta_0^{\epsilon}$ . This implies (*ii*) and we are done.  $\Box$ 

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

#### 6.2. Part II: Approximating sequence

Now let us consider the sequence of approximate solutions  $\theta^n$  determined by

$$\begin{cases} \partial_t \theta^{n+1} + \Lambda^{\kappa} \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u^n = (-R_2 \theta^n, R_1, \theta^n) \text{ in } \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^{n+1} \big|_{t=0} = \theta_0 \text{ in } \mathbb{R}^2, \end{cases}$$
(6.8)

for n = 1, 2, ..., and where  $\theta^0$  satisfies the heat equation

$$\begin{cases} \partial_t \theta^0 + \Lambda^{\kappa} \theta^0 = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}, \\ \theta^0 \big|_{t=0} = \theta_0 \text{ in } \mathbb{R}^2. \end{cases}$$
(6.9)

It is well-known that  $\theta^n$  is Gevrey regular for  $n \ge 0$ . In particular, we may define

$$\theta^n(s) := G_\gamma \theta^n, \text{ and } \widetilde{u}^n(s) := G_\gamma u^n(s), \tag{6.10}$$

where we choose  $\gamma = \gamma(s) := \lambda s^{\alpha/\kappa}$ . It is shown in [11] that there exists a subsequence of  $(\theta^n)_{n\geq 0}$  that converges in  $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)$  to some function  $\theta \in C([0,T); \dot{B}^{\sigma_c}_{p,q})$ , where  $\sigma_c := 1+2/p-\kappa$ , and which satisfies (1.1) in the sense of distribution, provided that either T or  $\|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}}$  is sufficiently small. Additionally, we will show that the approximating sequence satisfies

$$\sup_{0 < t < T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}} \text{ and } \lim_{T \to 0} \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} = 0, \quad (6.11)$$

for any  $0 < \beta < \kappa/2$  and  $n \ge 0$ , where the suppressed constant above is independent of n. Whenceforth, to prove Theorem 3.1 it will suffice to obtain a priori bounds for  $\|\theta^n(\cdot)\|_{X_T}$ , independent of n (see (6.5)). We will also make crucial use of the estimates (6.11) in the proof of Theorem 3.1 (see (6.25))

To prove (6.11), we follow [52]. First observe that  $\theta^0 = e^{-t\Lambda^{\kappa}}\theta_0$ . Then by Lemma 6.2 we have

$$t^{\beta/\kappa} \|\theta^0\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}} \text{ and } \lim_{T \to 0} \sup_{0 < t < T} t^{\beta/\kappa} \|e^{-t\Lambda^{\kappa}} \theta^0\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} = 0$$

We proceed by induction. Assume that (6.11) holds for some n > 0.

We apply  $\triangle_j$  to (6.8) to obtain

$$\partial_t \theta_j^{n+1} + \Lambda^{\kappa} \theta_j^{n+1} + \Delta_j (u^n \cdot \nabla \theta^{n+1}) = 0.$$
(6.12)

Then we take the  $L^2$  inner product of (6.12) with  $|\theta_j|^{p-2}\theta_j$  and use the fact that  $\nabla \cdot u^n = 0$  to write

Please cite this article in press as: A. Biswas et al., On Gevrey regularity of the supercritical SQG equation in critical Besov spaces, J. Funct. Anal. (2015), http://dx.doi.org/10.1016/j.jfa.2015.08.010

A. Biswas et al. / On Gevrey regularity for supercritical SQG

$$\frac{1}{p}\frac{d}{dt}\|\theta_{j}^{n+1}\|_{L^{p}}^{p} + \int_{\mathbb{R}^{2}} \Lambda^{\kappa}\theta_{j}^{n+1}|\theta_{j}^{n+1}|^{p-2}\theta_{j}^{n+1} dx$$

$$= -\int_{\mathbb{R}^{2}} [\Delta_{j}, u^{n}] \nabla \theta_{j}^{n+1}|\theta_{j}^{n+1}|^{p-2}\theta_{j}^{n+1} dx.$$
(6.13)

Note that we used the fact that

$$\int_{\mathbb{R}^2} u^n \cdot \nabla \theta_j^{n+1} |\theta_j|^{p-2} \widetilde{\theta}_j^{n+1} \, dx = 0, \tag{6.14}$$

which one obtains by integrating by parts and invoking the fact that  $\nabla \cdot u^n = 0$  for all n > 0. Now, we apply Lemmas 4.3, 4.2, and Hölder's inequality, so that after dividing by  $\|\theta_j\|_{L^p}^{p-1}$ , (6.13) becomes

$$\frac{d}{dt}\|\theta_j^{n+1}\|_{L^p} + C2^{\kappa j}\|\theta_j^{n+1}\|_{L^p} \lesssim \|[\Delta_j, u^n]\nabla\theta^{n+1}\|_{L^p}.$$

Let  $\beta < \kappa/2$ . By Corollary 3.6 with  $s = \sigma_c + \beta$  and  $t = 2/p - \kappa + \beta$  we get

$$\frac{d}{dt} \|\theta_j^{n+1}\|_{L^p} + C2^{\kappa j} \|\theta_j^{n+1}\|_{L^p} \lesssim 2^{-((\sigma_c + \beta) - (\kappa - \beta))j} c_j \|\theta^n\|_{\dot{B}^{\sigma_c + \beta}_{p,q}} \|\theta^{n+1}\|_{\dot{B}^{\sigma_c + \beta}_{p,q}}$$

Note that we have used boundedness of the Riesz transform. Thus, multiplying by  $2^{(\sigma_c+\beta)j}$ , then applying Gronwall's inequality gives

$$\begin{aligned} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} &\lesssim \left(\sum_{j} \left(e^{-C2^{\kappa j}t} 2^{(\sigma_{c}+\beta)j} \|\Delta_{j}\theta_{0}\|_{L^{p}}\right)^{q}\right)^{1/q} \\ &+ \left(\int_{0}^{t} \sum_{j} \left(e^{-C2^{\kappa j}(t-s)} 2^{(\kappa-\beta)j} c_{j} \|\theta^{n}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \|\theta^{n+1}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \, ds\right)^{q}\right)^{1/q}. \end{aligned}$$

In particular, this implies

$$\begin{split} t^{\beta/\kappa} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ \lesssim t^{\beta/\kappa} \left( \sum_{j} \left( e^{-C2^{\kappa j}t} 2^{(\sigma+\beta)j} \|\Delta_{j}\theta_{0}\|_{L^{p}} \right)^{q} \right)^{1/q} \\ + t^{\beta/\kappa} \left( \int_{0}^{t} s^{-2\beta/\kappa} (t-s)^{-(1-\beta/\kappa)} ds \right) \left( \sum_{j} c_{j}^{q} \right)^{1/q} \end{split}$$

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

$$\times \left( \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^n(s)\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} \right) \left( \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^{n+1}(s)\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} \right), \tag{6.15}$$

where we have used the fact that

$$x^b e^{-ax^c} \lesssim a^{-b/c}. \tag{6.16}$$

Since  $\beta < \kappa/2, (c_j)_{j \in \mathbb{Z}} \in \ell^q$  and

$$\int_{0}^{t} \frac{1}{s^{2\beta/\kappa} (t-s)^{1-\beta/\kappa}} \, ds \lesssim t^{-\beta/\kappa},$$

we actually have

$$\sup_{0 < t < T} t^{\beta/\kappa} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \\
\lesssim \sup_{0 < t < T} t^{\beta/\kappa} \left( \sum_{j} \left( e^{-C2^{\kappa j}t} 2^{(\sigma+\beta)j} \|\Delta_{j}\theta_{0}\|_{L^{p}} \right)^{q} \right)^{1/q} \\
+ \left( \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^{n}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \right) \left( \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \right). \quad (6.17)$$

In fact, (6.16) also implies

$$M(t) := t^{\beta/\kappa} \left( \sum_{j} \left( e^{-C2^{\kappa j} t} 2^{(\sigma_c + \beta)j} \| \Delta_j \theta_0 \|_{L^p} \right)^q \right)^{1/q} \lesssim \| \theta_0 \|_{\dot{B}^{\sigma_c}_{p,q}}.$$
(6.18)

From Lemma 6.2 we know that

$$e^{-C2^{\kappa j}t} \|\Delta_j \theta_0\|_{L^p} \lesssim \|e^{-c't\Lambda^{\kappa}} \Delta_j \theta_0\|_{L^p},$$

for some c' > 0, where  $v_j = e^{-c't\Lambda^{\kappa}} \triangle_j \theta_0$  solves the heat equation

$$\begin{cases} \partial_t v + c' \Lambda^{\kappa} v = 0\\ v(x,0) = \triangle_j \theta_0(x) \end{cases}$$

Hence

$$M(t) \lesssim \sup_{0 < t < T} t^{\beta/\kappa} \| e^{-c't\Lambda^{\kappa}} \theta_0 \|_{\dot{B}^{\sigma_c+\beta}_{p,q}},$$

so that arguing as in Lemma 6.2, we may deduce that

$$\lim_{T \to 0} \sup_{0 < t < T} M(t) = 0.$$
(6.19)

 $Please \ cite \ this \ article \ in \ press \ as: \ A. \ Biswas \ et \ al., \ On \ Gevrey \ regularity \ of \ the \ supercritical \ SQG \ equation \ in \ critical \ Besov \ spaces, \ J. \ Funct. \ Anal. \ (2015), \ http://dx.doi.org/10.1016/j.jfa.2015.08.010$ 

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Recall that by hypothesis, we have

$$\lim_{T \to 0} \sup_{0 < t < T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{\sigma_c+\beta}_{p,q}} = 0.$$

Then returning to (6.17), by hypothesis, we may choose T sufficiently small so that

$$\sup_{0 < t < T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{1+2/p-\kappa+\beta}_{p,q}} < 1/2.$$

This implies that

$$\sup_{0 < t < T} t^{\beta/\kappa} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \sup_{0 < t < T} M(t).$$

Finally, letting  $T \to 0$  and invoking (6.19) completes the induction.

6.3. Part III: A priori bounds in Gevrey spaces

It will be convenient to introduce the space

$$X_T := \{ v \in C((0,T); \dot{B}_{p,q}^{\sigma_c + \beta}(\mathbb{R}^2)) : \|v\|_{X_T} < \infty \},$$
(6.20)

where,

$$\|v\|_{X_T} := \sup_{0 < t < T} t^{\beta/\kappa} \|G_{\gamma} v(\cdot, t)\|_{\dot{B}^{\sigma_c + \beta}_{p,q}},$$
(6.21)

and  $0 < \beta < \kappa/2$  additionally satisfies (6.24) below.

Now we will demonstrate a priori bounds for  $\|\theta^n(\cdot)\|_{X_T}$ , independent of n. First apply  $G_{\gamma} \triangle_j$  to (6.8). Using the fact that  $G_{\gamma}, \triangle_j, \nabla$  are Fourier multipliers (and hence, commute), we obtain

$$\partial_t \widetilde{\theta}_j^{n+1} + \Lambda^{\kappa} \widetilde{\theta}_j^{n+1} + G_{\gamma} \triangle_j (u^n \cdot \nabla \theta^{n+1}) = \lambda^{\kappa/\alpha} \gamma^{1-\kappa/\alpha} \Lambda^{\alpha} \widetilde{\theta}_j^{n+1}, \tag{6.22}$$

where we have used the fact that  $\gamma := \lambda t^{\alpha/\kappa}$ . Now apply Lemmas 4.3, 4.2, and Hölder's inequality, as well as Lemma 6.1 to obtain

$$\frac{d}{dt} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}} + C2^{\kappa j} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}}$$
$$\lesssim \lambda^{\kappa/\alpha} \gamma^{1-\kappa/\alpha} \|\Lambda^{\alpha} \theta_{j}^{n+1}\|_{L^{p}} + \lambda^{\kappa/\alpha} \|\Lambda^{\kappa} \widetilde{\theta}_{j}^{n+1}\|_{L^{p}} + \|[G_{\gamma} \triangle_{j}, u^{n}] \nabla \theta^{n+1}\|_{L^{p}}.$$

We choose  $\lambda > 0$  small enough so that Lemma 4.1 implies

$$\frac{d}{dt} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}} + C2^{\kappa j} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}} 
\lesssim \gamma^{1-\kappa/\alpha} 2^{\alpha j} \|\theta_{j}^{n+1}\|_{L^{p}} + \|[G_{\gamma} \triangle_{j}, u^{n}] \nabla \theta^{n+1}\|_{L^{p}}.$$
(6.23)

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Now choose  $\alpha < \kappa$ ,  $0 < \beta < \min\{\alpha, \kappa/2\}$  and  $\delta > 0$  such that

$$\alpha < \delta + \beta < \kappa < \frac{1}{2} + \frac{1}{p} + \beta.$$
(6.24)

Then by Theorem 3.5 with  $s = \sigma_c + \beta$  and  $t = 2/p - \kappa + \beta$ , we have

$$\begin{split} \frac{d}{dt} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}} &+ 2^{\kappa j} \|\widetilde{\theta}_{j}^{n+1}\|_{L^{p}} \\ &\lesssim \gamma^{1-\kappa/\alpha} 2^{\alpha j} \|\theta_{j}^{n+1}\|_{L^{p}} \\ &+ 2^{-((\sigma_{c}+\beta)-(\kappa-\beta))j} C_{j} \gamma^{(\alpha-\delta)/\alpha} 2^{(\alpha-\delta)j} \|\widetilde{\theta}^{n}\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \|\widetilde{\theta}^{n+1}\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \\ &+ 2^{-((\sigma_{c}+\beta)-(\kappa-\beta))j} C_{j} \|\widetilde{\theta}^{n}\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \|\widetilde{\theta}^{n+1}\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}}. \end{split}$$

Now by Gronwall's inequality, for  $t \ge 0$  we have

$$\begin{split} & 2^{(\sigma_{c}+\beta)j} \|\widetilde{\theta}_{j}^{n+1}(t)\|_{L^{p}} \\ & \lesssim 2^{\beta j} e^{-C2^{\kappa j}t} 2^{\sigma j} \|\Delta_{j}\theta_{0}\|_{L^{p}} \\ & + \int_{0}^{t} \gamma(s)^{1-\kappa/\alpha} 2^{\alpha j} e^{-C(t-s)2^{\kappa j}} 2^{(\sigma_{c}+\beta)j} \|\theta_{j}^{n+1}(s)\|_{L^{p}} \, ds \\ & + C_{j} \int_{0}^{t} \gamma(s)^{(\alpha-\delta)/\alpha} 2^{(\alpha-\delta+\kappa-\beta)j} e^{-C(t-s)2^{\kappa j}} \|\widetilde{\theta}^{n}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \|\widetilde{\theta}^{n+1}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \, ds \\ & + C_{j} \int_{0}^{t} 2^{(\kappa-\beta)j} e^{-C(t-s)2^{\kappa j}} \|\widetilde{\theta}^{n}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} \|\widetilde{\theta}^{n+1}(s)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}}. \end{split}$$

Substituting  $\gamma(s) = \lambda s^{\alpha/\kappa}$ , applying the decay properties of the heat kernel  $e^{-C(t-s)2^{\kappa j}}$ , Minkowski's inequality, and by definition of the space  $X_T$ , we arrive at

$$\begin{split} \|\widetilde{\theta}^{n+1}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}} &\lesssim t^{-\beta/\kappa} \|\theta_{0}\|_{\dot{B}^{\sigma_{c}}_{p,q}} \\ &+ \left(\int_{0}^{t} s^{-(1-(\alpha-\beta)/\kappa)}(t-s)^{-\alpha/\kappa} \, ds\right) \left(\sup_{0 < t \le T} t^{\beta/\kappa} \|\theta^{n+1}(t)\|_{\dot{B}^{\sigma_{c}+\beta}_{p,q}}\right) \\ &+ \left(\int_{0}^{t} s^{(\alpha-\delta-2\beta)/\kappa}(t-s)^{-(\alpha-\delta+\kappa-\beta)/\kappa} \, ds\right) \|\theta^{n}\|_{X_{T}} \|\theta^{n+1}\|_{X_{T}} \\ &+ \left(\int_{0}^{t} s^{-2\beta/\kappa}(t-s)^{-(\kappa-\beta)/\kappa} \, ds\right) \|\theta^{n}\|_{X_{T}} \|\theta^{n+1}\|_{X_{T}} \end{split}$$

Please cite this article in press as: A. Biswas et al., On Gevrey regularity of the supercritical SQG equation in critical Besov spaces, J. Funct. Anal. (2015), http://dx.doi.org/10.1016/j.jfa.2015.08.010

31

A. Biswas et al. / On Gevrey regularity for supercritical SQG

Since  $\beta < \min\{\alpha, \kappa/2\}$ ,  $\alpha < \beta + \delta$ , and  $\alpha < \kappa$ , we deduce after an application of (6.11) that

$$\|\theta^{n+1}\|_{X_T} \le C_1 \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}} + C_2 \|\theta^n\|_{X_T} \|\theta^{n+1}\|_{X_T},$$
(6.25)

for some constants  $C_1, C_2 > 1$ . By Lemma 6.2 we have

$$\|\theta^0\|_{X_T} \le C_3 \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}} \le 2(C_1 \lor C_3) \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}},\tag{6.26}$$

for some constant  $C_3 > 1$ . Let  $C_4 := 2(C_1 \vee C_3)$  and assume that  $\|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}} \leq (2C_2C_4)^{-1}$ . If  $\|\theta^n\|_{X_T} \leq C_4 \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}}$  for some n > 0, then from (6.25), we get

$$\frac{1}{2} \|\theta^{n+1}\|_{X_T} \le C_1 \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}}.$$
(6.27)

Therefore, by induction  $\|\theta^n\|_{X_T} \leq C_4 \|\theta_0\|_{\dot{B}^{\sigma_c}_{p,q}}$  for all  $n \geq 0$ .

For arbitrary  $\theta_0 \in \dot{B}_{p,q}^{\sigma_c}$ , we can deduce uniform bounds for  $\{\theta^n\}_{n\geq 0}$  by induction similarly. To this end, we first observe that by Lemma 6.2, there exists  $T_1 > 0$  such that  $\|\theta^0\|_{X_{T_1}} \leq C$ , where  $C < (2C_2)^{-1}$ . We can also choose  $T_0 = T_0(\theta_0)$  such that  $\sup_{0 < t < T_0} M(t) \leq C(2C_1)^{-1}$ , where M(t) is defined as in (6.18). Now let  $T^* := T \wedge T_0$ . It follows that  $\|\theta^0\|_{X_{T^*}} \leq C$ .

For n > 0, observe that similar to (6.25), we also have the estimate

$$\|\theta^{n+1}\|_{X_{T^*}} \le C_1 \left( \sup_{0 < t < T^*} M(t) \right) + C_2 \|\theta^n\|_{X_{T^*}} \|\theta^{n+1}\|_{X_{T^*}}.$$
(6.28)

If  $\|\theta^k\|_{X_{T^*}} \leq C$ , for all  $0 < k \leq n$ , then applying this to (6.28) and using the fact that  $C < (2C_2)^{-1}$ , we have

$$\|\theta^{n+1}\|_{X_{T^*}} \le 2C_1 \left(\sup_{0 < t < T^*} M(t)\right).$$

Since  $\sup_{0 < t < T^*} M(t) \le C(2C_1)^{-1}$  we therefore have

$$\|\theta^{n+1}\|_{X_{T^*}} \le C,$$

which completes the induction.

Finally, define the spaces  $Y_T$  and  $Z_T$  by

$$Y_T := \{ v \in C([0,T); B_{p,q}^{\sigma_c} : \|v\|_{Y_T} := \sup_{0 \le t < T} \|G_{\gamma}v\|_{B_{p,q}^{\sigma_c}} < \infty \}$$

and

$$Z_T := \{ v \in X_T \cap Y_T : \|v\|_{Z_T} := \max\{\|v\|_{X_T}, \|v\|_{Y_T}\} < \infty \}.$$

A. Biswas et al. / On Gevrey regularity for supercritical SQG

To obtain estimates in the class  $Z_T$ , one must first prove an analog of Lemma 6.2(i) for the space  $Y_T$  to take care of the case n = 0. This follows easily from the proof of Lemma 6.2 by setting  $\beta = 0$ . Then for the case n > 0, one returns to (6.23) and applies Theorem 3.5 with  $s = 1 + 2/p - \kappa + \beta$  and  $t = 2/p - \kappa$ , which forces the additional constraint  $1/2 + 1/p + \beta/2 > \kappa$ . One can then obtain uniform bounds on  $\|\theta^n\|_{Y_T}$  by following steps similar to those made for estimating  $\|\theta^n\|_{X_T}$ , and taking advantage of the fact that  $\|\theta^n\|_{X_T}$  is already uniformly bounded for all  $n \ge 0$ . This finishes the proof of the theorem.

### Acknowledgment

The authors would like to thank Professors Ciprian Demeter, Michael Jolly, Edriss Titi, and Alexander Volberg for useful discussions in the course of this research. We especially thank the referees for their suggestions and careful review of the manuscript.

### Appendix A

We recall in the proofs of Theorems 3.4 and 3.5, we made crucial use of the concavity of the function  $\|\xi\|^{\alpha}$ , where  $\alpha < 1$ . In particular, we used the following fact, whose proof we supply now.

**Lemma A.1.** Let  $\alpha < 1$  and  $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$  be given by

$$f(\xi,\eta) := \|\xi\|^{\alpha} + \|\eta\|^{\alpha} - \|\xi + \eta\|^{\alpha}.$$
 (A.1)

If  $\|\xi\|/\|\eta\| \ge c$  for some c > 0, then there exists  $\epsilon > 0$ , depending only on c, such that  $f(\xi, \eta) \ge \epsilon \|\eta\|^{\alpha}$ .

**Proof.** Observe that

$$f(\xi,\eta) = \|\eta\|^{\alpha} \left( \left\| \frac{\xi}{\|\eta\|} \right\|^{\alpha} + 1 - \left\| \frac{\xi}{\|\eta\|} + \frac{\eta}{\|\eta\|} \right\|^{\alpha} \right).$$

Also observe that if R is a rotation matrix, then  $f(R\xi, R\eta) = f(\xi, \eta)$ . Thus, we may assume that  $\|\xi\| \ge c$  and that  $\eta = e_1$ , where  $e_1 := (1, 0)$ . Now observe that

$$f(\xi,\eta) = (\xi_1^2 + \xi_2^2)^{\alpha/2} + 1 - ((\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2)^{\alpha/2}$$
$$= (\xi_1^2 + \xi_2^2)^{\alpha/2} + 1 - ((\xi_1 + 1)^2 + \xi_2^2)^{\alpha/2}.$$

Let  $x := \|\xi\|$ . Then

$$f(\xi,\eta) = g_{\xi_1}(x) := x^{\alpha} + 1 - (x^2 + 1 + 2\xi_1)^{\alpha/2},$$

where  $x \ge c$ . Thus, we may assume  $\xi_2 = 0$  and  $|\xi_1| \ge c$ . In particular, we may assume that  $x = \xi_1$ . An elementary calculation finally shows that  $g(x) := |x|^{\alpha} + 1 - |x+1|^{\alpha} \ge g(c) > 0$  since  $x \ge c$ .  $\Box$ 

Now we provide the proof of our multiplier theorem, Theorem 3.7.

**Proof.** By Proposition A.2, we may assume that for each fixed  $\xi \in \mathbb{R}^d$ ,  $m(\xi, \eta)$  is supported in  $[1/2 \leq ||\eta|| \leq 2] \subset [0, 4]^d$  as a function of  $\eta$ . Thus, we may take the Fourier transform in the variables  $\eta_1, \ldots, \eta_d$ , i.e.,

$$m(\xi,\eta) \sim \sum_{k \in \mathbb{Z}^d} \hat{m}_k(\xi) e^{ik \cdot \eta} \chi(\eta), \tag{A.2}$$

where  $\hat{m}_k(\xi) := \hat{m}(\xi, k)$  is the k-th Fourier coefficient of m and  $\chi(\eta) = 1$  for  $1/2 \le ||\eta|| \le 2$  and is supported on  $[1/4 \le ||\eta|| \le 4]$ . In fact, we write  $m(\xi, \eta)$  as

$$m(\xi,\eta) \sim \hat{m}_0(\xi)\chi(\eta) + \left(\sum_{k \in Z_0} + \dots + \sum_{k \in Z_{d-1}}\right) \hat{m}_k(\xi)e^{ik\cdot\eta}\chi(\eta),$$
(A.3)

where  $Z_i \subset \mathbb{Z}^d$  is defined by

 $Z_j := \{k \in \mathbb{Z}^d : k_i = 0 \text{ for exactly } j \text{ many indices } i \text{ and } k_{i'} \neq 0 \text{ for } i' \neq i\}.$  (A.4)

Observe that  $Z_j$  is isomorphic to C(d, d-j) copies of  $(\mathbb{Z} \setminus \{0\})^{(d-j)}$ , where  $\mathbb{Z}^0 := \{0\}$ .

Using multi-index notation, observe that for each  $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ , integration by parts gives

$$\hat{m}_k(\xi) = \int e^{-ik \cdot \eta} m(\xi, \eta) \ d\eta = c_\alpha (-ik)^{-\alpha} \widetilde{m}_{k,\alpha}(\xi),$$

for all  $\alpha \in \mathbb{N}^d$ , where

$$\widetilde{m}_{k,\alpha}(\xi) := \int e^{-ik \cdot \eta} \partial_{\eta}^{\alpha}(m(\xi,\eta)\chi(\eta)) \ d\eta.$$

By (3.5), it follows that  $m_0(\xi)$  is a Hörmander–Mikhlin multiplier. On the other hand, (3.5) and the fact that  $\chi$  is supported in  $[1/4 \le ||\eta|| \le 4]$  implies

$$\left|\partial_{\xi}^{\beta}\widetilde{m}_{k,\alpha}(\xi)\right| \lesssim \sum_{\alpha_{1}+\alpha_{2}=\alpha} \int \left|\partial_{\xi}^{\beta}\partial_{\eta}^{\alpha_{1}}m(\xi,\eta)\partial_{\eta}^{\alpha_{2}}\chi(\eta)\right| d\eta$$
$$\lesssim_{\beta,\alpha,d} \|\xi\|^{-|\beta|} \int_{[1/4\lesssim\|\eta\|\lesssim 4]} \|\eta\|^{-|\alpha_{1}|} d\eta$$
$$\lesssim_{\beta,\alpha,d} \|\xi\|^{-|\beta|}. \tag{A.5}$$

#### A. Biswas et al. / On Gevrey regularity for supercritical SQG

Thus  $\widetilde{m}_{k,\alpha}$  is also a Hörmander–Mikhlin multiplier for all  $k \in \mathbb{Z}^d$  and  $\alpha \in \mathbb{N}^d$ . Moreover, note that the suppressed constant in (A.5) is independent of k.

Now for each j = 1, ..., d, choose a multi-index  $a_j \in Z_j \cap \mathbb{N}^d$  so that  $\sum_{k \in Z_j} k^{-a_j} < \infty$ . Finally, observe that

$$T_m(f,g) = T_{m_0}(f)T_{\chi}(g) + \sum_{j=1}^d \sum_{k \in Z_j} T_{m_k}(f)T_{\chi_k}(g)$$
$$= T_{m_0}(f)T_{\chi}(g) + \sum_{j=1}^d \sum_{k \in Z_j} c_{\alpha_j}k^{-a_j}(T_{\widetilde{m}_{k,a_j}}f)(T_{\chi}\tau_{-k}g)$$

where  $\chi_k(\eta) := \chi(\eta)e^{ik\cdot\eta}$  and  $T_{m_k}, T_{\chi_k}$  denote linear multiplier operators with symbols  $m_k, \chi_k$ , respectively, and  $\tau_v$  denotes the translation by v operator, i.e.  $(\tau_v h)(x) = h(x-v)$ . Therefore, by Minkowski's inequality, Hölder's inequality, and the Hörmander–Mikhlin multiplier theorem we have

$$||T_m(f,g)||_{L^r} \lesssim_a ||f||_{L^p} ||\chi||_{L^1} ||g||_{L^q},$$

where we have used Young's convolution inequality and translation invariance of dx, and the suppressed constant depends on  $\sup_j \left( \sum_{k \in \mathbb{Z}_j} k^{-a_j} \right)$ .  $\Box$ 

The next proposition shows that Marcinkiewicz multipliers are dilation invariant. Thus, we may (isotropically) rescale the support of m without penalty.

**Proposition A.2.** Let 1/r = 1/p + 1/q and  $T_m : L^p \times L^q \to L^r$  be a bounded bilinear multiplier operator whose multiplier, m, satisfies  $m \in L^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ 

$$\left|\partial_{\xi}^{\beta_{1}}\partial_{\eta}^{\beta_{2}}m(\xi,\eta)\right| \lesssim_{\beta,d} \|\xi\|^{-|\beta_{1}|} \|\eta\|^{-|\beta_{2}|},\tag{A.6}$$

for all  $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$  and multi-indices  $\beta_1, \beta_2 \in \mathbb{N}^d$ . Then  $T_{m_{\lambda}}$  is also bounded with the same operator norm, where  $m_{\lambda}$  is given by

$$m_{\lambda}(\xi,\eta) := m(\lambda\xi,\lambda\eta).$$

**Proof.** We first show that  $m_{\lambda}$  also satisfies (A.6). Observe that

$$\partial_{\xi}^{\beta_1}\partial_{\eta}^{\beta_2}m_{\lambda}(\xi,\eta) = \lambda^{|\beta_1| + |\beta_2|} (\partial_{\xi}^{\beta_1}\partial_{\eta}^{\beta_2}m)(\lambda\xi,\lambda\eta).$$

Then since m satisfies (A.6) we have

$$\left|\partial_{\xi}^{\beta_1}\partial_{\eta}^{\beta_2}m_{\lambda}(\xi,\eta)\right| \lesssim \lambda^{|\beta_1|+|\beta_2|} \|\lambda\xi\|^{-|\beta_1|} \|\lambda\eta\|^{-|\beta_2|}.$$

#### A. Biswas et al. / On Gevrey regularity for supercritical SQG

Now we prove the claim. Indeed, let  $f \in L^p, g \in L^q$ , and  $\lambda > 0$ . Then

$$\begin{split} T_{m_{\lambda}}(f,g)(x) &= \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} e^{ix \cdot (\xi+\eta)} m_{\lambda}(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) \ d\xi \ d\eta \\ &= \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} e^{ix \cdot (\xi+\eta)} m(\lambda\xi,\lambda\eta) \hat{f}(\xi) \hat{g}(\eta) \ d\xi \ d\eta \\ &= \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} e^{i(x/\lambda) \cdot (\xi'+\eta')} m(\xi',\eta') \lambda^{-d} \hat{f}(\xi'/\lambda) \lambda^{-d} \hat{g}(\eta'/\lambda) \ d\xi' \ d\eta' \\ &= \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} e^{i(x/\lambda) \cdot (\xi+\eta)} m(\xi,\eta) \widehat{f_{\lambda}}(\xi) \widehat{g_{\lambda}}(\eta) \ d\xi \ d\eta \\ &= T_{m}(f_{\lambda},g_{\lambda})(x/\lambda) = (T_{m}(f_{\lambda},g_{\lambda}))_{1/\lambda}(x). \end{split}$$

This implies

$$\begin{aligned} \|T_{m_{\lambda}}(f,g)\|_{L^{r}} &= \lambda^{d/r} \|T_{m}(f_{\lambda},g_{\lambda})\|_{L^{r}} \\ &\lesssim \lambda^{d/r} \|f_{\lambda}\|_{L^{p}} \|g_{\lambda}\|_{L^{q}} = \lambda^{d/r} \lambda^{-d/p} \lambda^{-d/q} \|f\|_{L^{p}} \|g\|_{L^{q}}. \end{aligned}$$

In particular,  $||T_{m_{\lambda}}|| \leq ||T_m||$ . On the other hand, one can similarly argue

$$\begin{aligned} \|T_m(f,g)\|_{L^r} &= \lambda^{-d/r} \|T_{m_{1/\lambda}}(f_{1/\lambda},g_{1/\lambda})\|_{L^r} \\ &\lesssim \lambda^{-d/r} \|f_{1/\lambda}\|_{L^p} \|g_{1/\lambda}\|_{L^q} = \lambda^{-d/r} \lambda^{d/p} \lambda^{d/q} \|f\|_{L^p} \|g\|_{L^q}. \end{aligned}$$

Therefore  $||T_m|| \leq ||T_{m_{1/\lambda}}||$ . This completes the proof.  $\Box$ 

#### References

- H. Abidi, T. Hmidi, On the global well-posedness of the critical quasi-geostrophic equation, SIAM J. Math. Anal. 40 (1) (2008) 167–185.
- H. Bae, Global well-posedness of dissipative quasi-geostrophic equations in critical spaces, Proc. Amer. Math. Soc. 136 (1) (2008) 257–261.
- [3] H. Bae, A. Biswas, E. Tadmor, Analyticity and decay estimates of the Navier–Stokes equations in critical Besov spaces, Arch. Ration. Mech. Anal. 3 (2012) 963–991.
- [4] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss., vol. 343, Springer-Verlag, 2011.
- [5] A. Biswas, Gevrey regularity for a class of dissipative equations with applications to decay, J. Differential Equations 253 (2012) 2739–2764.
- [6] A. Biswas, Gevrey regularity for the supercritical quasi-geostrophic equation, J. Differential Equations 257 (2014) 1753–1772.
- [7] A. Biswas, D. Swanson, Existence and generalized Gevrey regularity of solutions to the Kuramoto– Sivashinsky equation in R<sup>n</sup>, J. Differential Equations 240 (1) (2007) 145–163.
- [8] A. Biswas, D. Swanson, Gevrey regularity to the 3-D Navier–Stokes equations with weighted  $\ell^p$  initial data, Indiana Univ. Math. J. 56 (3) (2007) 1157–1188.
- [9] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 171 (3) (2010) 1903–1930.

- [10] D. Chae, J. Lee, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, Comm. Math. Phys. 233 (2003) 297–311.
- [11] Q. Chen, C. Miao, Z. Zhang, A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation, Comm. Math. Phys. 271 (2007) 821–838.
- [12] R. Coifman, Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilineéaires, Ann. Inst. Fourier 28 (3) (1978) 177–202.
- [13] P. Constantin, D. Cordoba, J. Wu, On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. 50 (2001) 97–107.
- [14] P. Constantin, G. Iyer, J. Wu, Global regularity for a modified critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. 58 (6) (2008) 2681–2692.
- [15] P. Constantin, A. Majda, E. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity 7 (1994) 1495–1533.
- [16] P. Constantin, A. Tarfulea, V. Vicol, Long time dynamics of forced critical SQG, Communications in Mathematical Physics 335 (1) (August 2014) 93–141.
- [17] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geom. Funct. Anal. 22 (2012) 1289–1321.
- [18] P. Constantin, J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal. 30 (5) (1999) 937–948.
- [19] P. Constantin, J. Wu, Regularity of Hölder continuous solutions of the supercritical quasigeostrophic equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (2008) 2681–2692.
- [20] P. Constantin, J. Wu, Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (1) (2009) 159–180.
- [21] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004) 511–528.
- [22] M. Dabkowski, Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation, Geom. Funct. Anal. 21 (2011) 1–13.
- [23] M. Dabkowski, A. Kiselev, L. Silvestre, V. Vicol, Global well-posedness of slightly supercritical active scalar equations, Anal. PDE 7 (1) (2014) 43–72.
- [24] H. Dong, Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness, Discrete Contin. Dyn. Syst. 26 (4) (2010) 1197–1211.
- [25] H. Dong, D. Li, On the 2D critical and supercritical dissipative quasi-geostrophic equation in Besov spaces, J. Differential Equations 248 (2010) 2684–2702.
- [26] A.B. Ferrari, E.S. Titi, Gevrey regularity for nonlinear analytic parabolic equations, Comm. Partial Differential Equations 23 (1&2) (1998) 1–16.
- [27] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, J. Funct. Anal. 87 (1989) 350–369.
- [28] H. Fujita, T. Kato, On the Navier–Stokes initial value problem. I, Arch. Ration. Mech. Anal. 16 (4) (1964) 269–315.
- [29] P. Gérard, Y. Guo, E.S. Titi, On the radius of analyticity of solutions to the cubic Szegö equation, Ann. Inst. Henri Poincaré 32 (1) (2015) 97–108.
- [30]Y. Giga, T. Miyakawa, Solutions in  $L^r$  of the Navier–Stokes initial value problem, Arch. Ration. Mech. Anal. 89 (3) (1985) 267–281.
- [31] L. Grafakos, N.J. Kalton, The Marcinkiewicz multiplier condition for bilinear operators, Studia Math. 146 (2) (2001) 115–156.
- [32] Z. Grujić, I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L<sup>p</sup>, J. Funct. Anal. 152 (1998) 447–466.
- [33] Z. Grujić, I. Kukavica, Space analyticity for the Navier–Stokes and related equations with initial data in L<sup>p</sup>, J. Funct. Anal. 152 (1998) 447–466.
- [34] Y. Guo, E.S. Titi, Persistency of analyticity for quasi-linear wave-equations: an energy-like approach, Bull. Inst. Math. Acad. Sin. (N.S.) (2013) 1–27.
- [35] T. Hmidi, S. Keraani, Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces, Adv. Math. 214 (2007) 618–638.
- [36] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys. 251 (2004) 365–376.
- [37] N. Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys. 255 (2005) 161–181.
- [38] N. Ju, Global solutions to the two dimensional quasi-geostrophic equation with critical or supercritical dissipation, Math. Ann. 334 (2006) 627–642.

Please cite this article in press as: A. Biswas et al., On Gevrey regularity of the supercritical SQG equation in critical Besov spaces, J. Funct. Anal. (2015), http://dx.doi.org/10.1016/j.jfa.2015.08.010

A. Biswas et al. / On Gevrey regularity for supercritical SQG

- [39] T. Kato, Strong  $L^p$  solutions of the Navier–Stokes initial value problem in  $\mathbb{R}^m$ , with applications to weak solutions, Math. Z. 187 (4) (1984) 471–480.
- [40] A. Kiselev, Some recent results on the critical surface quasi-geostrophic equation: a review, Proc. Sympos. Appl. Math. 67 (Part 1) (2009) 105–122.
- [41] A. Kiselev, F. Nazarov, Variation on a theme of Caffarelli and Vasseur, J. Math. Sci. 166 (1) (2010) 31–39.
- [42] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasigeostrophic equation, Invent. Math. 167 (2007) 445–453.
- [43] I. Kukavica, On the dissipative scale for the Navier–Stokes equation, Indiana Univ. Math. J. 47 (3) (1998) 1129–1154.
- [44] I. Kukavica, V. Vicol, On the radius of analyticity of solutions to the three-dimensional Euler equations, Proc. Amer. Math. Soc. 137 (2) (2009) 669–677.
- [45] I. Kukavica, V. Vicol, On the analyticity and Gevrey-class regularity up to the boundary for the Euler equations, Nonlinearity 24 (2011) 765–796.
- [46] A. Larios, E.S. Titi, On the Higher-order global regularity of the inviscid Voigt-regularization of three-dimensional hydrodynamic models, Discrete Contin. Dyn. Syst. Ser. B 14 (2) (2010) 603–627.
- [47] P.G. Lemarié-Rieusset, Recent developments in the Navier–Stokes problem, Chapman & Hall/CRC Res. Notes Math. 431 (2002).
- [48] C.D. Levermore, M. Oliver, Analyticity of solutions for a generalized Euler equation, J. Differential Equations 133 (1997) 329–339.
- [49] V.R. Martinez, On Gevrey regularity of equations of fluid and geophysical fluid dynamics with applications to 2D and 3D turbulence, ProQuest LLC, Ann Arbor, MI, Ph.D. thesis, Indiana University, 2014.
- [50] C. Miao, L. Xue, On the regularity of a class of generalized quasi-geostrophic equations, J. Differential Equations 251 (2011) 2789–2821.
- [51] C. Miao, L. Xue, Global well-posedness for a modified critical dissipative quasi-geostrophic equation, J. Differential Equations 252 (2012) 792–818.
- [52] H. Miura, Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space, Comm. Math. Phys. 267 (2006) 141–157.
- [53] M. Oliver, E.S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in  $\mathbb{R}^n$ , J. Funct. Anal. 172 (2000) 1–18.
- [54] M. Oliver, E.S. Titi, On the domain of analyticity for solutions of second order analytic nonlinear differential equations, J. Differential Equations 174 (2001) 55–74.
- [55] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
- [56] S.G. Resnick, Dynamical problems in non-linear advective partial differential equations, August 1995, pp. 1–86, Ph.D. thesis.
- [57] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Ser. Nonlinear Anal. Appl., vol. 3, Walter de Gruyter, 1996.
- [58] F.B. Weissler, The Navier–Stokes initial value problem in  $L^p$ , Arch. Ration. Mech. Anal. 74 (3) (1980) 219–230.
- [59] J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, SIAM J. Math. Anal. 36 (3) (2005) 1014–1030.
- [60] J. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier– Stokes equations in Besov spaces, Comm. Math. Phys. 263 (2005) 803–831.
- [61] J. Wu, The two-dimensional quasi-geostrophic equation with critical or supercritical dissipation, Nonlinearity 18 (2005) 139–154.