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Dissipation Length Scale Estimates for Turbulent Flows: A Wiener Algebra Approach

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Abstract In this paper, a lower bound estimate on the uniform radius of spatial analyticity is established for solutions to the incompressible, forced Navier–Stokes system on an n-torus. This estimate matches previously known estimates provided that a certain bound on the initial data is satisfied. In particular, it is argued that for two-dimensional (2D) turbulent flows, the initial data is guaranteed to satisfy this hypothesized bound

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on a significant portion of the 2D global attractor, in which case, the estimate on the radius matches the best known one found in Kukavica (1998). A key feature in the approach taken here is the choice of the Wiener algebra as the phase space, i.e., the Banach algebra of functions with absolutely convergent Fourier series, whose structure is suitable for the use of the so-called Gevrey norms. We note that the method can also be applied with other phase spaces such as that of the functions with square-summable Fourier series, in which case the estimate on the radius matches that of Doering and Titi (1995). It can then similarly be shown that for three-dimensional (3D) turbulent flows, this estimate holds on a significant portion of the 3D weak attractor.

Keywords Navier–Stokes equations · Turbulence · Radius of analyticity

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1 Introduction

The conventional theory of turbulence posits the existence of certain universal length scales of paramount importance. For instance, according to Kolmogorov, there exists a *dissipation length scale*, λ_d, beyond which the viscous effects dominate the nonlinear coupling. This length scale can be characterized by the exponential decay of the energy density. Consequently, one expects the dissipation wave number, $\kappa_d = \lambda_d^{-1}$, to majorize the inertial range where energy consumption is largely governed by the nonlinear effects and dissipation can be ignored.

In Foias (1995) and Doering and Titi (1995) it is shown that, as characterized by Gevrey norms, the (uniform) radius of spatial analyticity, here denoted λ_a, provides a lower bound for the dissipation length scale, i.e., $\lambda_a \leq \lambda_d$. The space analyticity radius has been well studied over the years, especially following the pioneering work of Foias and Temam (1989), where a novel Gevrey norm is used to establish the analyticity of solutions to the Navier–Stokes equations (NSEs) in both space and time. An advantage of this approach is that it avoids having to make cumbersome recursive estimates on derivatives. Consequently, it has become a standard tool in estimating the analyticity radius for various equations (cf. Ferrari and Titi 1998; Oliver and Titi 2001, 2000; Levermore and Oliver 1997; Biswas and Swanson 2007; Biswas 2012; Kukavica and Vicol 2009; Larios and Titi 2010).

Kolmogorov's theory for three-dimensional (3D) turbulence asserts that

$$\lambda_d \sim \lambda_\varepsilon := (\nu^3/\varepsilon)^{1/4} , \tag{1.1}$$

where ν is viscosity and ε is the mean *energy dissipation rate* per unit mass. For 3D decaying turbulence in a periodic box of length L, it is shown in Doering and Titi (1995) that

$$\lambda_a \sim \kappa_0^{-1}(\kappa_0\tilde{\lambda}_\varepsilon)^4 , \tag{1.2}$$

where $\kappa_0 := 2\pi/L$ and $\tilde{\lambda}_\varepsilon$ is as in (1.1), except that the energy dissipation rate is a supremum in time rather than an averaged quantity. We can show that under the 2/3-power law assumption [see (3.16)] on the energy spectrum for a forced, turbulent flow, this estimate is valid for the *true* Kolmogorov length scale defined with the mean energy dissipation rate [see (3.11), (3.14)] by means of an ensemble average with respect to an invariant measure. Ultimately, we can conclude that

$$\lambda_a \gtrsim_p \kappa_0^{-1}(\kappa_0\lambda_\varepsilon)^4 \tag{1.3}$$

holds with probability $1 - p$ with respect to this invariant measure, where the suppressed constant in the inequality tends to 0 as $p \to 0$. Similarly, a heuristic scaling argument by Kraichnan for two-dimensional (2D) turbulence leads to

$$\lambda_d \sim \lambda_\eta := (\nu^3/\eta)^{1/6}\ , \tag{1.4}$$

where η is the mean *enstrophy dissipation rate* per unit mass. We show that if the 2D power law (3.21) for the energy spectrum holds, then

$$\lambda_a \gtrsim_p \kappa_0^{-1}(\kappa_0\lambda_\eta)^2 \tag{1.5}$$

holds with probability $1 - p$ with respect to some invariant measure.

These improved estimates actually follow from more general bounds on the radius of analyticity, which require the solution to satisfy certain "smallness" conditions. Those conditions are met under the power law assumptions when averaged with respect to an invariant measure. The same bound was achieved in Kukavica (1998) in 2D up to a logarithmic correction on the whole global attractor using complex analytic techniques, interpolating between L^p norms of the initial data and the complexified solution, and invoking the theory of singular integrals. The approach in Kukavica (1998) was actually a modification of the approach in Grujić and Kukavica (1998), where it was shown that $\lambda_d \gtrsim \nu(\sup_{t\le T^*/2}\|u(t)\|_{L^\infty})^{-1}$. It is interesting to ask whether these estimates could be obtained by working exclusively in frequency space using Fourier techniques rather than in physical space with the L^∞ norm. Indeed, this is an impetus of our work.

The technique applied here combines the use of Gevrey norms with the semigroup approach of Weissler (1980). Motivated by recent developments, we work over a subspace of the Wiener algebra, whose norm is a Sobolev–Gevrey-type norm in ℓ^1 [see (2.11)]. This norm and approach were applied in Biswas and Swanson (2007) to study the spatial analyticity and Gevrey regularity of solutions to the NSEs. However, the resulting estimate on the spatial radius of analyticity was not optimal for large data. This approach is refined here to obtain a sharper estimate for such data. The advantage of working in the Wiener algebra, $\mathcal{W}$, i.e., the Banach algebra of functions whose Fourier series converge absolutely, was explored in Oliver and Titi (2001), where a sharp estimate on the radius of analyticity was obtained, for instance, for real steady states of the nonlinear Schrödinger equations. More recently, these ℓ^1-based Gevrey norms were also applied to the Szegö equation in Gérard et al. (2013) and the quasilinear wave equation in Guo and Titi (2013). In Gérard et al. (2013), an essentially sharp estimate on the radius is obtained as well. While these works used energylike approaches, the effectiveness and robustness of $\mathcal{W}$ as a working space to study analyticity have become increasingly clear.

Our approach has several advantages. First, our method is quite elementary. Since $\mathcal{W}$ is embedded in L^∞, we essentially recover the results of Grujić and Kukavica (1998) and Kukavica (1998) without resorting to complex-analytic techniques and the theory of singular integrals; at the same time, we allow for rougher initial data. Secondly, this approach also applies to the case $1 < p < \infty$, thereby unifying the results of Doering and Titi (1995), Foias and Temam (1989), Grujić and Kukavica (1998), and Kukavica (1998). Thirdly, no logarithmic corrections appear in our estimates initially; they only appear when specializing to the context of 3D or 2D turbulence [see (3.25)]. Finally, the method is rather robust and applies to a wide class of active and passive scalar equations with dissipation, including the quasigeostrophic (QG) equations. Note that in the case of QG equations with supercritical dissipation, the method will only accommodate subanalytic Gevrey regularity (see Martinez 2014).

2 Preliminaries

The Navier–Stokes system in $\Omega := [0, L]^n$ for $n > 1$ is given by

$$\begin{cases} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = F, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where $u_0 : \Omega \to \mathbb{R}^n$ and $F : \Omega\times[0,T) \to \mathbb{R}^n$ are given and $p : \Omega\times[0,T) \to \mathbb{R}$ and $u : \Omega\times[0,T) \to \mathbb{R}^n$ are unknown. We assume that u_0, u, p, F are all L-periodic and mean-zero, and that u_0 is divergence-free.

We will use the so-called wave-vector form of (2.1), which is simply (2.1) written in terms of its Fourier coefficients

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(k,t) = -\nu\kappa_0^2|k|^2\hat{u}(k,t) + B[\vec{u},\vec{u}](k,t) + \hat{f}(k,t), \\ k\cdot\hat{u}(k,t) = 0, \\ \hat{u}(k,0) = \hat{u}_0(k), \end{cases} \tag{2.2}$$

where $k \in \mathbb{Z}^n$, $\vec{u} : \mathbb{Z}^n \times [0,T) \to \mathbb{C}^n$ such that $\vec{u}(t) = (\hat{u}(k,t))_{k\in\mathbb{Z}^n}$ and $f = \mathcal{P}F$, where $\mathcal{P}$ is the Helmholtz–Leray orthogonal projection, i.e., projection onto divergence-free vector fields,

$$\mathcal{P}(\hat{u}(k)e^{i\kappa_0 k\cdot x}) = \left(\hat{u}(k) - \left(\frac{k}{|k|}\cdot\hat{u}(k)\right)\frac{k}{|k|}\right)e^{i\kappa_0 k\cdot x}, \quad (k\in\mathbb{Z}^n). \tag{2.3}$$

Recall also that the mean-zero condition forces $\hat{u}(0,t) = 0$ for all t. The bilinear term B has Fourier coefficients given by

$$B[\vec{u}, \vec{v}](k,t)e^{i\kappa_0 k\cdot x} := i\kappa_0 \mathcal{P}\left(\sum_{\ell\in\mathbb{Z}^n\setminus\{\vec{0}\}} (k\cdot\hat{u}(\ell,t))\hat{v}(k-\ell,t)e^{i\kappa_0 k\cdot x}\right). \tag{2.4}$$

Note that $\vec{B}[\vec{u}, \vec{v}]$ will denote the sequence $(B[\vec{u}, \vec{v}](k))_{k\in\mathbb{Z}^n}$.

Observe that

$$|\widehat{\mathcal{P}u}(k)| \lesssim |\hat{u}(k)| \tag{2.5}$$

and that the following basic convolution estimate holds:

$$|B[\vec{u}, \vec{v}](k)| \lesssim \kappa_0|k|(|\vec{u}| * |\vec{v}|)(k) \text{ for all } k \in \mathbb{Z}^n. \tag{2.6}$$

Since we will be working with (2.2), we choose an appropriate sequence space as our ambient space. Define

$$\mathcal{K} := \{(\hat{u}(k))_{k\in\mathbb{Z}^n} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \hat{u}(0)=0, \hat{u}(k)=\hat{u}(-k)^*, k\cdot\hat{u}(k)=0\}, \tag{2.7}$$

where $\hat{u}(k)^* = (\overline{\hat{u}_1(k)}, \ldots, \overline{\hat{u}_n(k)})$. For $\sigma \in \mathbb{R}$ define

$$V_\sigma := \{(\hat{u}(k))_{k\in\mathbb{Z}^n} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \|\vec{u}\|_\sigma < \infty\} \cap \mathcal{K}, \tag{2.8}$$

where

$$\|\vec{u}\|_\sigma := \kappa_0^\sigma \sum_{k\in\mathbb{Z}^n} |k|^\sigma |\hat{u}(k)| \tag{2.9}$$

and $\vec{u}$ denotes an element of $(\mathbb{C}^n)^{\mathbb{Z}^n}$. Observe that when $\sigma = 0$, the norm on V_σ agrees with that on the Wiener algebra, i.e.,

$$(\nu\kappa_0)^{-1}\|\vec{u}\|_0 = \|u\|_{\mathcal{W}}, \tag{2.10}$$

where u is a continuous function whose Fourier coefficients are given by $\hat{u}(k)$. In fact, we have $V_\sigma \subset \mathcal{W}\cap\mathcal{K} \subset V_{-\sigma}$ for all $\sigma \geq 0$.

For $\vec{u} \in V_\sigma$ we define the *(analytic) Gevrey norm* of $\vec{u}$ by

$$\|\vec{u}\|_{\lambda,\sigma} := \kappa_0^\sigma \sum_{k\in\mathbb{Z}^n} e^{\lambda\kappa_0|k|}|k|^\sigma|\hat{u}(k)| \tag{2.11}$$

for $\lambda \geq 0$. Observe that λ has the physical dimension of *length*.

For a time-dependent sequence $\vec{u}(\,\cdot)$ such that $\vec{u}(t) \in V_\sigma$, for all $t \geq 0$, we define the *(analytic) Gevrey norm* of $\vec{u}(t)$ by

$$\|\vec{u}(t)\|_{\lambda(t),\sigma} := \kappa_0^\sigma \sum_{k\in\mathbb{Z}^n} e^{\lambda(t)\kappa_0|k|}|k|^\sigma|\hat{u}(k,t)|, \tag{2.12}$$

where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and sublinear, i.e., $\lambda(s+t) \leq \lambda(s) + \lambda(t)$ for all $s, t \geq 0$. Observe that

$$\|u(t)\|_{\mathcal{W}} \lesssim_\sigma \lambda(t)^{\varphi(\sigma)} \frac{\kappa_0^{-\sigma}}{\nu\kappa_0} \|\vec{u}(t)\|_{\lambda(t),\sigma} \tag{2.13}$$

for all $\sigma \in \mathbb{R}$ and $t > 0$, where $\varphi(\sigma) = \sigma$ if $\sigma < 0$ and 0 otherwise.

It is well known that the Gevrey norm characterizes analyticity, a fact stated more precisely in the following proposition (cf. Levermore and Oliver 1997; Katznelson 2004).

Proposition 1 *Let* $\sigma \in \mathbb{R}$.

(1) *If* $\|\vec{u}\|_{\lambda,\sigma} < \infty$, *then* u *admits an analytic extension on* $\{x + iy : |y| < \lambda\}$;
(2) *If* u *has an analytic extension on* $\{x + iy : |y| < \lambda\}$, *then* $\|\vec{u}\|_{\lambda',\sigma} < \infty$ *for all* $\lambda' < \lambda$.

In particular, if a function has a finite Gevrey norm, then the Fourier modes decay exponentially. Indeed, if $\|\vec{u}\|_{\lambda,\sigma} < \infty$, then

$$|\hat{u}(k)| \leq e^{-\lambda|k|}|k|^{-\sigma}\|\vec{u}\|_{\lambda,\sigma}. \tag{2.14}$$

Definition 1 If u is analytic, then we define

$$\lambda_{\max} = \sup\{\lambda' > 0 : \|\vec{u}\|_{\lambda',\sigma} < \infty\} \tag{2.15}$$

to be the the maximal (uniform) radius of spatial analyticity of u. Moreover, due to (2.14) we have $\lambda_d \geq \lambda_{\max}$.

Remark 2 For convenience, we adopt the following conventions for the rest of the paper.

(1) We will usually write $\vec{u}$ simply as u, which is a function whose Fourier series has modes $\hat{u}(k)$, for $k \in \mathbb{Z}^n$.
(2) By $u(t)$, $u(k)$, or (when the context is clear) simply u we shall mean the time-dependent sequence $\vec{u}(t) = (\hat{u}(k,t))_{k\in\kappa_0\mathbb{Z}^n}$, unless otherwise specified.
(3) We will use $\lesssim$ to suppress extraneous absolute constants or physical parameters. In some instances, the dependence of these constants will be indicated as subscripts on $\lesssim$.
(4) We will also use the notation $\sim$ to denote that the two-sided relation $\lesssim$ and $\gtrsim$ holds.

For $1 \leq q \leq \infty$ and $0 < T_f \leq \infty$ we define

$$M_0 := \frac{\kappa_0^{-\sigma}}{\nu\kappa_0}\|u_0\|_\sigma, \tag{2.16}$$

$$M_f := \begin{cases} \frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3}\left(\nu\kappa_0^2 \int_0^{T_f} \|f(s)\|_{\lambda(s),\sigma}^q \, \mathrm{d}s\right)^{1/q}, & 1 \le q < \infty \\ \frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3} \sup_{0\le t\le T_f} \|f(t)\|_{\lambda(t),\sigma} & q = \infty \end{cases} \tag{2.17}$$

and

$$M := M_0 + M_f. \tag{2.18}$$

For any dimension $n > 0$ the Grashof number is defined as

$$G := \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3} \sup_{0\le t\le T_f} \|f(t)\|_{L^2}. \tag{2.19}$$

Observe that M and G are dimensionless. One can show that when f is time-independent and has only finitely many modes, i.e., $f = P_{\bar{\kappa}} f$, where

$$P_{\bar{\kappa}} f := \sum_{|k|\le\bar{\kappa}/\kappa_0} \hat{f}(k) e^{i\kappa_0 k\cdot x}, \tag{2.20}$$

then M_f is comparable to G up to a constant depending only on κ_0, $\bar{\kappa}$, a fixed parameter τ, and λ_f, where λ_f satisfies

$$\sup_{|y|\le\lambda_f} \|f(\cdot + iy)\|_{L^2} < \infty; \tag{2.21}$$

see Proposition 23 in the appendix.

Now suppose that the data u_0 and f are given such that $M < \infty$. Let A be the Stokes operator, $A := -\mathcal{P}\Delta$, where $\mathcal{P}$ is defined as in (2.3). Then the heat kernel, $e^{\nu t A}$, is the Fourier multiplier defined by

$$\widehat{e^{\nu t A} u}(k) := e^{-\nu t \kappa_0^2 |k|^2} \hat{u}(k) \tag{2.22}$$

or, equivalently, $e^{\nu t A}\vec{u} = (e^{-\nu t\kappa_0^2|k|^2}\hat{u}(k))_{k\in\mathbb{Z}^n}$. We will use two notions of solutions to (2.2).

Definition 2 For $0 < T \le \infty$ a *mild solution* to (2.2) is any function $\vec{u} \in C([0, T]; \mathcal{K})$ such that

$$\int_0^t e^{-\nu(t-s)\kappa_0^2|k|^2} |B[\vec{u}, \vec{u}](k, s)| \, \mathrm{d}s < \infty \tag{2.23}$$

for all $k \in \mathbb{Z}^n$ and

$$\vec{u}(t) = e^{-\nu t A}\vec{u}_0 + \int_0^t e^{-\nu(t-s)A}\vec{f}(s)\,\mathrm{d}s - \int_0^t e^{-\nu(t-s)A}\vec{B}[\vec{u},\vec{u}](s)\,\mathrm{d}s \tag{2.24}$$

for all $0 \le t \le T$.

Definition 3 For $0 < T \le \infty$ a *weak solution* to (2.2) is any function $\vec{u} \in C([0,T];\mathcal{K})$ such that

$$B[\vec{u},\vec{u}](k,t) < \infty \tag{2.25}$$

for all $k \in \mathbb{Z}^n$ and a.e. $t \in [0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(k,t) = -\nu\kappa_0^2|k|^2\hat{u}(k,t) - B[\hat{u},\hat{u}](k,t) + \hat{f}(k,t) \tag{2.26}$$

holds for all $k \in \mathbb{Z}^n$ and a.e. $t \in [0,T]$.

The fact that Definition 3 is equivalent to the common definition of a weak solution for a periodic flow can be found in Temam (1977).

Finally, we define the regularity that we ultimately seek to establish.

Definition 4 A mild or weak solution $\vec{u} \in C([0,T];\mathcal{K})$ of (2.2) is *Gevrey regular* if there exists $\sigma \in \mathbb{R}$ and sublinear $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sup_{0\le t\le T} \|\vec{u}(t)\|_{\lambda(t),\sigma} < \infty. \tag{2.27}$$

3 Main Theorems

We first state a result for a general force.

Theorem 3 *Let $1 < q \le \infty$ and $-1 < \sigma \le 0$ satisfy $q'|\sigma| < 2$, where $1/q' := 1-1/q$, and let M, T_f be defined as in (2.18). Suppose that u_0 and f are given such that $M < \infty$. Then for some $0 < T^* \le T_f$ there exists a mild solution $u \in C([0,T^*];V_\sigma)$ to (2.2), which is also a Gevrey regular weak solution, with a radius of analyticity at time T^* satisfying*

$$\lambda_a \gtrsim \kappa_0^{-1} M^{-1/(1-|\sigma|)}, \tag{3.1}$$

Moreover, there exists a constant C^* such that if $M \le C^*$, then one may take $T^* = T_f$. In this case, the solution exists for all $0 \le t \le T_f$ and the radius of analyticity at time t satisfies

$$\lambda_a \gtrsim \sqrt{\nu t}. \tag{3.2}$$

In the case where the forcing is time-independent and has finitely many modes, we can express the estimate on the radius of analyticity in terms of the Grashof number, provided a "smallness" condition on the solution holds.

Theorem 4 *Suppose that f is time-independent and satisfies $f = P_{\bar{\kappa}} f$. If*

$$\|u_0\|_{\mathcal{W}} \lesssim G^{1/2}, \tag{3.3}$$

then for some $0 < T^* < (\nu\kappa_0^2)^{-1}$ there exists a unique weak solution $u \in C([0, T^*], V_0)$ to (2.1) such that u is Gevrey regular and the radius of analyticity at time T^* satisfies

$$\lambda_a \gtrsim_{\bar{\kappa},\kappa_0} \kappa_0^{-1} \min\{1, G^{-1/2}\}. \tag{3.4}$$

The following theorem is applicable only in dimension three and is obtained by using ℓ^2 as the phase space in place of the Wiener algebra. Because of this, we omit its proof (cf. Martinez 2014), but it will be used in the following section to demonstrate an application to three-dimensional turbulence.

Theorem 5 *Suppose $\Omega = [0, L]^3$ and that f is time-independent and satisfies $f = P_{\bar{\kappa}} f$. If*

$$\frac{\kappa_0^{3/2}}{\nu\kappa_0^2}\|A^{1/2}u_0\|_{L^2(\Omega)} \lesssim G^{3/4}, \tag{3.5}$$

where $A = -\Delta$ with periodic boundary conditions, is the Stokes operator, then for some $0 < T^* < (\nu\kappa_0^2)^{-1}$, there exists a unique weak solution $u \in C([0, T^*], (H^1_{per}(\Omega))^2)$ to (2.1) such that u is Gevrey regular and the radius of analyticity at time T^* satisfies

$$\lambda_a \gtrsim_{\bar{\kappa},\kappa_0} \kappa_0^{-1} \min\{1, G^{-3/2}\}. \tag{3.6}$$

Remark 6 One can also have $\sigma > 0$ in Theorem 3. In fact, a more general version of Theorem 3 is proved in Sect. 7 (see Theorem 20).

The estimate on λ_a in Theorem 3 can be compared to that in Biswas and Swanson (2007) when $q = 2$. However, in that work the authors' choice of $\lambda(t)$ (as in Definition 4) yielded instead the estimate

$$\lambda_a \gtrsim \kappa_0^{-1} M^{-2/(1-|\sigma|)}, \tag{3.7}$$

which is less sharp than the corresponding estimate in (3.1) when M is large.

One should also note that if the constant, C^*, in Theorem 3 is too small, then the global attractor in two dimensions becomes trivial (cf. Dascaliuc et al. 2008; Marchioro 1987). Physically, this corresponds to the case of decaying turbulence. Nevertheless, if M is sufficiently small, then $T_f = \infty$ is allowed, in which case the solution exists globally in time with a radius that grows without bound in time as $\sqrt{\nu t}$.

The uniqueness of weak solutions to (2.1) is guaranteed in two dimensions but in three dimensions is still an open question. There are, however, cases where uniqueness is guaranteed in any dimension (see Temam 1977, pp. 298–299). In particular, as long as $\sigma \geq 0$, the solution of Theorem 3 is unique in the class of weak solutions.

In the case where the force is identically zero, one can employ energy techniques as in Doering and Titi (1995) and Foias and Temam (1989) and obtain

$$\lambda_a \geq C \frac{\kappa_0^{-1}}{\|u_0\|_{\mathcal{W}}}, \tag{3.8}$$

where λ_a represents the radius of analyticity at some time T^* strictly less than the maximal time of existence. The constant here can be explicitly identified as $C = \log(1+\gamma)/\sqrt{\gamma}$, where γ is the nontrivial solution to

$$(2\gamma)^{-1}\log(1+\gamma) - (1+\gamma)^{-1} = 0.$$

Note that (3.8) is precisley the estimate in (3.1) (up to an absolute constant). The energy approach, however, encounters technical difficulties when one includes forcing on infinitely many scales. The reader is referred to Martinez (2014) for additional details.

In Martinez (2014), estimates are also made in ℓ^p for $1 < p < \infty$. In particular, when $n = 3, p = 2, \sigma = 1$, the result of Doering and Titi (1995) is generalized to include forcing on all scales, and the estimate on the radius is the same as that derived there (up to an absolute constant). In Sect. 3.1.1, we use this result to present an application to 3D turbulence (see Theorem 7), where one works with the 3D weak attractor. For background on the weak attractor, see Dascaliuc et al. (2009) or Foias et al. (2001).

Finally, the techniques used to prove Theorem 4 apply equally well to the vorticity formulation of Navier–Stokes, the case of fractional dissipation, and a wide class of active and passive scalar equations, including 2D dissipative QG equations Martinez 2014. These techniques also apply to the case $\Omega = \mathbb{R}^n$ Biswas 2012. For more results on the subcritical QG equation, see, for instance, Constantin et al. 2001, where analyticity is established for arbitrary initial data in H^2, or Dong and Li (2008), where a local smoothing effect is exploited to establish analyticity, or Biswas (2012), where analytic Gevrey regularity is established for several other equations as well. For results on the analyticity of solutions for critical QG equations, see Dong (2010) and Kiselev (2009). For results on the regularity of passive scalar equations see Silvestre (2012) or Silvestre et al. (2012). The classical Hilbert space techniques of Foias and Temam (1989) have also been successfully applied to the Euler equations Kukavica and Vicol 2009; Levermore and Oliver 1997.

3.1 Application to Turbulent Flows

In this subsection, we show how our results in Theorems 4 and 5 improve the known estimates for λ_d for turbulent flows. While their "smallness" assumptions may not hold on all of the 2D global (3D weak) attractor, in the context of turbulence, one can expect these conditions to hold *on average*, in a precise sense.

The statistical theory of turbulence concerns relations between quantities that are averaged, either with respect to time or over an ensemble of flows, e.g., results from repeated experiments. It is remarkable that these two seemingly different approaches are in fact related.

The mathematical equivalent of a large time average is rigorously expressed in terms of Banach limits. Following Foias et al. (2001), define the space H by

$$H := \{(\hat{u}(k))_{k\in\mathbb{Z}^n} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \|\vec{u}\|_{\ell^2} < \infty\} \cap \mathcal{K}. \tag{3.9}$$

Let Φ be a real-valued weakly continuous function on H. Then for any weak solution u of (2.2) on $[0, \infty)$ there exists a probability measure μ for which

$$\langle \Phi \rangle := \int_H \Phi(u)\, \mathrm{d}\mu(u) = \mathrm{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \Phi(u(t))\, \mathrm{d}t, \tag{3.10}$$

where Lim is a Hahn–Banach extension of the classical limit. The measure μ is called a *time-average measure* of u. Note that neither Lim nor μ is unique. The use of Lim overcomes the technical difficulty that the limit in the usual sense may not exist. If u is a weak solution to the 2D NSEs, then by the regularity of such solutions, one can work in the strong topology on H. Moreover, by uniqueness, one can show that μ is in fact *invariant* with respect to the corresponding semigroup, i.e., $\mu(E) = \mu(S(t)^{-1}E)$ for all $t \geq 0$, for all measurable sets $E \subset H$. Thus, a time-average measure is also a so-called *stationary statistical solution* of the NSEs. For a more detailed background see Foias et al. (2001).

We now specialize to the cases of 3D and 2D turbulence and interpret the main theorems in those settings.

3.1.1 3D Turbulence

The mean energy dissipation rate per unit mass is defined as

$$\varepsilon := \nu\kappa_0^3 \langle \|\nabla u\|_{L^2}^2 \rangle\ . \tag{3.11}$$

In three dimensions, Kolomogorov argued that because one can ignore nonlinear effects in the dissipation range, the length scale indicating where dissipation is the dominant effect should depend solely on ε and ν. By a simple dimensional argument, one then arrives at

$$\lambda_\varepsilon = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}. \tag{3.12}$$

In other words, according to Kolmogorov, for turbulent flows in three dimensions, $\lambda_d \sim \lambda_\varepsilon$, with λ_ε given in (3.12). We will now describe the best known rigorous result in this direction.

In Doering and Titi (1995), the radius of analyticity was estimated in terms of $\varepsilon_{\sup}$ as

$$\lambda_a \gtrsim \frac{(\nu\kappa_0)^3}{\varepsilon_{\sup}}, \tag{3.13}$$

where

$$\varepsilon_{\sup} := \nu\kappa_0^3 \sup_{0\le t\le T^*/2} \|\nabla u(t)\|_{L^2}^2 \tag{3.14}$$

represents the largest instantaneous energy dissipation rate (per unit mass) up to time $T^*/2$, and T^* is the maximal time of existence of a regular solution. A heuristic argument is given to support $\varepsilon_{\sup} \sim \varepsilon$ as in Doering and Titi (1995); then (3.13) becomes

$$\lambda_d \gtrsim \kappa_0^{-1}(\kappa_0\tilde{\lambda}_\varepsilon)^4\ , \quad \text{where} \quad \tilde{\lambda}_\varepsilon = \left(\frac{\nu^3}{\varepsilon_{\sup}}\right)^{1/4}. \tag{3.15}$$

It is not presently known whether $\varepsilon_{\sup}$ remains finite beyond T^*. Hence, it is not possible to obtain an estimate of the smallest length scale for an arbitrary weak solution. In fact, it is not possible to extend these estimates on the weak attractor either since it is not known whether or not a trajectory, i.e., a weak solution defined for all $t \in \mathbb{R}$, is regular. However, it is well accepted that statements regarding length scales in turbulence actually concern "averages" and not specific trajectories (cf. Foias and Prodi 1976; Foias et al. 2002, 2005; Balci et al. 2010 or Foias et al. 2001; Frisch 1995 for introductory approaches). Indeed, this is the thrust of our current discussion.

In addition to the dissipation range and wave number, another basic tenet in the Kolmogorov theory of turbulence is the so-called power law for the energy spectrum. More specifically, let $\bar{\kappa}$ denote the wave number in which energy is injected into the flow, i.e., $f = P_{\bar{\kappa}} f$. Denote the Kolmogorov wave number $\kappa_\varepsilon := 1/\lambda_\varepsilon$. Then the range of wave numbers $[\bar{\kappa}, \kappa_\varepsilon]$ is known as the inertial range in which the effect of viscosity is negligible. The nonlinear (inertial) term simply transfers the energy injected into the flow through the inertial range at a rate of ε. Moreover, defining the quantity

$$\mathrm{e}_{\kappa_1,\kappa_2} := \kappa_0^3\langle\|(P_{\kappa_2} - P_{\kappa_1})u\|_{L^2}^2\rangle,$$

the well-celebrated Kolmogorov power law asserts that a turbulent flow must satisfy the relation

$$\mathrm{e}_{\kappa,2\kappa} \sim \varepsilon^{2/3}/\kappa^{2/3}, \ \text{for } \kappa \in [\bar{\kappa}, \kappa_\varepsilon]. \tag{3.16}$$

Additionally, it is also known that if the Grashof number is sufficiently small, then the flow will not be turbulent and the attractor in this case will consist of only one point. In view of this discussion, we *define* a flow to be turbulent if the Kolmogorov power law holds and the Grashof number is sufficiently large, i.e.,

$$G \gtrsim \left(\frac{\bar{\kappa}}{\kappa_0}\right)^{3/2}. \tag{3.17}$$

It is shown in Dascaliuc et al. (2009) that for such a flow one necessarily has the bounds

$$\frac{\nu^2}{\kappa_0}\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{5/2} G \lesssim \langle \|u\|_{L^2}^2\rangle \lesssim \frac{\nu^2}{\kappa_0}\left(\frac{\kappa_0}{\bar{\kappa}}\right) G, \tag{3.18}$$

$$\nu^2\kappa_0\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{11/4} G^{3/2} \lesssim \langle \|A^{1/2}u\|_{L^2}^2\rangle \lesssim \nu^2\kappa_0\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{1/2} G^{3/2}. \tag{3.19}$$

The following is the main result of this section. While the power of 4 is the same as that in Doering and Titi (1995), here we use the *mean energy dissipation rate.*

Theorem 7 *Let μ be a time-average measure for a 3D turbulent flow, and let $0 < p < 1$. There exists a set $S \subset \mathcal{A}_w$, with $\mu(S) \geq 1 - p$, such that*

$$\lambda_d(u) \gtrsim_p \kappa_0^{-1}(\kappa_0\lambda_\varepsilon)^4 \text{ for all } u \in S.$$

Proof By definition of ε, λ_ε and the relation (3.19), we have

$$\kappa_0\lambda_\varepsilon \sim \kappa_0\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim \kappa_0\left(\frac{\nu^3}{\nu\kappa_0^3\langle\|A^{1/2}u\|_{L^2}^2\rangle}\right)^{1/4} \sim \kappa_0\left(\frac{1}{\kappa_0^4 G^{3/2}}\right)^{1/4} \sim G^{-3/8}.$$

In other words, $(\kappa_0\lambda_\varepsilon)^{8/3} \sim G^{-1}$. On the other hand, by Chebyshev's inequality and (3.19) we have

$$\mu\left\{u \in \mathcal{A}_w : \nu^{-2}\kappa_0^{-1}\|A^{1/2}u\|_{L^2}^2 \gtrsim p^{-1}G^{3/2}\right\} \leq p$$

for any $0 < p < 1$. Since the support of μ is contained in $\mathcal{A}_w$ (cf. Foias et al. 2001), Theorem 5 therefore implies that

$$\lambda_d(u) \gtrsim_p \kappa_0^{-1}G^{-3/2} \sim \kappa_0^{-1}(\kappa_0\lambda_\varepsilon)^{-4}$$

holds with probability $1 - p$ on $\mathcal{A}_w$ with respect to μ, as desired. □

Remark 8 Note that there are other ways to identify a small length scale in the flow. Another way is through the dimension of the attractor, $d_{\mathcal{A}}$, which is related to the number of degrees of freedom in the sense of Landau (cf. Landau and Lifshitz 1959). In this direction, Gibbon and Titi show in Gibbon and Titi (1997) that

$$\lambda_d \gtrsim \bar{\lambda}_\varepsilon^{1.6}, \tag{3.20}$$

with $\bar{\epsilon}$ defined as in (3.15). In contrast, the estimate of the dissipation length scale in Theorem 7 is associated with the *exponential decay* of the Fourier spectrum, and again, our estimate is in terms of the actual Kolmogorov length scale λ_ε rather than $\bar{\lambda}_\varepsilon$.

In Bartucelli et al. (1993), so-called ladder estimates are used to identify small length scales in two and three dimensions. However, in three dimensions their estimates involve the quantity $\|\nabla u\|_{L^\infty}$, as in the work of Henshaw et al. (1990).

3.1.2 2D Turbulence

In the Kraichnan theory of 2D turbulence, enstrophy $\|A^{1/2}u\|_{L^2}^2$ is also dissipated, and it does so at a mean rate per unit mass given by

$$\eta = \nu\kappa_0^2\langle\|Au\|_{L^2}^2\rangle.$$

Two key wave numbers are

$$\kappa_\eta := \left(\frac{\eta}{\nu^3}\right)^{1/6} \sim \left(\frac{\langle\|Au\|_{L^2}^2\rangle}{L^2\nu^2}\right)^{1/6}, \quad \kappa_\sigma := \left(\frac{\langle\|Au\|_{L^2}^2\rangle}{\langle\|A^{1/2}u\|_{L^2}^2\rangle}\right)^{1/2},$$

where A is the Stokes operator.

It is shown in Dascaliuc et al. (2008) that if the well-recognized power law

$$e_{\kappa,2\kappa} = \langle\|P_{2\kappa}Q_\kappa u\|_{L^2}^2\rangle \sim \frac{\eta^{2/3}}{\kappa^2} \tag{3.21}$$

holds over the *inertial range* $[\underline{\kappa}_i, \bar{\kappa}_i]$, and if

$$\underline{\kappa}_i \le 4\kappa_\eta, \quad \langle\|A^{1/2}P_{\underline{\kappa}_i}u\|_{L^2}^2\rangle \lesssim \langle\|A^{1/2}Q_{\underline{\kappa}_i}u\|_{L^2}\rangle, \quad G \gtrsim (\bar{\kappa}/\kappa_0)^2, \tag{3.22}$$

then

$$\nu^2\kappa_0^2\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{-1} G \lesssim \langle\|A^{1/2}u\|_{L^2}^2\rangle \lesssim \nu^2\kappa_0^2\left(\frac{\bar{\kappa}}{\kappa_0}\right) G(\ln G)^{3/2}, \tag{3.23}$$

$$\nu^2\kappa_0^4\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{-3/2}\frac{G^{3/2}}{(\ln G)^{3/2}} \lesssim \langle\|Au\|_{L^2}^2\rangle \lesssim \nu^2\kappa_0^4\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{3/2} G^{3/2}(\ln G)^{3/4}. \tag{3.24}$$

This is to say that *on average* $\|A^{1/2}u\|_{L^2}$ is of order $\nu\kappa_0 G^{1/2}$ on the global attractor. As in the 3D case, we can make this precise in terms of probabilities.

First, observe that by the "time-averaged" Brézis–Gallouët inequality (Proposition 24),

$$(\nu\kappa_0)^2\langle\|u_0\|_{\mathcal{W}}^2\rangle \lesssim \langle\|A^{1/2}u_0\|_{L^2}^2\rangle\left(1 + \ln\left(\kappa_\sigma^2/\kappa_0^2\right)\right).$$

Hence, (3.23) and (3.24) imply that

$$\langle\|u_0\|_{\mathcal{W}}^2\rangle \lesssim \mathcal{L}G,$$

where

$$\mathcal{L} := (\bar{\kappa}/\kappa_0)(\ln G)^{3/2}[1 + \ln((\bar{\kappa}/\kappa_0)^{5/2}G^{1/2}(\ln G)^{3/4})].$$

As before, Chebyshev's inequality then implies

$$\mu\left\{u \in \mathcal{A} : \|u\|_{\mathcal{W}} \lesssim \sqrt{\frac{\mathcal{L}}{p}}G^{1/2}\right\} \geq 1 - p \tag{3.25}$$

for any $0 < p < 1$, provided that both (3.21) and (3.22) hold. Therefore, we can conclude by Theorem 4 that

$$\mu\left\{u \in \mathcal{A} : \lambda_a \gtrsim \kappa_0^{-1}G^{-1/2}\right\} \geq 1 - p, \tag{3.26}$$

where the constant inside depends only on p, $\bar{\kappa}/\kappa_0$, and logarithms of G. Since, by (3.24),

$$\lambda_\eta = \left(\frac{\nu^3}{\eta}\right)^{1/6} \leq \frac{1}{\kappa_0}\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{1/4} G^{-1/4},$$

we have the following theorem.

Theorem 9 *Let μ be a time-average measure for a 2D turbulent flow, and let $0 < p < 1$. There exists a set $S \subset \mathcal{A}$ with $\mu(S) \geq 1 - p$ such that*

$$\lambda_d(u) \gtrsim_p \kappa_0^{-1}(\kappa_0\lambda_\eta)^2 \text{ for all } u \in S.$$

4 Outline of Proofs of Main Theorems

Following Biswas and Swanson (2007), our approach is to use a contraction mapping argument. Fix $0 < T \leq \infty$, $\sigma > -1$, and $\beta \geq 0$. Define the spaces

$$X := \{u(\,\cdot\,) \in C([0, T]; V_\sigma) : \|u\|_X < \infty\}, \tag{4.1}$$

$$Y := \{u(\,\cdot\,) \in C((0, T]; V_{\sigma+\beta}) : \|u\|_Y < \infty\}, \tag{4.2}$$

$$Z := X \cap Y, \tag{4.3}$$

where X, Y, Z are equipped with the norms

$$\|u\|_X := \frac{\kappa_0^{-\sigma}}{\nu\kappa_0} \cdot \sup_{0\leq t\leq T} \|u(t)\|_{\sqrt{\nu t},\sigma}, \tag{4.4}$$

$$\|u\|_Y := \nu^{\beta/2}\frac{\kappa_0^{-\sigma}}{\nu\kappa_0} \cdot \sup_{0<t\leq T} (t \wedge (\nu\kappa_0^2)^{-1})^{\beta/2}\|u(t)\|_{\sqrt{\nu t},\sigma+\beta}, \tag{4.5}$$

$$\|u\|_Z := \max\{\|u\|_X, \|u\|_Y\}, \tag{4.6}$$

and $a \wedge b := \min\{a, b\}$. Then X, Y, Z are Banach spaces with $Z \hookrightarrow X, Y$ continuously. Observe, moreover, that these norms are dimensionless.

By the Duhamel principle, the solution u that we seek will be a fixed point of the operator S defined by

$$(Su(\cdot))(t) := \underbrace{e^{-\nu t A} u_0 + \int_0^t e^{-\nu(t-s)A} f(s)\, \mathrm{d}s}_{\Phi(t)} - \underbrace{\int_0^t e^{-\nu(t-s)A} B[u(s), u(s)]\, \mathrm{d}s}_{w(t)} . \tag{4.7}$$

In particular, we establish the existence of such a function u in the closed subset $E \subset Z$ given by

$$E := \{u \in Z : \|u - \Phi\|_Z \le C\}, \tag{4.8}$$

for some $C > 0$, which satisfies $\|\Phi\|_Y \le C$. To do so, we will invoke the following existence theorem whose proof can be found in Biswas and Swanson (2007).

Theorem 10 *Suppose that $\Phi \in Z$ and that $\|\Phi\|_Y \le C$ for some $C > 0$. If $w \in Z$ and $\|w\|_Z \le (1/3)\|v\|_Y$ whenever $u \in E$ and $v \in Z$, for w given by either*

$$w(t) = \int_0^t e^{-\nu(t-s)A} B[u(s), v(s)]\, \mathrm{d}s \quad or \quad w(t) = \int_0^t e^{-\nu(t-s)A} B[v(s), u(s)]\, \mathrm{d}s, \tag{4.9}$$

then there exists a unique $u \in E$ such that

$$u = \Phi - \int_0^t e^{-\nu(t-s)A} B[u(s), u(s)]\, \mathrm{d}s. \tag{4.10}$$

The hypotheses of Theorem 10 are verified in Sects. 6 and 7. In particular, in Sect. 6 we show that $\Phi \in Z$ and $\|\Phi\|_Z \le C$ for some $C > 0$. Consequently, this shows that E is nonempty. We also show in that section that $w \in Z$ whenever $u \in E$ and $v \in Z$. Finally, in Sect. 7 we deduce sufficient conditions for when $\|w\|_Z \le (1/3)\|v\|_Y$.

5 Estimates with Heat Semigroup

In this section we list some preliminary estimates. These estimates concern how the heat kernel, e^{tA}, controls the Gevrey multiplier, $e^{\lambda(t)A^{1/2}}$. The main idea is that the dissipation effect from the heat kernel is stronger than the amplification effect from the Gevrey multiplier. This idea will also be used to control the nonlinear term. However, for the nonlinear term one must exploit in a crucial way the Banach algebra structure of $\mathcal{W}$ in the form of a convolution inequality (Proposition 14). We sketch this below in Proposition 15. The proofs of all of these estimates can be found in Sects. 5–7 of Biswas and Swanson (2007), where all physical dimensions are normalized. We have rescaled them here with the relevant physical parameters, and constants as well. For additional details, see Martinez (2014).

Proposition 11 *Let $\nu > 0$ and $\beta, \lambda \geq 0$, and let $\sigma \in \mathbb{R}$. Then*

$$(\nu t)^{\beta/2} \|e^{-\nu t A} u\|_{\lambda,\sigma+\beta} \lesssim C_{12}(\beta) \|u\|_{\lambda,\sigma} \tag{5.1}$$

holds for $t > 0$, where

$$C_{12}(\beta) = \beta^{\beta/2}.$$

Proposition 12 *Let $\nu > 0$ and $\sigma \in \mathbb{R}$. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be sublinear. Then*

$$\|e^{-\nu(t-s)A} u\|_{\lambda(t),\sigma} \leq C_{13}(s,t) \|e^{-(\nu/2)(t-s)A} u\|_{\lambda(s),\sigma} \tag{5.2}$$

for all for $0 \leq s < t$, where

$$C_{13}(s,t) = \exp\left(\frac{1}{2\nu} \frac{\lambda(t-s)^2}{(t-s)}\right). \tag{5.3}$$

Remark 13 Observe that Proposition 12 identifies a suitable sublinear function $\lambda(t)$ with which to establish Gevrey regularity, namely, $\lambda(t) = \alpha\sqrt{\nu t}$, for some scalar $\alpha \geq 0$ [see (4.4) and (4.5)]. Note that in this case (5.3) becomes

$$C_{13}(s,t) = C_{13}(\alpha) = e^{\alpha^2/2}. \tag{5.4}$$

For convenience, we set $\alpha = 1$.

The following proposition states that the Gevrey norm defined in (2.11) induces a Banach algebra with respect to convolution.

Proposition 14 *Let $\lambda, \gamma \geq 0$. Then*

$$\|u * v\|_{\lambda,\gamma} \lesssim \kappa_0^{-\gamma} \|u\|_{\lambda,\gamma} \|v\|_{\lambda,\gamma}. \tag{5.5}$$

This allows us to establish the following estimate on the nonlinear term.

Proposition 15 *Let $\lambda, \gamma \geq 0$. Then for any $\delta \in \mathbb{R}$*

$$\|e^{-\nu t A} B[u,v]\|_{\lambda,\delta} \lesssim C_{16}(\delta,\gamma) \kappa_0^{1+\delta-2\gamma} (\nu \kappa_0^2 t)^{-\max\{0,(1/2)(1+\delta-\gamma)\}} \|u\|_{\lambda,\gamma} \|v\|_{\lambda,\gamma}, \tag{5.6}$$

where

$$C_{16}(\delta,\gamma) = (1+\delta-\gamma)^{\max\{0,(1/2)(1+\delta-\gamma)\}}.$$

Proof Let $\alpha = (1/2)(1 + \delta - \gamma)$. We estimate as follows:

$$\begin{aligned}
\|e^{-\nu t A} B[u, v]\|_{\lambda,\delta} &\leq \sum_{\substack{k=\kappa_0 k' \\ k' \in \mathbb{Z}^n \setminus \{0\}}} e^{-\nu t|k|^2} e^{\lambda|k|} |k|^{\delta} |B[u, v](k)| \\
&\lesssim \sum e^{-\nu t|k|^2} e^{\lambda|k|} |k|^{\delta+1} (|u| * |v|(k)) \\
&= \|e^{-\nu t|k|^2} |k|^{1+\delta-\gamma}\|_{\ell^\infty} \sum e^{\lambda|k|} |k|^{\gamma} (|u| * |v|(k)) \\
&\leq \left(\frac{1+\delta-\gamma}{2e}\right)^{\max\{0,\alpha\}} (\nu t)^{-\max\{0,\alpha\}} \||u| * |v|\|_{\lambda,\gamma} \\
&\lesssim C_{16}(\delta, \gamma) \kappa_0^{1+\delta-2\gamma} (\nu \kappa_0^2 t)^{-\max\{0,\alpha\}} \|u\|_{\lambda,\gamma} \|v\|_{\lambda,\gamma},
\end{aligned}$$

where in the second inequality we apply (2.5) and (2.6), while in the last inequality we apply Proposition 14. □

Remark 16 There is an ℓ^p-analog of Proposition 15 for $1 < p < \infty$. However, one must restrict the parameter γ according to the dimension n and index p. This restriction is due to the fact that in general, ℓ^p lacks the structure of a Banach algebra for $p > 1$ (cf. Biswas and Swanson 2007).

6 Estimating Φ and w

First we estimate the term

$$\Phi(t) := e^{-\nu t A} u_0 + \int_0^t e^{-\nu(t-s)A} f(s)\, \mathrm{d}s \tag{6.1}$$

in order to show that $\Phi \in Z$ [see (4.3) and (4.7)].

Lemma 17 *Let $1 < q \leq \infty$ and $1/q' = 1 - 1/q$. Let $\sigma \in \mathbb{R}$ and M be given as in (2.18). Then for $0 \leq \beta < 2/q'$ and $0 < T \leq T_f$,*

(i) $\|\Phi\|_X \lesssim C_{18}^{(i)}(q) M$, *where*

$$C_{18}^{(i)}(q) = (1/q')^{1/q'};$$

(ii) $\|\Phi\|_Y \lesssim C_{18}^{(ii)}(q, \beta, \lambda) M$, *where*

$$C_{18}^{(ii)}(q, \beta) = C_{12}(\beta) C_{23}(\beta q'/2, 0)^{1/q'} (q')^{\beta/2};$$

(iii) $(\nu t)^{\beta/2} \|\Phi(t)\|_{\sqrt{\nu t}, \sigma+\beta} \leq C(t)$ *for* $0 < t \leq T_f$, *with* $\lim_{t\to 0^+} C(t) = 0$, *if* $\beta > 0$.

Proof Fix $T \leq T_f$, and let $0 \leq t \leq T$. Observe that

$$\|\Phi(t)\|_{\sqrt{\nu t},\sigma} \lesssim \underbrace{\|e^{-\nu t A}u_0\|_{\sqrt{\nu t},\sigma}}_{(A)} + \underbrace{\int_0^t \|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma}\,\mathrm{d}s}_{(B)}.$$

We estimate (A) by applying Proposition 12, with $s = 0$, and using the fact that $e^{-\nu t A}$ is a contractive semigroup for $t > 0$ so that

$$\|e^{-\nu t A}u_0\|_{\sqrt{\nu t},\sigma} \lesssim \|e^{-(\nu/2)tA}u_0\|_\sigma \leq \|u_0\|_\sigma. \tag{6.2}$$

Now we estimate (B). Observe again that by contractivity and Proposition 12,

$$\|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma} \lesssim \|e^{-(\nu/2)(t-s)A}f(s)\|_{\sqrt{\nu s},\sigma}. \tag{6.3}$$

In fact, we can further bound (6.3) by

$$\|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma} \lesssim e^{-(\nu/2)(t-s)\kappa_0^2}\|f(s)\|_{\sqrt{\nu s},\sigma}. \tag{6.4}$$

Now fix $1 < q < \infty$. Integrating both sides of (6.3) and applying the Hölder inequality gives

$$\int_0^t \|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma}\,\mathrm{d}s$$

$$\lesssim (2/q')^{1/q'}(\nu\kappa_0^2)^{-1}\left(\nu\kappa_0^2\int_0^{T_f}\|f(s)\|^q_{\sqrt{\nu s},\sigma}\,\mathrm{d}s\right)^{1/q}, \tag{6.5}$$

where q, q' are Hölder conjugates. Adding (6.2) and (6.5), normalizing the physical dimensions, then taking the supremum proves (i). For $q = \infty$, make an L^1-L^∞ Hölder estimate in (6.3) instead.

To prove (ii), instead let $0 < t \leq T$. Then as before, we have

$$\|\Phi(t)\|_{\sqrt{\nu t},\sigma+\beta} \lesssim \underbrace{\|e^{-\nu t A}u_0\|_{\sqrt{\nu t},\sigma+\beta}}_{(A')} + \underbrace{\int_0^t \|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma+\beta}\,\mathrm{d}s}_{(B')}. \tag{6.6}$$

Using Proposition 11 we estimate (A') as

$$\begin{aligned}\|e^{-\nu t A}u_0\|_{\sqrt{\nu t},\sigma+\beta} &\lesssim \|e^{-(\nu/2)tA}u_0\|_{\sigma+\beta}\\ &\lesssim C_{12}(\nu t/2)^{-\beta/2}\|u_0\|_\sigma\\ &\leq C_{12}(\nu/2)^{-\beta/2}(t \wedge ((\nu\kappa_0^2)/2)^{-1})^{-\beta/2}\|u_0\|_\sigma. \end{aligned} \tag{6.7}$$

Similarly, assuming $1 < q < \infty$, we can estimate (B') as

$$\|e^{-\nu(t-s)A} f(s)\|_{\sqrt{\nu t},\sigma+\beta} \lesssim C_{12} e^{-(\nu/q')(t-s)\kappa_0^2} (\nu(t-s)/q')^{-\beta/2} \|f(s)\|_{\sqrt{\nu s},\sigma}. \tag{6.8}$$

Now integrate both sides of (6.8) and apply the Hölder inequality and then Proposition 22 to obtain

$$(B') \lesssim C_{12} \int_0^t \frac{e^{-(\nu/q')(t-s)\kappa_0^2}}{(\nu(t-s)/q')^{\beta/2}} \|f(s)\|_{\sqrt{\nu s},\sigma} \, \mathrm{d}s \tag{6.9}$$

$$\leq C_{12} C_{23}^{1/q'} \cdot (\nu/q')^{-\beta/2} (t \wedge (\nu\kappa_0^2)^{-1})^{1/q'-\beta/2} (\nu\kappa_0^2)^{-1/q} \frac{\kappa_0^\sigma}{\nu^{-2}\kappa_0^{-3}} M_f, \tag{6.10}$$

where

$$C_{23}(c,d) = \mathcal{B}(1-c, 1-d) = \int_0^1 t^{-c}(1-t)^{-d} \, \mathrm{d}t. \tag{6.11}$$

An elementary calculation shows that $\mathcal{B}(1-c, 1) = \frac{1}{1-c}$, which in particular implies that

$$C_{23}((\beta q'/2), 0) > 1. \tag{6.12}$$

Also, observe that for any $c \geq 1$

$$(t \wedge (\nu\kappa_0^2)^{-1}) \leq (t \wedge ((\nu\kappa_0^2)/c)^{-1}) \leq c(t \wedge (\nu\kappa_0^2)^{-1}). \tag{6.13}$$

Therefore, adding (6.7) and (6.10), then applying (6.12) and (6.13), we obtain

$$\nu^{\beta/2} \frac{\kappa_0^{-\sigma}}{\nu\kappa_0} (t \wedge (\nu\kappa_0^2)^{-1})^{\beta/2} \|\Phi(t)\|_{\sqrt{\nu t},\sigma+\beta}$$
$$\lesssim C_{12} C_{23}^{1/q'} \left(\frac{\kappa_0^{-\sigma}}{\nu\kappa_0} \|u_0\|_\sigma + (t \wedge (\nu\kappa_0^2)^{-1})^{1/q'} (\nu\kappa_0^2)^{1/q'} M_f \right). \tag{6.14}$$

Using the fact that $(t \wedge (\nu\kappa_0^2)^{-1}) \leq (\nu\kappa_0^2)^{-1}$ and then taking the supremum over $0 < t \leq T$ completes the proof of (ii) for $1 < q < \infty$.

If $q = \infty$, then instead make an L^1-L^∞ Hölder estimate in (6.9), so that (6.10) becomes

$$\int_0^t \|e^{-\nu(t-s)A} f(s)\|_{\sqrt{\nu t},\sigma+\beta} \, \mathrm{d}s$$
$$\lesssim C_{12} C_{23} \cdot (\nu/2)^{-\beta/2} (t \wedge (\nu\kappa_0^2/2)^{-1})^{1-\beta/2} \frac{\kappa_0^\sigma}{\nu^{-2}\kappa_0^{-3}} M_f.$$

Then apply (6.13) again.

Finally, we prove (iii). By Proposition 12, we have

$$(\nu t)^{\beta/2}\|\Phi(t)\|_{\sqrt{\nu t},\sigma+\beta} \lesssim (\nu t)^{\beta/2}\|e^{-(\nu/2)tA}u_0\|_{\sigma+\beta}+(\nu t)^{\beta/2}\left(\int_0^t \|e^{-(\nu/2)(t-s)A}f(s)\|_{\sqrt{\nu s},\sigma+\beta}\,ds\right).$$

Now consider the projection P_κ onto modes $|k| \le \kappa/\kappa_0$, with $Q_\kappa = I - P_\kappa$. Observe that

$$\|e^{-(\nu/2)tA}u_0\|_{\sigma+\beta} \le \|e^{-(\nu/2)tA}Q_\kappa u_0\|_{\sigma+\beta} + \|e^{-(\nu/2)tA}P_\kappa u_0\|_{\sigma+\beta} \lesssim C_{12}(\nu t)^{-\beta/2}\|Q_\kappa u_0\|_\sigma + \|P_\kappa u_0\|_{\sigma+\beta}.$$

Similarly,

$$(\nu t)^{\beta/2}\|e^{-(\nu/2)(t-s)A}f(s)\|_{\sqrt{\nu s},\sigma+\beta} \lesssim C_{12}\|Q_\kappa f(s)\|_{\sqrt{\nu s},\sigma} + (\nu t)^{\beta/2}\|P_\kappa f(s)\|_{\sqrt{\nu s},\sigma+\beta}.$$

Since κ is arbitrary, sending $t \to 0^+$ completes the proof. □

Corollary 18 *Under the same hypotheses as Lemma 17, suppose, moreover, that*

$$M_0 \lesssim (T\nu\kappa_0^2)^{1/q'}M_f, \tag{6.15}$$

where $T \le T_f$. Then

(i) $\|\Phi\|_X \lesssim C_{18}^{(i)}(q)(T\nu\kappa_0^2)^{1/q'}M_f$,
(ii) $\|\Phi\|_Y \lesssim C_{18}^{(ii)}(q,\beta)(T\nu\kappa_0^2)^{1/q'}M_f$.

Proof First, recall (6.4) from the proof of Lemma 17 (i):

$$\|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma} \lesssim e^{-(\nu/2)(t-s)\kappa_0^2}\|f(s)\|_{\sqrt{\nu s},\sigma}.$$

Since $s \le t$, we have $e^{-(\nu/4)(t-s)\kappa_0^2} \le 1$. Thus, integrating (6.4) and applying Hölder's inequality implies

$$\frac{\kappa_0^{-\sigma}}{\nu\kappa_0}\int_0^t \|e^{-\nu(t-s)A}f(s)\|_{\sqrt{\nu t},\sigma}\,ds \lesssim (1/q')^{1/q'}(T\nu\kappa_0^2)^{1/q'}M_f. \tag{6.16}$$

After normalizing, we add (6.2) to complete the proof of (i).

On the other hand, recall (6.14) in the proof of Lemma 17 (ii), which we rewrite as

$$\nu^{\beta/2}\frac{\kappa_0^{-\sigma}}{\nu\kappa_0}(t\wedge(\nu\kappa_0^2)^{-1})^{\beta/2}\|\Phi(t)\|_{\sqrt{\nu t},\sigma+\beta}$$
$$\lesssim C_{18}^{(ii)}\left(M_0+(T\wedge(\nu\kappa_0^2)^{-1})^{1/q'}(\nu\kappa_0^2)^{1/q'}M_f\right) \tag{6.17}$$

for all $0 < t \leq T$. Therefore, (6.15) and the fact that $(T\wedge(\nu\kappa_0^2)^{-1}) \leq T$ prove (ii). □

The following lemma provides the necessary estimate for

$$w(t) := \int_0^t e^{-\nu(t-s)A}B[u(s),v(s)]\,\mathrm{d}s. \tag{6.18}$$

Lemma 19 *Let $\sigma > -1$. Let $0 \leq \beta < 1$ such that $\gamma = \sigma+\beta \geq 0$. Then*

$$\|w\|_Z \lesssim C_{20}(\beta)(\nu\kappa_0^2)^{(1-\beta)/2}(T\wedge(\nu\kappa_0^2)^{-1})^{(1-\beta)/2}\|u\|_Y\|v\|_Y,$$

where

$$C_{20}(\beta)=\max\{C_{23}((1-\beta)/2,\beta),C_{23}(1/2,\beta)\}.$$

Its proof follows exactly that of Proposition 8.5 in Biswas and Swanson (2007). For additional details see Martinez (2014).

7 Proofs of Main Theorems

Proof of Theorem 3 Let $1 < q \leq \infty$, $\sigma > -1$, and $\sigma_- := \max\{-\sigma, 0\}$. Suppose that $q'\sigma_- < 2$. Define $\beta = \beta(\sigma)$ by

$$\beta(\sigma) = \sigma_-. \tag{7.1}$$

Observe that $0 \leq \beta < \min\{2/q', 1\}$ holds for all $1 < q \leq \infty$. Let X, Y, Z be given by (4.1), (4.2), (4.3), respectively. Let Φ be defined by (6.1). Then, by Lemma 17, we have $\Phi \in Z$ and

$$\|\Phi\|_Y \leq C_{18}^{(ii)}M.$$

Thus, the set $E \subset X$ given by (4.8) becomes

$$E = \{u \in Z : \|u-\Phi\|_Z \leq C_{18}^{(ii)}M\}.$$

Obviously, Lemma 19 implies that $w \in Z$ whenever $u \in E$ and $v \in Z$, where w is given by (6.18). Hence, by Theorem 10, it suffices to show that $\|w\|_Z \leq (1/3)\|v\|_Y$ whenever $u \in E$ and $v \in Z$. We determine sufficient conditions for this to hold.

By Lemma 19, we have

$$\|w\|_Z \lesssim C_{20}(\beta)(\nu\kappa_0^2)^{(1-\beta)/2}(T \wedge (\nu\kappa_0^2)^{-1})^{(1-\beta)/2}\|u\|_Y\|v\|_Y$$

for any $u, v \in Y$ and, in particular, for any $u \in E$. By definition of E, $\|u\|_Y \leq 2C_{18}^{(ii)}M$ whenever $u \in E$, so that

$$\|w\|_Z \lesssim C_{20}C_{18}^{(ii)}(\nu\kappa_0^2)^{(1-\beta)/2}(T \wedge (\nu\kappa_0^2)^{-1})^{(1-\beta)/2}M\|v\|_Y.$$

Thus, to satisfy $\|w\|_Z \leq (1/3)\|v\|_Y$, it suffices to have

$$C \cdot C_{20}C_{18}^{(ii)}(\nu\kappa_0^2)^{(1-\beta)/2}T^{(1-\beta)/2}M \leq 1/3$$

for some sufficiently large absolute constant $C > 0$. In other words, if

$$T^* = (C^*)^{2/(1-\beta)}(\nu\kappa_0^2)^{-1}M^{-2/(1-\beta)},$$

where C^* is given by

$$C^* := (1/(3C))(C_{20}C_{18}^{(ii)})^{-1}, \tag{7.2}$$

for some large $C > 0$, then there exists a unique $u \in E$ such that $u = \Phi - w$, whose radius of analyticity at time T^* is at least

$$\lambda_a \gtrsim \kappa_0^{-1}M^{-1/(1-\beta)}.$$

In particular, since $u \in X$, with $\lambda(s) = \sqrt{\nu s}$, u is Gevrey regular.

On the other hand, if we instead assume that

$$M \lesssim C^*,$$

then the solution u exists up to time $T^* = T_f$. Hence, $\lambda_a \gtrsim \sqrt{\nu T_f}$.

The proof that u is also a weak solution follows exactly as in Biswas and Swanson (2007, pp. 1184–1185). This completes the proof. □

Proof of Theorems 4 and 5 Let M_0 and M_f be given by (2.16) and (2.17), respectively. Let β be given by (7.1). Assume that

$$M_0 \lesssim C_* M_f^{(1-\beta)/(1-\beta+2/q')}, \tag{7.3}$$

where

$$C_* = (C^*)^{(2/q')/(1-\beta+2/q')}, \tag{7.4}$$

and C^* is given by (7.2). Let

$$T^* = (C_*)^{q'}(\nu\kappa_0^2)^{-1}M_f^{-2/(1-\beta+2/q')}. \tag{7.5}$$

Now let E be given by

$$E = \{u \in Z : \|u - \Phi\|_Z \leq C_{18}^{(ii)}C_*M_f^{(1-\beta)/(1-\beta+2/q')}\}.$$

Since (7.3) holds, by Corollary 18 (with $T = T^*$), we know $\Phi \in Z$ such that

$$\|\Phi\|_Y \lesssim C_{18}^{(ii)}C_*M_f^{(1-\beta)/(1-\beta+2/q')}.$$

We can now verify directly the condition $\|w\|_Z \leq (1/3)\|v\|_Y$ for $u \in E$ and $v \in X$. Indeed, proceeding as in the proof of Theorem 3, we know that by Lemma 19,

$$\|w\|_Z \lesssim C_{20}(\nu\kappa_0^2)^{(1-\beta)/2}T^{(1-\beta)/2}\|u\|_Y\|v\|_Y$$

for all $T \leq T^*$ whenever $u, v \in Y$. Now observe that for $u \in E$ we have $\|u\|_Y \leq 2C_{18}^{(ii)}C_*M_f^{(1-\beta)/(1-\beta+2/q')}$. Hence, by definition of (7.4) and (7.5),

$$\begin{aligned}\|w\|_Z &\leq C \cdot C_{20}(\nu\kappa_0^2)^{(1-\beta)/2}(T^*)^{(1-\beta)/2}C_{18}^{(ii)}C_*M_f^{(1-\beta)/(1-\beta+2/q')}\|v\|_Y \\ &\leq C \cdot C_{20}C^*C_{18}^{(ii)}\|v\|_Y \\ &= (1/3)\|v\|_Y,\end{aligned}$$

where $C > 0$ is some large absolute constant.

Thus, Theorem 10 furnishes a unique $u \in E$ such that the radius of analyticity at time T^* satisfies

$$\lambda_a \gtrsim \kappa_0^{-1}M_f^{-1/(1-\beta+2/q')}. \tag{7.6}$$

provided that $M_f \gtrsim 1$. As before, u is also a weak solution.

For Theorem 4, set $\sigma = 0$ and $q = 2$, so that $\beta = 0$. Then

$$\lambda_a \gtrsim \kappa_0^{-1}M_f^{-1/2}, \tag{7.7}$$

provided that

$$M_0 \lesssim M_f^{1/2}. \tag{7.8}$$

Finally, let $\tau := (\nu\kappa_0^2)^{-1}$ and $\lambda_f := \kappa_0^{-1}$. Observe that for any $0 \leq s \leq \tau$

$$\sqrt{\nu s} \leq \sqrt{\nu(\nu\kappa_0^2)^{-1}} = \kappa_0^{-1}.$$

Applying Proposition 23 with this choice of τ and λ_f to (7.7) establishes the desired lower bound in Theorem 4. Since $V_0 \subset \ell^2$ and $C([0, T^*]; V_0) \subset L^\infty([0, T^*]; \ell^2)$, the uniqueness of u as a weak solution follows from a criterion of Lions (cf. Temam 1977, pp. 298–299). □

We have, in fact, just proven the following more general theorem.

Theorem 20 *Let* $1 < q \leq \infty$ *with* $1/q' = 1 - 1/q$, *and* $\sigma > -1$. *Let* β *be given by* (7.1) *and* M_0, M_f *be given by* (2.16) *and* (2.17), *respectively. Suppose that* f *satisfies* $M_f < \infty$. *If*

$$M_0 \lesssim M_f^{(1-\beta)/(1-\beta+2/q')}, \tag{7.9}$$

then there exists $T^* < T_f$ *and a mild solution* $u \in C([0, T^*]; V_\sigma)$ *to* (2.2) *such that* u *is also a Gevrey regular weak solution, with a radius of analyticity at time* T^* *satisfying*

$$\lambda_a \gtrsim \kappa_0^{-1} M_f^{-1/(1-\beta+2/q')}. \tag{7.10}$$

Remark 21 Observe that Theorem 20 gives some freedom over the assumption on M_0. For instance, if $\sigma \geq 0$ and $1 \leq q' < 2$, then λ_a at time T^*, given by (7.5), will satisfy the improved estimate

$$\lambda_a \gtrsim \kappa_0^{-1} M_f^{-1/(1+2/q')}, \tag{7.11}$$

provided that

$$M_0 \lesssim M_f^{1/(1+2/q')}. \tag{7.12}$$

If f is time-independent with finitely many modes, then by Proposition 23 we can replace M_f with G. It would be interesting to know whether (7.12) could be established *on average* on the global attractor in two dimensions in the spirit of Dascaliuc et al. (2008) for $\sigma = 0$ and some $1 \leq q' < 2$, for example, without invoking Brézis–Gallouët and the estimates established by Dascaliuc et al. (2008). Indeed, if $q' = 1$, then Theorem 20 yields the estimate $\lambda_a \gtrsim G^{-1/3}$, which would recover the estimate for λ_d predicted by the Kraichnan theory of 2D turbulence (see Kraichnan 1967).

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8 Appendix

We require the following elementary inequality.

Proposition 22 *Let $b \geq 0$ and $0 \leq c, d < 1$. Then for all $t > 0$*

$$\int_0^t \frac{e^{-b(t-s)}}{(t-s)^c (s \wedge b^{-1})^d} \, \mathrm{d}s \leq C_{23}(c,d)(t \wedge b^{-1})^{1-c-d}, \tag{8.1}$$

where $C_{23}(c,d) = \max\{\mathcal{B}(1-c, 1-d), \Gamma(1-c)\}$, where Γ is the gamma function and $\mathcal{B}$ is the beta function.

Proof First, if $b = 0$, then set $(x \wedge b^{-1}) = x$.

Observe that

$$\int_0^t \frac{e^{-b(t-s)}}{(t-s)^c (s \wedge b^{-1})^d} \, \mathrm{d}s \leq \int_0^t \frac{1}{(t-s)^c s^d} \, \mathrm{d}s = t^{-c-d} \int_0^t \left(1 - \frac{s}{t}\right)^{-c} \left(\frac{s}{t}\right)^{-d} \, \mathrm{d}s.$$

Making the change of variables $\sigma = s/t$ and assuming that $bt \leq 1$, we have

$$\begin{aligned} t^{-c-d} \int_0^t (1 - \frac{s}{t})^{-c} (\frac{s}{t})^{-d} \, \mathrm{d}s &\leq t^{1-c-d} \int_0^1 (1-\sigma)^{-c} \sigma^{-d} \, \mathrm{d}\sigma \\ &= t^{1-c-d} \int_0^1 (1-\sigma)^{(1-c)-1} \sigma^{(1-d)-1} \, \mathrm{d}\sigma \\ &= \mathcal{B}(1-c, 1-d)(t \wedge b^{-1})^{1-c-d}, \end{aligned}$$

where $\mathcal{B}$ is given by (6.11).

On the other hand, if $bt > 1$, observe that

$$\begin{aligned} \int_0^t \frac{e^{-b(t-s)}}{(t-s)^c (s \wedge b^{-1})^d} \, \mathrm{d}s &= b^d \int_0^t (t-s)^{-c} e^{-b(t-s)} \, \mathrm{d}s \\ &= b^d \int_0^t (t-s)^{-c} e^{-b(t-s)} \, \mathrm{d}s \\ &= b^{d-1} \frac{1}{b^{-c}} \int_0^{bt} \sigma^{-c} e^{-\sigma} \, \mathrm{d}\sigma \\ &\leq (b^{-1})^{1-c-d} \int_0^\infty \sigma^{(1-c)-1} e^{-\sigma} \, \mathrm{d}\sigma \\ &= \Gamma(1-c)(t \wedge b^{-1})^{1-c-d}. \end{aligned}$$

□

Now we prove Proposition 23, which establishes the equivalency (up to a constant) of M_f [see (2.17)] and the Grashof number, G [see (2.19)].

Proposition 23 *Let $n > 1$. Suppose that f is time-independent and satisfies $f = P_{\bar{\kappa}} f$. Let λ_f be given such that*

$$\sup_{|y|\le\lambda_f} \|f(\,\cdot + iy)\|_{L^2} < \infty \tag{8.2}$$

and $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $\lambda(s) \le \lambda_f$, whenever $0 \le s \le \tau$, for some $\tau > 0$. Then

$$M_f \sim_{\sigma,\bar{\kappa},\lambda_f,\tau} G, \tag{8.3}$$

where the suppressed constants are explicitly identified in (8.6) and (8.7).

Proof of Proposition 24 Let $z = x + iy$, with $x \in [0, L]^n$ and $|y| \le \lambda(s)$. Then we can write $f(z) = \sum_{|k|\le\bar{\kappa}/\kappa_0} \hat{f}(k)e^{i\kappa_0 k\cdot z}$. Observe that since $\kappa_0 = 2\pi/L$,

$$\begin{aligned}\|f(\,\cdot + iy)\|_{L^2}^2 &= \sum_{|k|,|\ell|\le\bar{\kappa}/\kappa_0} \hat{f}(k)\overline{\hat{f}(\ell)}e^{\kappa_0(k+\ell)\cdot y} \int_{[0,L]^n} e^{i\kappa_0(k-\ell)\cdot x}\,\mathrm{d}x\\ &= (2\pi)^n\kappa_0^{-n} \sum_{|k|\le\bar{\kappa}/\kappa_0} |\hat{f}(k)|^2 e^{2\kappa_0 k\cdot y}.\end{aligned}$$

This implies that

$$e^{-2\bar{\kappa}\lambda_f}\kappa_0^{-n/2}\|e^{\lambda(s)A^{1/2}} f\|_{\ell^2} \lesssim \|f(\,\cdot + iy)\|_{L^2} \lesssim \kappa_0^{-n/2}\|e^{\lambda(s)A^{1/2}} f\|_{\ell^2}$$

for all $|y| \le \lambda(s)$. Hence

$$\frac{1}{\nu^2\kappa_0^3}\|e^{\lambda(s)A^{1/2}} f\|_{\ell^2} \sim_{\bar{\kappa},\lambda_f} \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3} \sup_{|y|\le\lambda(s)} \|f(\,\cdot + iy)\|_{L^2}.$$

Now recall the following elementary facts:

- $\|f\|_{\ell^q} \le \|f\|_{\ell^p} \lesssim_{p,q,\bar{\kappa}} \|f\|_{\ell^q}$ for $1 \le p < q < \infty$;
- $\|f\|_{\ell^p} \le \kappa_0^{-\sigma}\|f\|_\sigma \le \left(\frac{\bar{\kappa}}{\kappa_0}\right)^\sigma \|f\|_{\ell^p}$ for $1 \le p \le \infty$.

These imply that

$$\frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3}\|f\|_{\lambda(s),\sigma} \sim_{\sigma,\bar{\kappa},\lambda_f} \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3}\|f(\,\cdot + iy)\|_{L^2} \tag{8.4}$$

for all $|y| \le \lambda(s)$. Obviously, if we set $y = 0$, then by the definition of the Grashof number [see (2.19)], we obtain

$$\frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3} \sup_{0\le s\le\tau} \|f\|_{\lambda(s),\sigma} \sim_{\sigma,\bar{\kappa},\lambda_f} G.$$

On the other hand, for $1 \leq q < \infty$, if we take the $L^q((0,\tau), \mathrm{d}s/(\nu\kappa_0^2)^{-1})$ norm of (8.4), then

$$M_f \sim_{\sigma,\bar{\kappa},\lambda_f,\tau} \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3} \|f(\,\cdot + iy)\|_{L^2} \tag{8.5}$$

for all $|y| \leq \lambda(s)$. Thus, by setting $y = 0$ in (8.5) and by the definition of (2.17), we deduce that

$$M_f \sim_{\sigma,\bar{\kappa},\lambda_f,\tau} G.$$

In particular, we have

$$C_{\lambda_f,\bar{\kappa},n} M_f \leq (\nu\kappa_0^2\tau)^{1/q} G \leq C_n M_f, \tag{8.6}$$

where $C_n := (2\pi)^n$ and

$$C_{\lambda_f,\bar{\kappa},n} := (2\pi)^{-n} \left(\sum_{|k|\leq\bar{\kappa}} 1 \right)^{-1/2} e^{-2\lambda_f\bar{\kappa}} \left(\frac{\kappa_0}{\bar{\kappa}}\right)^{\sigma}. \tag{8.7}$$

□

In Sect. 3.1.2, we made use of a time-averaged Brézis–Gallouët-type inequality in two dimensions. Our proof of the Brézis–Gallouët-type inequality mimics that in Doering and Gibbon (1967) with an additional step to accomodate time averages [see (8.9)]. We note that an explicit form for the constant appearing in the Brézis–Gallouët inequality can be found in Bartucelli and Gibbon (2001).

Proposition 24 *Let $L > 0$ and $\Omega = [0, L]^2$. Let $\mathcal{A}$ be the global attractor of (2.2) with time-independent forcing f satisfying $P_{\bar{\kappa}} f = f$. Then there exists an absolute constant $C > 0$ such that*

$$(\nu\kappa_0)^2 \langle \|u\|_{\mathcal{W}}^2 \rangle \leq C \langle \|A^{1/2}u\|_{L^2(\Omega)}^2 \rangle \left[1 + \ln\left(\kappa_0^{-2} \frac{\langle \|Au\|_{L^2(\Omega)}^2 \rangle}{\langle \|A^{1/2}u\|_{L^2(\Omega)}^2 \rangle} \right) \right] \tag{8.8}$$

for all $u \in \mathcal{A}$, where A is the Stokes operator and $\langle\,\cdot\rangle$ denotes an ensemble average in the sense of (3.10).

Proof Let $u_k := |\hat{u}(k)|$ for all $k \in \mathbb{Z}^n$. Fix $\lambda > 0$ to be chosen later. Observe that

$$\sum_{k\in\mathbb{Z}^d} u_k = \underbrace{\sum_{|k|\leq\lambda} |k|^{-1}|k|u_k}_{A} + \underbrace{\sum_{|k|>\lambda} |k|^{-2}|k|^2 u_k}_{B}.$$

Estimate A with Cauchy–Schwarz to obtain

$$A \leq \left(\sum_{|k| \leq \lambda} |k|^{-2} \right)^{1/2} \left(\sum_{|k| \leq \lambda} |k|^2 u_k^2 \right)^{1/2}.$$

Observe that

$$\sum_{|k| \leq \lambda} |k|^{-2} \leq C \int_1^\lambda r^{-1} \, \mathrm{d}r = C \log \lambda.$$

On the other hand, we estimate B as follows:

$$B \leq \left(\sum_{|k| > \lambda} |k|^{-4} \right)^{1/2} \left(\sum_{|k| > \lambda} |k|^4 u_k^2 \right)^{1/2}.$$

Observe that

$$\sum_{|k| > \lambda} |k|^{-4} \leq C \int_\lambda^\infty r^{-3} \, \mathrm{d}r = \frac{C}{2} \lambda^{-2}.$$

Combining A and B, so far we have

$$\|\vec{u}\|_{\ell^1} \leq C^{1/2} (\log \lambda)^{1/2} \||\ \cdot |\vec{u}\|_{\ell^2} + \sqrt{\frac{C}{2}} \lambda^{-1} \||\ \cdot |^2 \vec{u}\|_{\ell^2},$$

An elementary calculation gives

$$\|\vec{u}\|_{\ell^1}^2 \leq C (\log \lambda) \||\ \cdot |\vec{u}\|_{\ell^2}^2 + C \lambda^{-2} \||\ \cdot |^2 \vec{u}\|_{\ell^2}^2.$$

Taking time-averages, monotonicity and linearity of generalized Banach limits imply

$$\langle \|\vec{u}\|_{\ell^1}^2 \rangle \leq C (\log \lambda) \langle \||\ \cdot |\vec{u}\|_{\ell^2}^2 \rangle + C \lambda^{-2} \langle \||\ \cdot |^2 \vec{u}\|_{\ell^2}^2 \rangle. \tag{8.9}$$

Now choose λ such that

$$\lambda^{-2} = \frac{\langle \||\ \cdot |\vec{u}\|_{\ell^2}^2 \rangle}{\langle \||\ \cdot |^2 \vec{u}\|_{\ell^2}^2 \rangle}.$$

Observe that $\lambda \geq 1$. Therefore, for some absolute constant $C > 0$,

$$\langle \|\vec{u}\|_{\ell^1}^2 \rangle \leq C \langle \|\vec{u}\|_{\ell^2}^2 \rangle \left[1 + \ln \left(\frac{\langle \||\ \cdot |^2 \vec{u}\|_{\ell^2}^2 \rangle}{\langle \||\ \cdot |\vec{u}\|_{\ell^2}^2 \rangle} \right) \right].$$

Rescaling with physical units and applying Parseval's identity completes the proof. □

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