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Existence time for the 3D Navier–Stokes equations in a generalized Gevrey class

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HIGHLIGHTS

- Obtain local existence for the 3D Navier–Stokes equations in a new functional class.
- This class is larger than the analytic and sub-analytic Gevrey classes.
- The norm evolution in this class satisfies an almost linear differential inequality.
- Existence time is better than reciprocal of any power of the norm of initial data.
- Yields better existence time for certain initial data compared to Sobolev classes.

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ABSTRACT

Gevrey class technique is a widely used tool for studying higher regularity properties of solutions to dissipative equations. *Maximal radius* in a Gevrey class determines a small length scale associated to the decay of the Fourier power spectrum and turbulence. In this paper, we consider existence theory for the three dimensional, incompressible Navier–Stokes equations, in a certain generalized Gevrey class of functions, which contains all the analytic and sub-analytic Gevrey classes, and is in turn contained in the space of all C^∞ functions. This class has been in focus recently, pertaining to the study of the attractor for the 2D Navier–Stokes equations, particularly in relation to a question posed by Peter Constantin as to whether or not zero is in the attractor of the 2D Navier–Stokes equations. We show that in the 3D case, the differential inequality that one obtains in this class is *almost linear*, and the corresponding existence time is better than the reciprocal of any algebraic power of the norm of the initial data (in this class). Subsequently, we compare the existence times with well-known ones in Sobolev classes for certain types of initial data. We also obtain a lower bound for the maximal radius in the generalized Gevrey class under consideration here. The first two authors are delighted to dedicate this paper to Professor Edriss Titi on the occasion of his 60th birthday. They admire Professor Titi's enormous research contributions to the field of fluid dynamics, as well as his mathematical acumen, novel ideas, and infectious enthusiasm for the subject, as evidenced by his numerous lucid, and enthralling, lectures. This research was completed after the passing of Professor Basil Nicolaenko. The first two authors are confident that he too would have shared their sentiments concerning Professor Titi.

1. Introduction

Since its introduction in the study of the Navier-Stokes equations in [1], Gevrey class technique has become a standard tool for studying higher order regularity and analyticity properties of solutions for a wide class of dissipative equations, including the incompressible Navier–Stokes equations (NSE) (see [2–8], and the references therein). In [9], a sharp estimate in the analytic Gevrey class was obtained and used to rigorously establish decay of the Fourier power spectrum for the NSE, and thus establish a Kolmogorov type small length scale for decaying turbulence. Other applications include (i) derivation of optimal bounds for the (time) decay of higher order derivatives of solutions to a wide class of dissipative equations including the NSE [3,10,11]; (ii) rigorous justification for exponential convergence of the Galerkin approximation to the true solution [12], and, (iii) establishing geometric regularity criteria for solutions and measuring complexity of fluid flows [13,14].

In this paper, we consider existence theory for the 3D incompressible, Navier–Stokes equations, in a certain *generalized Gevrey*

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class of functions. The generalized Gevrey class was first introduced in [15] and is termed the *Constantin-Chen Gevrey class* in [16]. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that (i) $\psi' > 0$ on $\mathbb{R}_+$ and (ii) $\psi'' < 0$ on $\mathbb{R}_+$, which implies that ψ is strictly increasing and concave. In terms of the Fourier coefficients, the Constantin-Chen Gevrey class, corresponding to the given function ψ, is defined to be the set of all (vector-valued) periodic functions u, with period L, such that $\sum_{k\in\mathbb{Z}^3} |\widehat{u}(k)|^2 e^{2\psi(|k|)} < \infty$. This encompasses the analytic and sub-analytic Gevrey classes, as well as the usual Sobolev classes.

(i) The choice of $\psi(|k|) = \alpha|k|(\alpha > 0)$ yields the analytic Gevrey class.
(ii) In case $\psi(|k|) = \alpha|k|^\beta, \alpha, 0 < \beta < 1$, the corresponding classes we obtain are the sub-analytic Gevrey classes.
(iii) In case $\psi(k) = \sigma\log(1 + |k|)$, we get the usual Sobolev class $\mathbb{H}^\sigma$.

Our focus here is the study of the class, denoted by $\mathfrak{S}$, that one obtains when

$$\psi(|k|) = \varphi(k) := \alpha(\ln[|k| + \gamma])^2, \alpha > 0.$$

The class $\mathfrak{S}$ contains all Gevrey classes (see Remark 2) and is contained in all Sobolev classes (i.e. all functions belonging to $\mathfrak{S}$ are C^∞). This class has recently played a prominent role in the investigation of the attractor $\mathcal{A}$ of the 2D NSE, in relation to a question posed by Peter Constantin, as to when zero belongs to the attractor $\mathcal{A}$. Pertaining to this question, it is shown in [16,17] that the class $\mathfrak{S}$ satisfies the so-called "all for one, one for all" law, namely that if $\mathcal{A} \cap \mathfrak{S}$ is non-empty, then in fact $\mathcal{A} \subset \mathfrak{S}$. This immediately leads one to conclude that if $0 \in \mathcal{A}$, then $\mathcal{A} \subset \mathfrak{S}$. Additionally, if the driving force for the 2D NSE does not belong to $\mathfrak{S}$, then zero cannot belong to the attractor.

Our goal here is to study existence theory in the class $\mathfrak{S}$. We note that in case one starts with an initial data in $\mathbb{H}^\sigma, \sigma > \frac{1}{2}$, instantaneously, the solution belongs to $\mathfrak{S}$ (in fact, to the much smaller analytic Gevrey class [1,10,18]). We show that the differential inequality that one obtains for the evolution of the norm in the class $\mathfrak{S}$ is *nearly linear*, as opposed to being a polynomial as in the usual Sobolev classes $\mathbb{H}^\sigma, \sigma > \frac{1}{2}$, or even in any analytic or sub-analytic Gevrey class [1,10,15]. More precisely, the best known differential inequality in $\mathbb{H}^\sigma$ [19] is given by

$$\begin{aligned}&\frac{d}{dt}\|u\|^2_{\mathbb{H}^\sigma} \lesssim (\|u\|^2_{\mathbb{H}^\sigma})^{\frac{2\sigma+1}{2\sigma-1}}, \frac{1}{2} < \sigma < \frac{5}{2}, \sigma \neq \frac{3}{2},\\ &\frac{d}{dt}\|u\|^2_{\mathbb{H}^\sigma} \lesssim 2^\sigma(\|u\|^2_{\mathbb{H}^\sigma})^{\frac{4\sigma+5}{4\sigma}}, \sigma > \frac{5}{2}.\end{aligned} \tag{1}$$

Observe that as $\sigma \to \infty$, the differential inequality in (1) is linear *in the limit* (although the constant 2^σ blows up), and the existence time that one obtains in $\mathbb{H}^\sigma$ is the reciprocal of an algebraic power of $\|u_0\|_{\mathbb{H}^\sigma}$, where u_0 denotes the initial data. By contrast, the differential inequality in the class $\mathfrak{S}$ that we obtain is, roughly speaking, of the form (up to a logarithmic correction)

$$\frac{d}{dt}\|u\|^2_{\mathfrak{S}} \lesssim \|u\|^2_{\mathfrak{S}}\, e^{\frac{\Gamma}{\sqrt{\ln(\|u\|^2_{\mathfrak{S}})}}}, \tag{2}$$

where Γ is an adequate constant. As long as $\|u\|_{\mathfrak{S}}$ is finite, the solution u is in C^∞ and therefore regular. Clearly, for sufficiently large values $\|u\|_{\mathfrak{S}}$ (which is the regime of interest to us), the differential inequality in (2) is better than any algebraic power greater than one. More interestingly, as one approaches a potential blow up time T_*, the differential inequality becomes linear in the limit $t \to T_*$. In this context, we note here the following curious fact: if one obtains (2) with an adequately *small value of* Γ, *then global regularity follows.* The existence time resulting from (2) is better than the reciprocal of any algebraic power of $\|u_0\|_{\mathfrak{S}}$; see Remark 3 for a comparison of existence times.

Subsequently, we show that the *maximal radius* attained by a regular solution with initial data in $\mathbb{H}^1$, has a lower bound which is reciprocal of the logarithm of the $\mathbb{H}^1$ norm of the initial data. This should be contrasted with an analogous result concerning the Gevrey class radius, the best known result in which case is reciprocal of square of the $\mathbb{H}^1$ norm of the initial data [9]; see also [20,21]. As shown in [9,20,21], the maximal radius in Gevrey classes is connected to turbulence and decay length scales. The maximal radius in $\mathfrak{S}$ is connected to the decay of the Fourier power spectrum; see Remark 1.

The organization of the paper is as follows. In Section 2, we introduce the notation and recall some requisite facts; in Section 3, we introduce the functional class $\mathfrak{S}$ and in Section 4, we state our main results. Section 5 is devoted to the proofs of the main results.

2. Notation and preliminaries

We consider the 3D, incompressible Navier–Stokes equations, on a spatial domain $\Omega = [0, 2\pi]^3$, given by

$$\begin{aligned}&\partial_t u + (u\cdot\nabla)u + \nabla p = 0,\\ &\nabla\cdot u = 0, u(x, 0) = u_0(x), x \in \Omega,\end{aligned} \tag{3}$$

where $u = u(x, t)$ denotes the fluid velocity at a location $x \in \Omega$ and time $t \geq 0$ and $p = p(x, t)$ is the fluid pressure. The system (3) is supplemented with the space periodic boundary condition (with space period L).

The L^2 inner product on Ω is denoted $\langle\cdot,\cdot\rangle$. Following the notational convention in [22–24], we denote $|u|$ to be the L^2 norm. We will also use $|\cdot|$ to denote the Euclidean length of a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. Whether $|\cdot|$ denotes the L^2 norm or Euclidean norm in $\mathbb{R}^n$ or C^n will be understood from the context.

For a function $u : \Omega \to \mathbb{R}^d$(or $\mathbb{C}^d$), its Fourier coefficients are defined by

$$\widehat{u}(k) = \frac{1}{L^3}\int_\Omega u(x)e^{-\imath\kappa_0 k\cdot x}\,(k \in \mathbb{Z}^d),\ \text{where } \kappa_0 = \frac{2\pi}{L}.$$

Then by the Parseval identity,

$$|u|^2 \simeq \kappa_0^{-3}\sum_{k\in\mathbb{Z}^3}|\widehat{u}(k)|^2,$$

where $a \simeq b$ means $a = cb$ where c is an absolute constant, not depending on L (or any other parameters in the problem).

By the well-known Galileian invariance of the NSE, if the initial data has space average zero, then it remains so for all future times, under the evolution. Therefore, we will always take the phase space to comprise of vector-valued functions, that are space periodic with period L, and have space average zero (over the spatial domain Ω). In terms of the Fourier coefficients, this amounts to the condition $\widehat{u}(0) = 0$ (which is then preserved under the evolution). We will denote

$$\dot{L}^2(\Omega) = \left\{u \in L^2(\Omega) : \int_\Omega u(x)\,dx = 0, \text{ or equivalently}, \widehat{u}(0) = 0\right\}.$$

The phase space H is given by

$$H = \left\{u \in \dot{L}^2(\Omega), \nabla\cdot u = 0\right\},$$

where the derivative is understood in the distributional sense. Using Fourier coefficients, the space H can alternatively be characterized by

$$H = \{u \in L^2(\Omega), \widehat{u}(0) = 0, k\cdot\widehat{u}(k) = 0, \widehat{u}(-k) = \overline{\widehat{u}}(k)\}.$$

Note that the space $(-\Delta)(H\cap\mathbb{H}^2) \subset H$, where $\mathbb{H}^s$ denotes the usual (L^2-based) Sobolev spaces. The Stoke's operator A, with domain $\mathcal{D}(A) = H \cap \mathbb{H}^2$, is defined to be

$$A = (-\Delta)|_{\mathcal{D}(A)}.$$

The Stoke's operator A is positive and self adjoint with a compact inverse. It therefore admits a unique, positive square root, denoted $A^{1/2}$ with domain V, where the space V is characterized by

$$V = \{u \in H : |A^{1/2}u|^2 = \sum_{k\in\widetilde{\mathbb{Z}}^d} |k|^2 |\widehat{u}(k)|^2 < \infty\},$$

where we will henceforth denote $\widetilde{\mathbb{Z}}^d := \mathbb{Z}^d \setminus \{0\}$. We denote the Leray–Helmholtz orthogonal projection operator by $\mathbb{P} : \dot{L}^2(\Omega) \to H$, where $\dot{L}^2(\Omega)$ is as defined above. It is well-known that $\mathbb{P}$ commutes with A, see e.g. [22–24].

Following the notational convention in [22–24], we will denote

$$\|u\| = |A^{1/2}u|, u \in V.$$

The spectrum of A comprises of eigenvalues $0 < 1 = \lambda_1 \le \lambda_2 \le \cdots$, where, for each i, $\lambda_i \in \{|k|^2 : k \in \widetilde{\mathbb{Z}}^d\}$. The set of eigenvectors $\{\mathbf{e}_i\}_{i=1}^{\infty}$, where $\mathbf{e}_i$ is an eigenvector corresponding to the eigenvalue λ_i, forms an orthonormal basis of H. We will denote by $H_N = Span\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

It is easy to see that the dual V' of V is given by

$$V' = \{v \in \mathcal{D} : \widehat{v}(k) = \overline{\widehat{v}(-k)}, \widehat{v}(0) = 0, \sum_{k\in\widetilde{\mathbb{Z}}^d} \frac{|\widehat{v}(k)|^2}{|k|^2} < \infty\},$$

where $\mathcal{D}$ denotes the space of distributions. The duality bracket between V and V' is given by

$${}_V\langle u, v\rangle_{V'} = \sum_{k\in\widetilde{\mathbb{Z}}^d} \widehat{u}(k)\overline{\widehat{v}(k)}, \quad u \in V, v \in V'.$$

It should be noted that

$$|A^{s/2}u|^2 = \sum_{k\in\widetilde{\mathbb{Z}}^d} |k|^{2s} |\widehat{u}(k)|^2$$

where $\widetilde{\mathbb{Z}}^d = \mathbb{Z}^d \setminus \{0\}$, $u \in \mathcal{D}(A^{s/2}) = \dot{L}^2(\Omega) \cap \dot{\mathbb{H}}^s$.

It is well known that $|A^{s/2}(\cdot)| \sim \|\cdot\|_{\mathbb{H}^s}$ on $\mathcal{D}(A^{s/2})$ and the Poincaré inequality holds, i.e.,

$$|A^{s/2}u| \sim |u|_{\mathbb{H}^s} \text{ and } |A^{s/2}u| \ge \kappa_0^s|u|, \quad (u \in \mathcal{D}(A^{s/2})). \tag{4}$$

Using the Sobolev and interpolation inequalities, we also have

$$\|u\|_{L^p} \lesssim |A^{s/2}u| \le |u|^{1-s}\|u\|^s, p = \frac{2d}{d-2s}, 1 \le d \le 4, \tag{5}$$

where d denotes the spatial dimension. In our case, $d = 3$ throughout. Often, we will also need the so-called Wiener algebra. This comprises of all functions

$$\mathcal{W} = \{u \in H : \|u\|_{\mathcal{W}} := \sum_k |\widehat{u}(k)| < \infty\}. \tag{6}$$

Clearly, from the expression of u in terms of its Fourier series $u(x) = \sum_{k\in\widetilde{\mathbb{Z}}^d} \widehat{u}(k)e^{\iota k\cdot x}$, it immediately follows that

$$\|u\|_{L^\infty} \le \|u\|_{\mathcal{W}}.$$

We will also need the Agmon inequality

$$\|u\|_{L^\infty} \le \|u\|_{\mathcal{W}} \lesssim |A^{s_1/2}u|^{\theta} |A^{s_2/2}u|^{1-\theta},$$
$$0 \le s_1 < \frac{3}{2} < s_2, \theta s_1 + (1-\theta)s_2 = \frac{3}{2}. \tag{7}$$

Let

$$B(u, v) = \mathbb{P}(u \cdot \nabla v) = \mathbb{P}\nabla \cdot (u \otimes v).$$

It follows from (5) that for $u, v \in V$, $B(u, v) \in V'$.

Applying the Leray–Helmholtz projection operator $\mathbb{P}$, we can write the Navier–Stokes equations in the functional form

$$\frac{d}{dt}u + \nu Au + B(u, u) = 0, u(0) = u_0, \tag{8}$$

where $u_0 \in H$. We recall that a weak solution of (8) (see [22–24] for definition of a weak solution) is said to be regular on $[0, T]$ is $\sup_{t\in[\epsilon,T]} \|u(t)\| < \infty$ for all $\epsilon > 0$. Moreover, we say that $[0, T_*)$ is the *maximal interval of regularity* if (i) either $T_* = \infty$, and for all $\epsilon > 0$ and $T < \infty$, $\sup_{t\in[\epsilon,T]} \|u(t)\| < \infty$ or (ii) $T_* < \infty$, and for all $\epsilon > 0$ and $T < T_*$, $\sup_{t\in[\epsilon,T]} \|u(t)\| < \infty$, and $\lim_{t\nearrow T_*} \|u(t)\| = \infty$.

Inequality Convention. $A \lesssim B$ means $A \le CB$ for an adequate, non-dimensional absolute constant C which does not depend on any parameters (i.e. either the physical parameters κ_0, ν or non-dimensional parameters such as γ, α. Similarly, $A \gtrsim B$ means $A \ge CB$ where C is a non-dimensional absolute constant. Observe that $A \simeq B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$. We will track all constants which depend on the physical parameters κ_0, ν or α, γ (up to the relational symbols $\lesssim, \gtrsim, \simeq$) but will not explicitly track absolute, non-dimensional constants.

3. The functional class $\mathfrak{S}$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by

$$\varphi(x) = [\ln(x+\gamma)]^2, \quad \gamma \ge e^2. \tag{9}$$

Note that the condition $\gamma \ge e^2$ ensures that φ is non-negative, increasing and concave (i.e., $\varphi'' \le 0$) on $\mathbb{R}_+$.

For $\alpha > 0$, define

$$|u|_\alpha := \left(\sum_{\widetilde{\mathbb{Z}}^n} \kappa_0^{-3} |\widehat{u}(k)|^2 e^{2\alpha\varphi(k)}\right)^{1/2} \text{ and}$$

$$\|u\|_\alpha := \left(\kappa_0 \sum_{\widetilde{\mathbb{Z}}^n} |k|^2 |\widehat{u}(k)|^2 e^{2\alpha\varphi(k)}\right)^{1/2},$$

where $\widetilde{\mathbb{Z}}^n = \mathbb{Z}^n \setminus \{0\}$. Clearly, with φ as in (9),

$$|u|_\alpha = |e^{\alpha\varphi(\kappa_0^{-1}A^{1/2})}u| \text{ and } \|u\|_\alpha = |A^{1/2}u|_\alpha.$$

Definition. For $\alpha > 0$ we denote the class $\mathfrak{S}_\alpha$ by

$$\mathfrak{S}_\alpha = \{u \in H : |u|_\alpha < \infty\}. \tag{10}$$

The class $\mathfrak{S}$ is defined as

$$\mathfrak{S} = \bigcup_{\alpha>0} \mathfrak{S}_\alpha.$$

Clearly, $\mathfrak{S}_\alpha \subset \mathfrak{S}_{\alpha'}$ if $\alpha \ge \alpha'$.

Definition. For $u \in \mathfrak{S}$, its *maximal radius* in $\mathfrak{S}$ is defined as

$$r_{\mathfrak{S}}(u) = \sup\{\alpha > 0 : u \in \mathfrak{S}_\alpha\} = \sup\{\alpha > 0 : |u|_\alpha < \infty\}. \tag{11}$$

We note here that the norm $|\cdot|_\alpha$ is stronger than all Sobolev norms. Indeed, we have the inequality

$$|(\kappa_0^{-1}A^{1/2} + \gamma)^\beta u|^2 \le e^{\frac{\beta^2}{2\alpha}} |u|_\alpha^2. \tag{12}$$

This inequality can be easily derived by computing the respective norms in terms of the Fourier coefficients, using Parseval equality. In fact, the converse is also true, i.e., if there exist $C = C(u)$ and $\alpha' > 0$ such that

$$|(\kappa_0^{-1}A^{1/2} + \gamma)^\beta u|^2 \le e^{\frac{\beta^2}{2\alpha'}} C(u) \,\forall\, \beta \in \mathbb{N},$$

then $u \in \mathfrak{S}_\alpha$ for an adequate α; see [16].

Remark 1. The maximal radius in the Gevrey class determines a small length scale associated with the decay of the Fourier power spectrum [9,21]. It is also connected to energy cascades in decaying turbulence [20]. As in the case of the radius in the Gevrey class, the quantity $\kappa_0^{-1} r_{\mathfrak{S}}(u)$ provides a decay length scale of the Fourier power spectrum. Let $\alpha < r_{\mathfrak{S}}(u)$. Then, $|u|_\alpha < \infty$. From the definition of the norm, it follows that

$$|\widehat{u}(k)| \lesssim \frac{|u|_\alpha}{e^{\alpha(\ln(|k|+\gamma))^2}} = \frac{|u|_\alpha}{(|k|+\gamma)^{\alpha\ln(|k|+\gamma)}}.$$

Since for $\alpha\ln(|k|+\gamma) \gtrsim m$ for any $m \in \mathbb{N}$ and for $|k|$ sufficiently large, we see that the decay of the Fourier power spectrum is faster than the reciprocal of any algebraic power of $|k|$.

Remark 2. The Gevrey classes are defined by

$$G_{\alpha,\sigma} = \{u \in H : |e^{\alpha\kappa_0^{-\sigma}A^{\sigma/2}}u|^2 = \sum_{k\in\widetilde{\mathbb{Z}}^3} e^{2\alpha|k|^\sigma}|\widehat{u}(k)|^2 < \infty\}.$$

The analytic Gevrey class corresponds to $\sigma = 1$. Since $\sup_{k\in\widetilde{\mathbb{Z}}^3}\frac{\varphi(|k|)}{|k|^\sigma} < \infty$, it follows that $G_{\alpha',\sigma} \subset \mathfrak{S}_\alpha$ for all $\alpha, \alpha', \sigma > 0$. On the other hand, due to (12), we have $\mathfrak{S}_\alpha \subset C^\infty(\Omega)$. Therefore,

$$G_{\alpha,\sigma} \subset \mathfrak{S} \subset C^\infty(\Omega)\ \forall\, \alpha, \sigma > 0.$$

It can be shown that each of the above inclusions is strict.

Since we will need to compare the norms $|(\kappa_0^{-1}A^{1/2}+\gamma I)^\beta u|$ and $|A^{\beta/2}u|$, let us note that

$$|A^{\beta/2}u| \le \kappa_0^\beta|(\kappa_0^{-1}A^{1/2}+\gamma I)^\beta u| \text{ and}$$
$$|(\kappa_0^{-1}A^{1/2}+\gamma I)^\beta u| \le \left(\frac{1+\gamma}{\kappa_0}\right)^\beta |A^{\beta/2}u|. \tag{13}$$

This inequality also follows by computing the norms in terms of the Fourier coefficients and using Parseval equality.

4. Main results

4.1. Existence time in $\mathfrak{S}$

Theorem 4.1. *Let* $\xi_0 = \frac{|u_0|_\alpha^2}{|u_0|^2}$ *and* $\gamma = \max\{e^2, e^{\frac{2}{\alpha}+1}-1\}$, *where* $\alpha > 0$. *There exists an adequate constant* $C_\alpha > 0$, *with* $C_\alpha \sim \Gamma_\alpha^2$ *for* $\alpha \lesssim 1$, *such that for all* t *satisfying*

$$0 < t < \frac{1}{C_\alpha\left(\kappa_0^2\nu + \frac{\kappa_0^3}{\nu}|u_0|^2\right)}\,\frac{1}{\xi_0^{\frac{\Gamma_\alpha}{\sqrt{\ln(\xi_0)}}}},$$

$$\text{where } \Gamma_\alpha = (\ln 2)\sqrt{2\alpha} + \frac{\frac{5}{2}}{\sqrt{2\alpha}}, \tag{14}$$

we have

$$|u(t)|_\alpha^2 \le |u(t)|^2 e^{\left[\frac{1}{\Gamma_\alpha}\ln\left(\frac{e^{\Gamma_\alpha\sqrt{\ln(\xi_0)}}}{1-Ce^{\Gamma_\alpha\sqrt{\ln(\xi_0)}}\left(\kappa_0^2\nu+\frac{\kappa_0^3}{\nu}|u_0|^2\right)t}\right)\right]^2}. \tag{15}$$

Remark 3.

(i) Consider initial data with its Fourier spectrum supported in a shell, i.e.,

$$u_0 = \sum_{k\in\mathbb{Z}^2, N_1\le|k|\le N_2} \widehat{u}_0(k)e^{i\kappa_0 k\cdot x} \text{ where } N_1 \sim N_2. \tag{16}$$

A straightforward computation shows that $\|u_0\|^2 \ge \kappa_0^2N_1^2|u_0|^2$. Therefore, the classical (due to Leray) existence time of solutions is

$$T(u_0) \lesssim \frac{\nu^3}{\|u_0\|^4} = \frac{\nu^3}{\kappa_0^4}\,\frac{1}{N_1^4}\,\frac{1}{|u_0|^4}.$$

On the other hand, a similar computation of the existence time in (14) leads to

$$T(u_0) \sim \frac{\nu}{\kappa_0^3}\,\frac{1}{(N_2+\gamma)^{4\alpha\ln 2+\frac{5}{2}}}\,\frac{1}{|u_0|^2}. \tag{17}$$

If $N_1 \sim N_2$ and α is sufficiently small, i.e., $4\alpha\ln 2 + \frac{5}{2} < 4$, then we obtain a better existence time in terms of the power of N_1.

(ii) The existence time in the Sobolev class $\mathbb{H}^\sigma$, for $\frac{1}{2} < \sigma < \frac{5}{2}$, $\sigma \ne \frac{3}{2}$, is given by [19]

$$T \sim \frac{\nu^{\frac{5-2\sigma}{2\sigma-1}}}{\|u_0\|_{\mathbb{H}^\sigma}^{\frac{4}{2\sigma-1}}}.$$

A computation of the existence time for data of the form (16) readily yields

$$T(u_0) \lesssim \frac{\nu^{\frac{5-2\sigma}{2\sigma-1}}}{(\kappa_0N_1)^{\frac{4\sigma}{2\sigma-1}}|u_0|^{\frac{4}{2\sigma-1}}}.$$

Since the exponent of N_1, namely, $\frac{4\sigma}{2\sigma-1} < \frac{5}{2}$ for σ in the range $\frac{1}{2} < \sigma < \frac{5}{2}$, for sufficiently small choice of α, (17) yields a better existence time.

(iii) To be fair, it should be noted that in case $\sigma > \frac{5}{2}$, the existence time in $\mathbb{H}^\sigma$ for the type of data considered in (16) is better. Indeed, the existence time for $\sigma > \frac{5}{2}$ is given by $T \sim \frac{1}{\|u_0\|_{\mathbb{H}^\sigma}^{\frac{5}{2\sigma}}|u_0|^{1-\frac{5}{2\sigma}}}$ [19]. For data as in (16), this immediately yields

$$T(u_0) \gtrsim \frac{1}{(\kappa_0N_2)^{\frac{5}{2}}|u_0|^2}.$$

Clearly this is better than (17) due to the fact that $\alpha\ln 2 > 0$. However, it should be noted that so far we have worked in the class $\mathfrak{S}_\alpha$, the definition of which is based on L^2-norm. Consideration of existence time in the class

$$\mathfrak{S}_{\alpha,\sigma} := \{u \in H : |A^{\sigma/2}u|_\alpha < \infty\}, \sigma > \frac{5}{2},$$

may be more appropriate while making a comparison in this case.

4.2. Radius in $\mathfrak{S}$

In this section, we start with initial data in $\mathbb{H}^1$ (i.e. V) and show that after a time T_*, the solution belongs to $\mathfrak{S}$ and its radius in $\mathfrak{S}$ is reciprocal of the logarithm of the $\mathbb{H}^1$ norm of the initial data. As shown in [9], the maximal radius in the analytic Gevrey class $\gtrsim \frac{1}{\|u_0\|^2}$.

Theorem 4.2. *Let* $u_0 \in V$ *and denote by* $[0, T)$ *the maximal interval of regularity. Then there exists a time* $T_* < T$ *such that*

$$r_{\mathfrak{S}}(u(T_*)) \ge \frac{1}{\ln\left(\frac{1+e^2}{\nu^2\kappa_0}\|u_0\|^2\right)},$$

where $r_{\mathfrak{S}}$ *is as defined in* (11).

5. Proofs of the main theorems

5.1. Preparatory results

We will start with the following lemma the proof of which is elementary and is therefore omitted.

Proposition 5.1. *Let* $a, b, c > 0$. *The function* $f(\lambda) = a\ln(b\lambda) - c\lambda$, $\lambda > 0$ *attains its supremum at* $\lambda_{max} = a/c$ *and moreover,* $\sup_{\lambda>0}f(\lambda) = a\ln(\frac{ab}{ce})$.

We will denote by $\mathbb{R}_+$ the set $[0, \infty)$.

Proposition 5.2. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable, increasing, concave function, i.e., $\psi' \geq 0$, $\psi'' \leq 0$ on $\mathbb{R}_+$. Then for all $x, y \in \mathbb{R}_+$, we have*

$$\psi(x+y) \leq \psi(x) + \psi(y) \text{ and } \\ \psi(x+y) - \psi(x) - \psi(y) \leq \psi(2x) - 2\psi(x) \quad \text{if } x \leq y. \tag{18}$$

Proof. Define $f(y) = \psi(x+y) - \psi(x) - \psi(y)$, $x, y \geq 0$. Since $\psi'' \leq 0$, it follows that ψ' is a decreasing function. Consequently, $f'(y) = \psi'(x+y) - \psi'(y) \leq 0$, i.e., f is decreasing. Thus, $f(y) \leq f(0)$ which yields

$$\psi(x+y) - \psi(x) - \psi(y) \leq \psi(x) - \psi(x) - \psi(0) \leq 0,$$

which yields the first inequality in (18). Moreover, for $x \leq y$, we have $f(y) \leq f(x)$, which yields the second inequality in (18). □

Proposition 5.3. *Let φ be as in (9). For $x \leq y \in \mathbb{R}_+$, we have*

$$\varphi(x+y) - \varphi(x) - \varphi(y) \leq -\varphi(x) + \Gamma_0 \ln(x+\gamma) + (\ln(2))^2, \\ \text{where } \Gamma_0 = 2\ln 2. \tag{19}$$

Proof. Observe that

$$\frac{\ln(2x+\gamma)}{\ln(x+\gamma)} \leq \frac{\ln[2(x+\gamma)]}{\ln(x+\gamma)} \leq \frac{\ln 2 + \ln(x+\gamma)}{\ln(x+\gamma)} = 1 + \frac{\ln 2}{\ln(x+\gamma)}.$$

Squaring both sides and using the fact that $\ln(x+\gamma) \geq 1$ for $x \geq 1$, $\gamma \geq e-1$, we obtain

$$\left[\frac{\ln(2x+\gamma)}{\ln(x+\gamma)}\right]^2 \leq 1 + \frac{1}{\ln(x+\gamma)}\left[2\ln 2 + \frac{(\ln(2))^2}{\ln(x+\gamma)}\right] \\ \leq 1 + \frac{\Gamma_0}{\ln(x+\gamma)} + \frac{(\ln(2))^2}{(\ln(x+\gamma))^2}, \quad \text{where } \Gamma_0 = 2\ln 2.$$

Consequently, with φ as defined in (9), we get

$$\varphi(2x) - 2\varphi(x) \leq -[\ln(x+\gamma)]^2 + \Gamma_0 \ln(x+\gamma) + (\ln(2))^2. \tag{20}$$

From the second inequality in (18), we immediately obtain (19). □

5.2. Interpolation inequalities in $\mathfrak{S}$

In view of (12), the norm $|\cdot|_\alpha$ is stronger than all Sobolev norms, and we can use it to interpolate $|(A^{1/2}+\gamma I)^{\Gamma\alpha} u|$ in the following way.

Proposition 5.4. *Let $\Gamma > 0$ and assume that*

$$\gamma \geq e^{\frac{\Gamma}{2}} - 1. \tag{21}$$

Then,

$$|(\kappa_0^{-1}A^{1/2} + \gamma I)^{\Gamma\alpha} u|^2 \leq |u|^2 e^{\Gamma\sqrt{2\alpha}\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}. \tag{22}$$

Proof. Let

$$y = e^{2\alpha\varphi(x+\gamma)} = (x+\gamma)^{2\alpha\ln(x+\gamma)}.$$

Elementary calculation yields

$$(x+\gamma)^{2\Gamma\alpha} = e^{\Gamma\sqrt{2\alpha}\sqrt{\ln y}} = e^{c_\alpha\sqrt{\ln y}}, \text{ where } c_\alpha = \Gamma\sqrt{2\alpha}.$$

Define a function g and a probability measure $d\mu$ on $\widetilde{\mathbb{Z}}^3$ by

$$g(y) = e^{c_\alpha\sqrt{\ln y}}, \quad d\mu(k) = \frac{|\widehat{u}(k)|^2}{|u|^2}, \ y = |k| + \gamma, k \in \widetilde{\mathbb{Z}}^3.$$

Provided g is a concave function (i.e., $g''(y)$ is negative) we have by Jensen's inequality that

$$\frac{|(\kappa_0^{-1}A^{1/2}+\gamma I)^{\alpha\Gamma}|^2}{|u|^2} = \sum \frac{(|k|+\gamma)^{2\alpha\Gamma}|\widehat{u}(k)|^2}{|u|^2} = \sum g(y) d\mu(k)$$

$$\leq g\left(\sum y d\mu(k)\right) = g\left(\sum e^{2\alpha\varphi(|k|)}|\widehat{u}(k)|^2 \frac{1}{|u|^2}\right)$$

$$= g\left(\frac{|u|_\alpha^2}{|u|^2}\right) = e^{c_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}, \text{ where recall that } c_\alpha = \Gamma\sqrt{2\alpha}. \tag{23}$$

It should be noted that Jensen's inequality is used to obtain the inequality in the second line above.

We will now check the concavity condition for the function g defined above. Note that

$$g''(y) = \frac{c_\alpha g(y)}{2}\left[\frac{c_\alpha}{2y^2(\ln(y))} - \frac{1}{y^2(\ln y)^{1/2}} - \frac{1}{y^2(\ln y)^{3/2}}\right] \\ < \frac{c_\alpha g(y)}{4y^2\ln y}\left[c_\alpha - 2(\ln y)^{1/2}\right].$$

For g to be concave, it is enough for the right hand side of the above inequality to be negative, i.e. $c_\alpha - 2(\ln y)^{1/2} \leq 0$. Since $y = (x+\gamma)^{2\alpha\ln(x+\gamma)}$ and $x \geq 1$, or equivalently since $y = e^{2\alpha[\ln(x+\gamma)]^2}$, for g to be concave, it is enough for γ to satisfy (21). □

For any $\gamma > 0$, the (unbounded) operator $\ln(\kappa_0^{-1}A^{1/2}+\gamma)$ is defined on a dense subspace of H by the relation

$$\widehat{v}(k) = \ln(|k|+\gamma)\widehat{w}(k), k \in \mathbb{Z}^3 \setminus \{0\}, \text{ and, } \widehat{v}(0) = 0,$$

where $v = \ln(\kappa_0^{-1}A^{1/2}+\gamma)w$. The domain of the operator $\ln(\kappa_0^{-1}A^{1/2}+\gamma)$ is given by

$$\mathcal{D}(\ln(\kappa_0^{-1}A^{1/2}+\gamma)) = \left\{w \in H : \sum[\ln(|k|+\gamma)]^2|\widehat{w}(k)|^2\right\} < \infty.$$

We will also need the following inequality in the sequel.

Proposition 5.5. *Assume that $\gamma + 1 \geq \sqrt{e}$. Then*

$$|\ln(\kappa_0^{-1}A^{1/2}+\gamma)w| \lesssim \ln\left(\frac{|(\kappa_0^{-1}A^{1/2}+\gamma)w|}{|w|}\right)|w| \\ \leq \ln\left((1+\gamma)\kappa_0^{-1}\frac{|A^{1/2}w|}{|w|}\right)|w|. \tag{24}$$

Proof. Observe that

$$\frac{|\ln(\kappa_0^{-1}A^{1/2}+\gamma)w|^2}{|w|^2} = \sum[\ln(|k|+\gamma)]^2\frac{|\widehat{w}(k)|^2}{|w|^2} = \sum g(y)d\mu(k)$$

where

$$y = (|k|+\gamma)^2, g(y) = [\ln(\sqrt{y})]^2 = \frac{1}{4}[\ln(y)]^2 \text{ and}$$

$$d\mu(k) = \frac{|\widehat{w}(k)|^2}{|w|^2}.$$

A straightforward computation yields

$$g'(y) = \frac{1}{2}\ln(y)\frac{1}{y} \text{ and } g''(y) = \frac{1}{2}\left\{\frac{1-\ln y}{y^2}\right\}.$$

If $(1+\gamma)^2 \geq e$, then $1 - \ln(y) \leq 0$ which implies g is concave. Applying Jensen's inequality, we readily obtain the first inequality in (39). The second follows from (13). □

5.3. Inequalities for the nonlinear term

The following estimates of the nonlinear term will be crucial to proceed. We have used here the orthogonality property $\langle B(u,v), v\rangle = 0$. Without this assumption, the estimate obtained is worse.

Proposition 5.6. *Let $\Gamma_0 = 2\ln 2$ and $1+\gamma \geq e^{\frac{2}{\alpha}+1}$. Then*

$$|\langle B(u,v), e^{2\alpha\varphi(A^{1/2})}v\rangle| \lesssim \kappa_0^{5/2}\max\{\alpha,1\}2^{\alpha\ln 2}\left(|u|_\alpha |v| e^{\frac{1}{2}\sqrt{2\alpha}(\Gamma_0+\frac{5}{2\alpha})\sqrt{\ln\left(\frac{|v|_\alpha^2}{|v|^2}\right)}}|v|_\alpha + |u|e^{\frac{1}{2}\sqrt{2\alpha}(\Gamma_0+\frac{5}{2\alpha})\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}|v|_\alpha^2 + |u|e^{\frac{1}{2}\sqrt{2\alpha}(\Gamma_0+\frac{5}{2\alpha})\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}|\ln(\kappa_0^{-1}A^{1/2}+\gamma)v|_\alpha |v|_\alpha\right). \quad (25)$$

When $u=v$, we have the estimate

$$|\langle e^{\alpha\varphi(A^{1/2})}B(u,u), e^{\alpha\varphi(A^{1/2})}u\rangle| \lesssim \kappa_0^{5/2}\max\{\alpha,1\}2^{\alpha\ln 2}\times \left(1+\frac{1}{\ln(1+\gamma)}\right)|u|e^{\frac{1}{2}\sqrt{2\alpha}(\Gamma_0+\frac{5}{2\alpha})\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}|\ln(\kappa_0^{-1}A^{1/2}+\gamma)u|_\alpha|u|_\alpha$$
$$\lesssim \kappa_0^{5/2}\max\{\alpha,1\}2^{\alpha\ln 2}\left(1+\frac{1}{\ln(1+\gamma)}\right)\times |u|e^{\frac{1}{2}\sqrt{2\alpha}(\Gamma_0+\frac{5}{2\alpha})\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\left[\ln\left((1+\gamma)\kappa_0^{-1}\frac{|A^{1/2}u|_\alpha}{|u|_\alpha}\right)\right]|u|_\alpha^2. \quad (26)$$

Proof. Note that

$$\langle e^{\alpha\varphi(A^{1/2})}B(u,v), e^{\alpha\varphi(A^{1/2})}v\rangle = \langle e^{\alpha\varphi(A^{1/2})}B(u,v) - B(u, e^{\alpha\varphi(A^{1/2})}v), e^{\alpha\varphi(A^{1/2})}v\rangle.$$

We will now estimate $|e^{\alpha\varphi(A^{1/2})}B(u,v) - B(u,e^{\alpha\varphi(A^{1/2})}v)|$. Note first that ∇ commutes with the operator $e^{\alpha\varphi(A^{1/2})}$. The kth Fourier coefficient of $e^{\alpha\varphi(A^{1/2})}B(u,v) - B(u,e^{\alpha\varphi(A^{1/2})}v)$ can thus be written as

$$(e^{\alpha\varphi(A^{1/2})}B(u,v) - B(u,e^{\alpha\varphi(A^{1/2})}v))(k) = \kappa_0\sum_h((k-h)\cdot\widehat{u}(h))[e^{\alpha\varphi(|k|)} - e^{\alpha\varphi(|k-h|)}]v(k-h).$$

Decompose the above sum as

$$\sum_h\{\cdots\} = \sum_{h\in\mathcal{S}_1}\{\cdots\} + \sum_{h\in\mathcal{S}_2}\{\cdots\} + \sum_{h\in\mathcal{S}_3}\{\cdots\} := I + II + III,$$

where, for notational simplicity, we denote

$$\mathcal{S}_1 = \{h : |k|/3 \leq |h| \leq 3|k|\},\ \mathcal{S}_2 = \{|h| < |k|/3\},$$
$$\mathcal{S}_3 = \{3|k| < |h|\}.$$

5.3.1. *Estimate of* $\|I\|_{L^2}$. Note that

$$|I| \leq \kappa_0\sum_{h\in\mathcal{S}_1}e^{\alpha\varphi(|k|)}|\widehat{u}(h)||k-h||v(k-h)| + \kappa_0\sum_{h\in\mathcal{S}_1}|\widehat{u}(h)|e^{\alpha\varphi(|k-h|)}|k-h||v(k-h)| := T_1 + T_2. \quad (27)$$

Observe that

$$|k-h| \leq |k| + |h| \leq 4|h|\ \forall\, h\in\mathcal{S}_1. \quad (28)$$

Thus,

$$T_2 \leq 4\kappa_0\sum_{h\in\mathcal{S}_1}|h||\widehat{u}(h)|e^{\alpha\varphi(|k-h|)}|v(k-h)|.$$

Consequently, by Agmon's inequality,

$$\|T_2\|_{L^2} \leq 4C\kappa_0|Au|^{1/2}|A^{3/2}u|^{1/2}|v|_\alpha. \quad (29)$$

We will now estimate T_1. Note first that since $|k| \leq |k-h| + |h|$ and φ is increasing,

$$T_1 \leq \kappa_0\sum_{h\in\mathcal{S}_1}e^{\alpha\varphi(|h|)}|\widehat{u}(h)|e^{\alpha\varphi(|k-h|)}|k-h| \times |v(k-h)|e^{\alpha(\varphi(|k-h|+|h|)-\varphi(|h|)-\varphi(|k-h|))}. \quad (30)$$

Decompose $\mathcal{S}_1 = \mathcal{S}_{11} + \mathcal{S}_{12}$ where

$$\mathcal{S}_{11} = \{h\in\mathcal{S}_1, |k-h| \leq |h|\},\ \mathcal{S}_{12} = \{h\in\mathcal{S}_1, |k-h| > |h|\}.$$

Using (18) followed by (20) in (30), we have

$$T_1 \lesssim \kappa_0 2^{\alpha\ln 2}\sum_{h\in\mathcal{S}_{11}}e^{\alpha\varphi(|h|)}|\widehat{u}(h)|(|k-h|+\gamma)^{\Gamma_0\alpha}|k-h||v(k-h)| + \kappa_0 2^{\alpha\ln 2}\sum_{h\in\mathcal{S}_{12}}|\widehat{u}(h)|(|h|+\gamma)^{\Gamma_0\alpha}|k-h|e^{\alpha\varphi(|k-h|)}|v(k-h)|$$
$$\leq \kappa_0 2^{\alpha\ln 2}\left(\sum_{h\in\mathcal{S}_{11}}e^{\alpha\varphi(|h|)}|\widehat{u}(h)|(|k-h|+\gamma)^{\Gamma_0\alpha}|k-h||v(k-h)| + 4\sum_{h\in\mathcal{S}_{12}}|\widehat{u}(h)||h|(|h|+\gamma)^{\Gamma_0\alpha}e^{\alpha\varphi(|k-h|)}|v(k-h)|\right), \quad (31)$$

where to obtain the last inequality in (31), we used (28). From (31), using Agmon's inequality, we immediately obtain

$$\|T_1\|_{L^2} \lesssim \kappa_0 2^{\alpha\ln 2}\left(|u|_\alpha|A(A^{1/2}+\gamma)^{\Gamma_0\alpha}v|^{1/2}|A^{3/2}(A^{1/2}+\gamma)^{\Gamma_0\alpha}v|^{1/2} + |A(A^{1/2}+\gamma)^{\Gamma_0\alpha}u|^{1/2}|A^{3/2}(A^{1/2}+\gamma)^{\Gamma_0\alpha}u|^{1/2}|v|_\alpha\right). \quad (32)$$

5.3.2. *Estimate of* $\|III\|_{L^2}$. This term proceeds in a similar manner but is some what simpler than the first. We first observe that easy applications of triangle inequality and the fact that φ is increasing, yield

$$\frac{2}{3}|h| \leq |k-h| \leq \frac{4}{3}|h| \text{ and } \varphi(|k|) \leq \varphi(|h|)\ \forall\, h\in\mathcal{S}_3. \quad (33)$$

We first note that $|III| \leq T_1 + T_2$ where T_i, $i=1,2$ are as given in (27). Using the second part of (33), we see that

$$\|T_1\|_{L^2} \lesssim \kappa_0|u|_\alpha|Av|^{1/2}|A^{3/2}v|^{1/2}.$$

Using the first part of (33), in a similar fashion, we obtain

$$\|T_2\|_{L^2} \lesssim \kappa_0|Au|^{1/2}|A^{3/2}u|^{1/2}|v|_\alpha.$$

This leads us to

$$\|III\|_{L^2} \lesssim \kappa_0|u|_\alpha|Av|^{1/2}|A^{3/2}v|^{1/2} + |Au|^{1/2}|A^{3/2}u|^{1/2}|u|_\alpha. \quad (34)$$

5.3.3. *Estimate of* $\|II\|_{L^2}$. We will see that this term will produce the worst estimate. Observe that

$$II(k) = \kappa_0\sum_{h\in\mathcal{S}_2}(k-h)\cdot\widehat{u}(h)[e^{\alpha\varphi(|k|)} - e^{\alpha\varphi(|k-h|)}]v(k-h)$$
$$= -\kappa_0\sum_{h\in\mathcal{S}_2}(k-h)\cdot\widehat{u}(h)[\int_0^1\frac{d}{d\tau}e^{\alpha\varphi(|k-\tau h|)}d\tau]v(k-h). \quad (35)$$

A computation with the explicit form of φ yields

$$\left|\frac{d}{d\tau}e^{\alpha\varphi(|k-\tau h|)}d\tau\right| \leq 2\alpha|h|\ln(|k-\tau h|+\gamma)\frac{e^{\alpha\varphi(|k-\tau h|)}}{(|k-\tau h|+\gamma)}. \quad (36)$$

We will now estimate each of the terms occurring on the right hand side of (36). For $h \in \mathcal{S}_2$ and $0 < \tau < 1$, we have

$$|k - \tau h| \geq |k| - \tau|h| \geq |k| - |k|/3 = \frac{2}{3}|k|.$$

On the other hand,

$$|k - h| \leq |k| + |h| \leq \frac{4}{3}|k| \; \forall\, h \in \mathcal{S}_2.$$

Consequently,

$$\frac{|k-h|}{(|k-\tau h| + \gamma)} \leq \frac{|k-h|}{|k-\tau h|} \leq (4/3)/(2/3) = 2. \tag{37}$$

We will now use the fact that φ is concave and monotonically increasing. Accordingly,

$$\begin{aligned}
&\varphi(|k - \tau h|) - \varphi(|h|) - \varphi(|k - h|)\\
&= \varphi(|k - h + (1-\tau)h|) - \varphi(|h|) - \varphi(|k-h|)\\
&\leq \varphi(|k-h| + (1-\tau)|h|) - \varphi(|h|) - \varphi(|k-h|)\\
&\leq \varphi(|k-h| + |h|) - \varphi(|h|) - \varphi(|k-h|) \leq \varphi(2|h|) - 2\varphi(|h|),
\end{aligned} \tag{38}$$

where to obtain (38), we first observe that for $h \in \mathcal{S}_2$, we must have

$$|k-h| \geq |k| - |h| \geq |k| - |k|/3 = \frac{2}{3}|k| > \frac{1}{3}|k| \geq |h|,$$

and then apply (18). Since $\ln(\cdot)$ is also a concave function, a similar computation as in (38) yields

$$\begin{aligned}
&\ln(|k - \tau h| + \gamma) \leq \ln(|k-h| + \gamma) + \ln(|h| + \gamma)\\
&\quad + \ln(2|h| + \gamma) - 2\ln(|h| + \gamma)\\
&\leq \ln(|k-h| + \gamma) + \ln(|h| + \gamma) + \ln(2|h| + 2\gamma) - 2\ln(|h| + \gamma)\\
&\leq \ln(|k-h| + \gamma) + \ln 2 \leq 2\ln(|k-h| + \gamma).
\end{aligned} \tag{39}$$

Finally, in view of (35)–(39), we obtain

$$\begin{aligned}
|II(k)| &\lesssim \kappa_0 2\alpha \sum_{h\in\mathcal{S}_2} |\widehat{u}(h)||h| e^{\alpha\varphi(|h|)} \ln(|k-h| + \gamma) e^{\alpha\varphi(|k-h|)}\\
&\quad \times |v(k-h)| e^{\alpha(\varphi(2|h|) - 2\varphi(|h|))}\\
&\lesssim \kappa_0 \alpha 2^{\alpha \ln 2} \sum_{h\in\mathcal{S}_2} (|h| + \gamma)^{\Gamma_0\alpha} |h||\widehat{u}(h)| \ln(|k-h| + \gamma)\\
&\quad \times e^{\alpha\varphi(|k-h|)}|v(k-h)|.
\end{aligned}$$

From this inequality, using Agmon's inequality, we immediately conclude that

$$\begin{aligned}
\|II\|_{L^2} &\lesssim \kappa_0 \alpha 2^{\alpha\ln 2} |A(A^{1/2} + \gamma)^{\Gamma_0\alpha} u|^{1/2} |A^{3/2}(A^{1/2} + \gamma)^{\Gamma_0\alpha} u|^{1/2}\\
&\quad \times |\ln(A^{1/2} + \gamma) v|_\alpha.
\end{aligned}$$

Gathering all the estimates and applying (13) and (22), we obtain (25).

The inequality in (26) follows from (25) by plugging in $u = v$ and applying (24). □

5.4. Proof of Theorem 4.1

Proof. Taking the L^2-inner product of (8) with $e^{2\alpha\varphi(A^{1/2})}u$ and using the inequality (26) for the nonlinear term, we arrive at the differential inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u|_\alpha^2 + \nu|A^{1/2}u|_\alpha^2\\
&\lesssim \kappa_0^{5/2} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}} |u|_\alpha^2 \ln\left((1+\gamma)\kappa_0^{-1}\frac{|A^{1/2}u|_\alpha}{|u|_\alpha}\right),
\end{aligned}$$

where

$$C_{\alpha,\gamma} = \max\{\alpha, 1\} 2^{\alpha\ln 2}\left(1 + \frac{1}{\ln(1+\gamma)}\right).$$

This in turn yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u|_\alpha^2 &\lesssim \kappa_0^{5/2} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}} |u|_\alpha^2\\
&\quad \times \left[\ln\left((1+\gamma)^2\kappa_0^{-2}\frac{|A^{1/2}u|_\alpha^2}{|u|_\alpha^2}\right) - c\nu\frac{|A^{1/2}u|_\alpha^2}{|u|_\alpha^2}\right],
\end{aligned} \tag{40}$$

where $c = \left(\kappa_0^{5/2} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\right)^{-1}$. Using Proposition 5.1, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u|_\alpha^2 &\lesssim \kappa_0^{5/2} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}} |u|_\alpha^2\\
&\quad \times \ln\left((1+\gamma)^2\nu^{-1}\kappa_0^{1/2} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\right),
\end{aligned} \tag{41}$$

where $C_{\alpha,\gamma}$ is as in (40). Using the fact that $\frac{d}{dt}|u|^2 = -\nu|A^{1/2}u|^2$ and (22), we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{|u|_\alpha^2}{|u|^2}\right) &= \frac{\frac{d}{dt}|u|_\alpha^2}{|u|^2} + \nu\frac{|u|_\alpha^2}{|u|^2}\frac{|A^{1/2}u|^2}{|u|^2}\\
&\leq \frac{\frac{d}{dt}|u|_\alpha^2}{|u|^2} + \nu\kappa_0^2 e^{\frac{1}{\alpha}\sqrt{2\alpha}\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\frac{|u|_\alpha^2}{|u|^2}
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\lesssim \kappa_0^2 e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\\
&\quad \times \left[\nu + \kappa_0^{1/2} C_{\alpha,\gamma}|u| \ln\left((1+\gamma)^2\frac{\sqrt{\kappa_0}}{\nu} C_{\alpha,\gamma}|u| e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}}\right)\right]\\
&\quad \times \frac{|u|_\alpha^2}{|u|^2},
\end{aligned} \tag{43}$$

where to obtain (42), we applied (22) with $\Gamma = \frac{1}{\alpha}$ and in order to obtain (43), we used (41).

Denote

$$\xi(t) = \frac{|u|_\alpha^2}{|u|^2}.$$

We thus have from (43) that

$$\begin{aligned}
&\Gamma_\alpha \frac{e^{-\Gamma_\alpha\sqrt{\ln(\xi(t))}}}{\sqrt{\ln\xi(t)}\,\xi(t)}\frac{d\xi(t)}{dt}\\
&\lesssim \kappa_0^2\Gamma_\alpha \frac{\left[\nu + \sqrt{\kappa_0} C_{\alpha,\gamma}|u_0| \ln\left((1+\gamma)^2\frac{\sqrt{\kappa_0}}{\nu} C_{\alpha,\gamma}|u_0| e^{\Gamma_\alpha\sqrt{\ln(\xi(t))}}\right)\right]}{\sqrt{\ln\xi(t)}},
\end{aligned} \tag{44}$$

where $C_{\alpha,\gamma}$ is as in (40). Using the fact that

$$(1+\gamma) = e^{\frac{2}{\alpha}+1} \text{ and } \sqrt{\ln\xi(t)} \geq \sqrt{2\alpha}\ln(1+\gamma), \tag{45}$$

it follows that

$$\frac{\kappa_0^2\Gamma_\alpha\nu}{\sqrt{\ln\xi(t)}} \leq \frac{\kappa_0^2\Gamma_\alpha\nu}{\sqrt{2\alpha}(\frac{2}{\alpha}+1)} \lesssim \kappa_0^2\nu. \tag{46}$$

Observe that for $\alpha \lesssim 1$, from (40), we have $C_{\alpha,\gamma} \sim 1$. Proceeding in a similar way using (45), for $\alpha \lesssim 1$, we obtain

$$\frac{\kappa_0^{5/2}\Gamma_\alpha C_{\alpha,\gamma}|u_0|\ln((1+\gamma)^2 C_{\alpha,\gamma})}{\sqrt{\ln(\xi(t))}} \lesssim \kappa_0^{5/2}|u_0|(\frac{2}{\alpha}+1), \tag{47}$$

where $C_{\alpha,\gamma}$ is as in (40). Similarly,

$$\frac{\kappa_0^{5/2}\Gamma_\alpha C_{\alpha,\gamma}|u_0|\ln(e^{\Gamma_\alpha\sqrt{\ln\left(\frac{|u|_\alpha^2}{|u|^2}\right)}})}{\sqrt{\ln(\xi(t))}} \lesssim \kappa_0^{5/2}\Gamma_\alpha^2|u_0|. \tag{48}$$

Finally,

$$\frac{\kappa_0^{5/2}\Gamma_\alpha C_{\alpha,\gamma}|u_0|\ln(\kappa_0^{1/2}\nu^{-1})|u_0|}{\sqrt{\ln(\xi(t))}} \lesssim \kappa_0^{5/2}|u_0|\ln(\kappa_0^{1/2}\nu^{-1}|u_0|). \tag{49}$$

Using Young's inequality on the right hand side of (46)–(48) and the fact that $\frac{x\ln(x)}{x^2} \lesssim C$ in (49), we obtain

$$\kappa_0^2\Gamma_\alpha\frac{\left[\nu+\sqrt{\kappa_0}C_{\alpha,\gamma}|u_0|\ln\left((1+\gamma)^2\frac{\sqrt{\kappa_0}}{\nu}C_{\alpha,\gamma}|u_0|e^{\Gamma_\alpha\sqrt{\ln(\xi(t))}}\right)\right]}{\sqrt{\ln\xi(t)}}$$
$$\lesssim C_\alpha\left(\kappa_0^2\nu+\frac{\kappa_0^3}{\nu}|u_0|^2\right),$$

where $C_{\alpha,\gamma}$ is as in (40) and

$C_\alpha \sim \Gamma_\alpha^2$ for $\alpha \lesssim 1$.

Combining this with (44), we readily obtain

$$\frac{d}{dt}\left(-e^{-\Gamma_\alpha\sqrt{\ln(\xi(t))}}\right) \le C_\alpha\left(\kappa_0^2\nu+\frac{\kappa_0^3}{\nu}|u_0|^2\right). \tag{50}$$

Integrate both sides to obtain

$$e^{-\Gamma_\alpha\sqrt{\ln(\xi(t))}} \ge e^{-\Gamma_\alpha\sqrt{\ln(\xi_0)}} - C_\alpha\left(\kappa_0^2\nu+\frac{\kappa_0^3}{\nu}|u_0|^2\right)t. \tag{51}$$

This yields that $\xi(t) := \frac{|u|_\alpha^2}{|u|^2} < \infty$ as long as the r.h.s. of (51) is strictly positive. More precisely, for all t satisfying (14), we have

$$\ln\xi(t) \le \left[\frac{1}{\Gamma_\alpha}\ln\left(\frac{e^{\Gamma_\alpha\sqrt{\ln(\xi_0)}}}{1-C_\alpha e^{\Gamma_\alpha\sqrt{\ln\xi_0}}\left(\kappa_0^2\nu+\frac{\kappa_0^3}{\nu}|u_0|^2\right)t}\right)\right]^2,$$

or equivalently, (15). □

5.5. Proof of Theorem 4.2

Proof. We will consider the evolution of the norm $|u(t)|_{\beta t}$ where $\beta > 0$ is a fixed. Using the first inequality in (18) and proceeding as in [FT], one can derive the following inequality:

$$|\langle B(u,u), Ae^{2\beta t\varphi(A^{1/2})}u\rangle| \lesssim |Au|_{\beta t}^{3/2}\|u\|_{\beta t}^{3/2}. \tag{52}$$

Taking the L^2-inner product of the (8) with $Ae^{2\beta t\varphi(A^{1/2})}$ and using (52), we readily obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_{\beta t}^2+\nu|Au|_{\beta t}^2\\
&=\langle B(u,u), Ae^{2\beta t\varphi(A^{1/2})}u\rangle+2\beta\|\ln(\kappa_0^{-1}A^{1/2}+\gamma)u\|_{\beta t}^2\\
&\lesssim |Au|_{\beta t}^{3/2}\|u\|_{\beta t}^{3/2}+2\beta\|\ln(\kappa_0^{-1}A^{1/2}+\gamma)u\|_{\beta t}^2\\
&\lesssim \frac{\nu}{2}|Au|_{\beta t}^2+\frac{C}{\nu^3}\|u\|_{\beta t}^6+2\beta\|u\|_{\beta t}^2\left[\ln\left(\kappa_0^{-1}(1+\gamma)\frac{|Au|_{\beta t}}{\|u\|_{\beta t}}\right)\right]^2,
\end{aligned} \tag{53}$$

where to obtain the inequality in (53), we used Young's inequality and (24). Rearranging (53), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{\beta t}^2+\nu|Au|_{\beta t}^2\\
&\lesssim \frac{1}{\nu^3}\|u\|_{\beta t}^6\\
&\quad+\|u\|_{\beta t}^2\left\{\beta\left[\ln\left(\kappa_0^{-2}(1+\gamma)^2\frac{|Au|_{\beta t}^2}{\|u\|_{\beta t}^2}\right)\right]^2-\nu\frac{|Au|_{\beta t}^2}{\|u\|_{\beta t}^2}\right\}.
\end{aligned} \tag{54}$$

Consider now the function (of the variable λ)

$$f(\lambda)=\beta[\ln(a\lambda)]^2-b\lambda.$$

In our setting,

$$\lambda=\frac{|Au|_{\beta t}^2}{\|u\|_{\beta t}^2},\ a=(1+\gamma)^2\kappa_0^{-2} \text{ and } b=\nu. \tag{55}$$

Note that

$$f'(\lambda)=2\beta\frac{\ln(a\lambda)}{\lambda}-b \text{ and } f''(\lambda)=\frac{2\beta}{\lambda}[1-\ln(a\lambda)].$$

Since $\frac{|Au|_{\beta t}^2}{\|u\|_{\beta t}^2} \ge \kappa_0^2$ and in our case, $\lambda=\frac{|Au|_{\beta t}^2}{\|u\|_{\beta t}^2}$, we proceed to find an upper estimate for $\sup_{\lambda\ge\kappa_0^2} f(\lambda)$. Observe that for $\lambda \ge \kappa_0^2$, a as in (55) and $(1+\gamma)\ge e$, we have,

$$\ln(a\lambda)\ge\ln(1+\gamma)^2=2\ln(1+\gamma)\ge 2.$$

Thus, $f''(\lambda)<0$ for $\lambda\ge\kappa_0^2$ and therefore, f' is one-to-one and strictly decreasing in $[\kappa_0^2,\infty)$. Moreover,

$$f'(\lambda)\to -b \text{ as } \lambda\to\infty \text{ and } f'(\kappa_0^2)=\frac{4\beta\ln(1+\gamma)}{\kappa_0^2}-\nu.$$

In case

$$\beta\le\frac{\nu\kappa_0^2}{4\ln(1+\gamma)}, \tag{56}$$

then $f'(\lambda)\le 0$ for all $\lambda\in[\kappa_0^2,\infty)$, i.e. f is decreasing on this interval and

$$\sup_{\lambda\ge\kappa_0^2} f(\lambda)=f(\kappa_0^2)\le 4\beta[\ln(1+\gamma)]^2\le\nu\kappa_0^2\ln(1+\gamma), \tag{57}$$

where the last inequality is obtained using (56).

On the other hand, in case

$$\beta>\frac{\nu\kappa_0^2}{4\ln(1+\gamma)}, \tag{58}$$

then there exists unique $\lambda_{max}\in(\kappa_0^2,\infty)$ such that

$$f'(\lambda_{max})=2\beta\frac{\ln(a\lambda_{max})}{\lambda_{max}}-b=0 \text{ and } \sup_{\lambda\ge\kappa_0^2} f(\lambda)=f(\lambda_{max}). \tag{59}$$

From (59) and the definition of f it follows that

$$\sup_{\lambda\ge\kappa_0^2} f(\lambda)\le f(\lambda_{max})\le\beta[\ln(a\lambda_{max})]^2. \tag{60}$$

We will now obtain an upper estimate of λ_{max}. Denoting $x=a\lambda$, from (59), it follows that $x_*=a\lambda_{max}$ is the unique solution of the equation (in x)

$$\frac{x}{\ln(x)}=\frac{2a\beta}{b}.$$

Since $\lambda\in(\kappa_0^2,\infty)$, we have $x\ge(1+\gamma)^2\ge e^2$. It is easy to see that $\sqrt{x}>\ln(x)$ for all $x\ge(1+\gamma)^2$. Thus, for $x>(1+\gamma)^2$, we have $\sqrt{x} = \frac{x}{\sqrt{x}} < \frac{x}{\ln(x)}$. Consequently, if x_{**} is the solution to the equation $\sqrt{x} = \frac{2a\beta}{b}$, while x_* is the solution of $\frac{x}{\ln(x)} = \frac{2a\beta}{b}$, then

$$x_* \le x_{**} = \frac{4a^2\beta^2}{b^2}.$$

Since $x_* = a\lambda_{max}$, this in turn implies

$$\lambda_{max} \le \frac{4a\beta^2}{b^2}.$$

In view of (60), it follows that

$$\sup_{\lambda \ge \kappa_0^2} f(\lambda) \le \beta\left[\ln(\frac{4a^2\beta^2}{b^2})\right]^2 = 4\beta\left[\ln(\frac{2a\beta}{b})\right]^2. \tag{61}$$

From (61) and (54), we obtain

$$\frac{d}{dt}\|u\|_{\beta t}^2 + \nu|Au|_{\beta t}^2 \lesssim \frac{\|u\|_{\beta t}^6}{\nu^3} + \delta\|u\|_{\beta t}^2, \tag{62}$$

where

$$\delta := \beta\left[\ln\left(\frac{(1+\gamma)^2}{\kappa_0^2\nu}\beta\right)\right]^2. \tag{63}$$

Using interpolation inequality $\|u\| \le |u|^{1/2}|Au|^{1/2}$, we obtain

$$\begin{aligned}\frac{d}{dt}\frac{\|u\|_{\beta t}^2}{|u|^2} &= \frac{1}{|u|^2}\frac{d}{dt}\|u\|_{\beta t}^2 - \frac{\|u\|_{\beta t}^2}{|u|^4}\frac{d}{dt}|u|^2 \\ &= \frac{1}{|u|^2}\frac{d}{dt}\|u\|_{\beta t}^2 + \nu\frac{\|u\|_{\beta t}^2\|u\|^2}{|u|^4} \\ &\le \frac{1}{|u|^2}\frac{d}{dt}\|u\|_{\beta t}^2 + \nu\frac{\|u\|_{\beta t}^2|Au|}{|u|^3} \\ &\le \frac{1}{|u|^2}\frac{d}{dt}\|u\|_{\beta t}^2 + \nu\frac{|Au|^2}{2|u|^2} + \nu\frac{\|u\|_{\beta t}^4}{|u|^4}.\end{aligned}$$

The estimate above, together with (62), yields

$$\frac{d}{dt}\frac{\|u\|_{\beta t}^2}{|u|^2} \lesssim \frac{1}{\nu^3}\frac{\|u\|_{\beta t}^6}{|u|^2} + \nu\frac{\|u\|_{\beta t}^4}{|u|^4} + \delta\frac{\|u\|_{\beta t}^2}{|u|^2}. \tag{64}$$

Let

$$\zeta = \kappa_0^{-2}\frac{\|u\|_{\beta t}^2}{|u|^2}.$$

Using the fact that $|u|^2 \le |u_0|^2$, from (64), we readily obtain the differential inequality

$$\frac{d}{dt}\zeta \lesssim \frac{\kappa_0^4|u_0|^4}{\nu^3}\zeta^3 + \nu\kappa_0^2\zeta^2 + \delta\zeta,$$

or equivalently,

$$\frac{\frac{d}{dt}\zeta}{\frac{\kappa_0^4|u_0|^4}{\nu^3}\zeta^3 + \delta\zeta} \lesssim 1 + \frac{\nu\kappa_0^2\zeta}{\frac{\kappa_0^4|u_0|^4}{\nu^3}\zeta^2 + \delta} \le 1 + \frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}}, \tag{65}$$

where the last inequality is obtained by maximizing the function $\zeta \to \frac{\nu\kappa_0^2\zeta}{\frac{\kappa_0^4|u_0|^4}{\nu^3}\zeta^2+\delta}$. An explicit integration of (65) immediately yields

$$\frac{\zeta^2}{\zeta^2 + \frac{\delta\nu^3}{\kappa_0^4|u_0|^4}} \le e^{2C\delta(1+\frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}})t}\frac{\zeta_0^2}{\zeta_0^2 + \frac{\delta\nu^3}{\kappa_0^4|u_0|^4}},$$

for some dimensionless, absolute constant $C > 0$. For notational simplicity, let us denote

$$\Psi_0^2 = \frac{\zeta_0^2}{\zeta_0^2 + \frac{\delta\nu^3}{\kappa_0^4|u_0|^4}}.$$

It then follows that, provided

$$t < \widetilde{T} := \frac{1}{2C\delta(1+\frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}})}\ln\left(1 + \frac{\delta\nu^3}{\zeta_0^2\kappa_0^4|u_0|^4}\right), \tag{66}$$

we have

$$\zeta^2 \le \frac{\delta\nu^3}{\kappa_0^4|u_0|^4}\Psi_0^2 e^{2C\delta(1+\frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}})t}\left(1 - \Psi_0^2 e^{2C\delta(1+\frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}})t}\right)^{-1} < \infty. \tag{67}$$

From elementary algebra, it follows from (66) and (67) that

$$\zeta^2 \le 2\zeta_0^2 + \frac{\delta\nu^3}{\kappa_0^4|u_0|4}$$

$$\text{for all } t \le T_* := \frac{1}{2C\delta(1+\frac{\nu^{5/2}}{|u_0|^2\sqrt{\delta}})}\ln\left(1 + \frac{\delta\nu^3}{2\zeta_0^2\kappa_0^4|u_0|^4}\right). \tag{68}$$

For T_* as in (68), provided we choose β such that

$$\beta > e\frac{\kappa_0^2\nu}{(1+\gamma)^2},$$

we have $\left[\ln\left(\frac{(1+\gamma)^2}{\kappa_0^2\nu}\beta\right)\right]^2 > 1$ which implies $\delta > \beta$. Therefore,

$$\beta T_* \ge \frac{1}{2C[\ln(\frac{(1+\gamma)^2}{\kappa_0^2\nu}\beta)]^2(1+\frac{\nu^2(1+\gamma)}{|u_0|^2\kappa_0})}\ln(1 + \frac{\beta\nu^3}{2\zeta_0^2\kappa_0^4|u_0|^4}). \tag{69}$$

Now choose β such that

$$\beta = \frac{(1+\gamma)^2}{\kappa_0^2\nu}\left(\frac{2\zeta_0^2\kappa_0^4|u_0|^4}{\nu^3}\right)^2.$$

With this choice of β, provided $\frac{\beta\nu^3}{2\zeta_0^2\kappa_0^4|u_0|^4} = \frac{(1+\gamma)^2}{\kappa_0^2\nu}\frac{2\zeta_0^2\kappa_0^4|u_0|^4}{\nu^3} > 1$, we deduce from (69) that

$$\beta T_* \gtrsim \frac{1}{\ln\left(\frac{(1+\gamma)}{\nu^2}\zeta_0\kappa_0|u_0|^2\right)}.$$

The result immediately follows by choosing $\gamma = e^2$. □

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References

[1] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, J. Funct. Anal. 87 (1989) 359–369.
[2] H. Bae, A. Biswas, Gevrey regularity for a class of dissipative equations with analytic nonlinearity, Methods Appl. Anal. 22 (4) (2015) 377–408.
[3] Animikh Biswas, Gevrey regularity for a class of dissipative equations with applications to decay, J. Differential Equations 253 (10) (2012) 2739–2764.
[4] Z. Bradshaw, Z. Grujić, I. Kukavica, Local analyticity radii of solutions to the 3D Navier–Stokes equations with locally analytic forcing, J. Differential Equations 259 (8) (2015) 3955–3975.
[5] C. Cao, M.A. Rammaha, E.S. Titi, The Navier–Stokes equations on the rotating 2-D sphere: Gevrey regularity and asymptotic degrees of freedom, Z. Angew. Math. Phys. 50 (3) (1999) 341–360.
[6] A.B. Ferrari, E.S. Titi, Gevrey regularity for nonlinear analytic parabolic equations, Comm. Partial Differential Equations 23 (1) (1998) 424–448.
[7] Z. Grujić, I. Kukavica, Space Analyticity for the Navier–Stokes and Related Equations with Initial Data in L_p, J. Funct. Anal. 152 (2) (1998) 447–466.
[8] Z. Grujić, I. Kukavica, Space analyticity for the nonlinear heat equation in a bounded domain, J. Differential Equations 154 (1) (1999) 42–54.

[9] C. Doering, E.S. Titi, Exponential decay rate of the power spectrum for solutions of the Navier-Stokes equations, Phys. Fluids 7 (1995) 1384–1390.
[10] M. Oliver, E.S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in $\mathbb{R}^n$, J. Funct. Anal. 172 (1) (2000) 1–18.
[11] H. Bae, A. Biswas, E. Tadmor, Analyticity and decay estimates of the Navier-Stokes equations in critical Besov spaces, Arch. Ration. Mech. Anal. 205 (3) (2012) 963–991.
[12] A. Doelman, E.S. Titi, Regularity of solutions and the convergence of the Galerkin method in the complex Ginzburg-Landau equation, Numer. Func. Opt. Anal. 14 (1993) 299–321.
[13] Z. Grujić, The geometric structure of the super-level sets and regularity for 3D Navier–Stokes equations, Indiana Univ. Math. J. 50 (3) (2001) 1309–1317.
[14] I. Kukavica, Level sets of the vorticity and the stream function for the 2-D periodic Navier- Stokes equations with potential forces, J. Differential Equations 126 (1996) 374–388.
[15] W. Chen, New a priori estimates in Gevrey class of regularity for weak solutions of 3D Navier-Stokes equations, Differ. Integral Equ. Appl. 7 (1994) 101–107.
[16] C. Foias, M.S. Jolly, Y. Yang, B. Zhang, On whether zero is in the global attractor of the 2D Navier–Stokes equations, Nonlinearity 27 (11) (2014) 2755–2770.
[17] C. Foias, M.S. Jolly, R. Lan, R. Rupam, Y. Yang, Time analyticity with higher norm estimates for the 2D Navier–Stokes equations, IMA J. Appl. Math. 80 (3) (2015) 766–810.
[18] A. Biswas, D. Swanson, Existence and generalized Gevrey regularity of solutions to the Kuramoto-Sivashinsky equation in $\mathbb{R}^n$, J. Differential Equations 240 (1) (2007) 145–163.
[19] J.C. Robinson, W. Sadowski, R.P. Silva, Lower bounds on blow up solutions of the three-dimensional Navier–Stokes equations in homogeneous Sobolev spaces, J. Math. Phys. 53 (11) (2012) 115618 15 pp.
[20] A. Biswas, C. Foias, On the maximal space analyticity radius for the 3D Navier–Stokes equations and energy cascades, Ann. Mat. Pura Appl. (4) 193 (3) (2014) 739–777.
[21] W.D. Henshaw, H.O. Kreiss, L.G. Reyna, Smallest scale estimates for the Navier-Stokes equations for incompressible fluids, Arch. Ration. Mech. Anal. 112 (1) (1990) 21–44.
[22] P. Constantin, C. Foias, Navier-Stokes Equations, University of Chicago Press, 1988.
[23] R. Temam, Navier-STokes Equations: Theory and Numerical Analysis, Revised ed., AMS/Chelsea, New York, 2001.
[24] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, in: CBMS Regional Conference Series, vol. 41, SIAM, Philadelphia, 1983.