Annali di Matematica (2014) 193:739–777
DOI 10.1007/s10231-012-0300-z

On the maximal space analyticity radius for the 3D Navier–Stokes equations and energy cascades

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Received: 18 March 2012 / Accepted: 16 October 2012 / Published online: 16 November 2012

Abstract In this paper, we study the maximal space analyticity radius associated with a regular solution of the Navier–Stokes equations and its connection to turbulence. In order to do this, we introduce a new auxiliary ODE for the evolution of the analyticity radius involving the Gevrey class norms. We further show that jumps in the maximal space analyticity radius are an intermittent event and are connected to inverse energy cascade. Our approach also leads to a new type of global regularity test for the Navier–Stokes equation.

Keywords Navier–Stokes equations · Analyticity · Gevrey regularity · Turbulence · Energy cascade · Intermittency

Mathemathics Subject Classification (2010) Primary 35Q30 · Secondary 35Q35 · 76D05

1 Introduction

It is well known that regular solutions of the Navier–Stokes equations (henceforth referred to as the NSE) are in fact analytic, in both space and time variables [15,29]. The space analyticity radius is an important physical object: at this length scale, the viscous effects take precedence over the (nonlinear) inertial effects. Below this length scale, the Fourier

In memory of Professor Giovanni Prodi, one of the great pioneers of the modern theory of the Navier–Stokes equations and their rigorous connections to turbulence theory, and an exceptional human being.

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spectrum decays exponentially [11,13,23,24]. Other applications of space analyticity occur in the study of long-term dynamics of solutions [30] and geometric regularity criteria for the NSE and to measure the spatial complexity of the flow (see [21,25,26]).

An effective approach to estimating the analyticity radius via Gevrey norms was introduced in [7,16]. Here, one can avoid cumbersome recursive estimation of higher-order derivatives. Since its introduction, Gevrey class technique has become an useful tool for studying analytic properties of solutions for a wide class of dissipative equations (see [3,5,12]). In this approach, one studies the evolution of the Gevrey norm defined by

$$\|\mathbf{u}\|_\alpha := \|A^{1/2}e^{\alpha A^{1/2}}\mathbf{u}\|_{L^2}, \quad (\alpha > 0).$$

In case $\|\mathbf{u}\|_\alpha < \infty$, $\mathbf{u}$ admits an analytic extension on the domain $\{x + \imath y : |y| < \alpha\} \subset \mathbb{C}^n$. On the other hand, if $\mathbf{u}$ admits an analytic extension on the domain $\{x + \imath y : |y| < \alpha\} \subset \mathbb{C}^n$, then $\|\mathbf{u}\|_{\alpha_1} < \infty$ for all $\alpha_1 < \alpha$. Consequently, if one defines

$$\beta_c = \sup\{\alpha_1 \geq 0 : \|\mathbf{u}\|_{\alpha_1} < \infty\},$$

then β_c can be viewed as the maximal (uniform in x) space analyticity radius. The set $\mathcal{G} := \bigcup_{\alpha>0} \{\mathbf{u} : \|\mathbf{u}\|_\alpha < \infty\}$ coincides with the class of real analytic functions in the periodic case and is a subset of it in case the phase space is $\mathbb{R}^n$. A detailed discussion of the connection between Gevrey norms and space analyticity radius can be found for instance in [27,30].

In [16], it was shown that for initial data belonging to the Sobolev space $\mathbb{H}^1$, the (lower estimate of the) radius of analyticity grows linearly in time up to a certain time T, which depends on the initial data. Several authors have subsequently improved this result; the main improvement resides in allowing larger classes of initial data [4,20]. It is important to emphasize that all the above-mentioned estimates of analyticity radius are lower estimates and the true analyticity radius may be much larger in specific examples. For example (see [14]), let $f(x,t)$ denote a periodic solution (with period 2π) of the heat equation in one space variable x and for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $g(\mathbf{x}, t) = f(k_1x_1 + k_2x_2 + k_3x_3, t)$. It is easy to check that $\mathbf{u}(\mathbf{x}, t) = (g(\mathbf{x}, t), g(\mathbf{x}, t), g(\mathbf{x}, t))$ is a solution of the three-dimensional, space periodic Navier–Stokes equations provided $k_1 + k_2 + k_3 = 0$, $k_i \in \mathbb{Z}, i = 1, 2, 3$. Since $f(\cdot, t)$ solves the heat equation, $\mathbf{u}(\cdot, t)$ is an entire function for all $t > 0$.

In this paper, we try to obtain a more intimate connection between the radius of analyticity and the dynamics of the Navier–Stokes equations. Let $\mathbf{u}(\cdot)$ be an arbitrary, but fixed, solution of the NSE with initial data $\mathbf{u}_0$ that is divergence free and space-periodic (with period L) with zero space average and belongs to the Sobolev space $\mathbb{H}^1$. We also assume that the solution $\mathbf{u}(\cdot)$ is regular on the interval $[0, t_\infty)$ with $t_\infty \leq \infty$. In other words, the solution $\mathbf{u}(\cdot)$ that we consider is not necessarily assumed to be globally regular. The main objective of this paper is to study the maximal analyticity radius $\beta_c(\cdot)$ as a function of time on the time interval $(0, t_\infty)$. Our main tool will be the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = F(t, \alpha(t)),\ t \geq t_0, \alpha(t_0) = \alpha_0 \text{ where } F(t,\alpha) = \frac{\delta}{\nu}\|\mathbf{u}(t)\|_\alpha^2, \tag{1}$$

where δ is a fixed parameter. The underlying physics dictates that α should have the dimension of length, while δ has no (physical) dimension. The ODE in (1) will henceforth be referred to as the auxiliary ODE associated with the solution $\mathbf{u}(\cdot)$.

We show that this ODE can be solved on a domain (connected, open set) $\mathcal{D}_\mathbf{u}$ of $\mathbb{R}^2_+$ ($= \{(t,\alpha) : t > 0, \alpha \geq 0\}$). The set $\mathcal{D}_\mathbf{u}$ can be defined as the maximal open set in $\mathbb{R}^2_+$ where the map $F(t,\alpha)$ is continuous and locally Lipschitz. Additionally, we show that this (open) set contains a neighborhood of $(0, t_\infty)$ and is simply connected, and its "upper" boundary is given by a lower-semicontinuous function of time. This function is precisely the maximal analyticity radius $\beta_c(\cdot)$ of the solution $\mathbf{u}(t)$ except possibly for a countable set (of time points). The existence time of (1) is now connected dynamically to the solution curve $\mathbf{u}(\cdot)$. These are the key differences with all previous approaches that we are aware of.

Since $F(t, \alpha)$ is locally Lipschitz on $\mathcal{D}_{\mathbf{u}}$, it follows from standard ODE theory that for initial data $(t_0, \alpha_0) \in \mathcal{D}_{\mathbf{u}}$, an unique solution to (1) exists on an open interval $(t_0 - \epsilon, t_0 + \epsilon)$. However, we also give an *ad hoc* proof that for points (t_0, α_0) belonging to the boundary of $\mathcal{D}_{\mathbf{u}}$ such that $F(t_0, \alpha_0) < \infty$, one can solve (1) uniquely on an adequate (one-sided) interval $[t_0, t_0 + \epsilon)$. Moreover, the solution $\alpha(t)$ of (1), which is clearly increasing in time, can be continued until it hits a boundary point of $\mathcal{D}_{\mathbf{u}}$. The left limit of $F(t, \alpha)$ along $\alpha(t)$ then must necessarily blow up at this boundary point. At these "blow-up" points, α reaches the "maximal analyticity" radius, and we expend considerable effort to extract information concerning the domain $\mathcal{D}_{\mathbf{u}}$ and the solution $\mathbf{u}(t)$. As we elaborate in the next paragraph, the regularity (or lack there of) of the boundary of $\mathcal{D}_{\mathbf{u}}$ is connected to certain physical properties of the flow.

The solution $\alpha(t)$ of (1) has several additional properties. For instance, if $t_1 < t_2$ is an arbitrary time interval, by suitably choosing δ large enough, we can make $\|\mathbf{u}(t)\|_{\alpha(t)}$ blowup in (t_1, t_2). This means that we can hit the boundary of $\mathcal{D}_{\mathbf{u}}$ by the solution curve $\alpha(t)$ between any two time points. Moreover, the norm $\|e^{\alpha(t)A^{1/2}}\mathbf{u}(t)\|_{L^2}$ increases monotonically in t along the solution curve. This fact yields also a new type of regularity test for the solutions of the 3D Navier–Stokes equations (see Remark 5). To the best of our knowledge, these results are new for the 3D NSE (however, see [28] for an earlier blow-up result in the complexified Hilbert space $L^2 \otimes \mathbb{C}$). As another consequence of those results, we show that if the Gevrey norm $\|\mathbf{u}\|_\alpha$ decreases on some interval, then there is an inverse cascade of energy.

In our analysis, the structure of the domain $\mathcal{D}_{\mathbf{u}}$, particularly its boundary, emerged as an important object. We show that certain features of the boundary, namely "Leray boundary points" and "Leray spikes" (see Sect. 7), are intermittent events and they are connected to inverse energy cascades. Relation between intermittency and complex singularities has been hypothesized before based on numerical and experimental evidence [17,34]. Here, we established some rigorous connections with the occurrence of spatial complex singularities. We hope that some experimental or numerical evidence of such inverse cascades associated with Leray boundary points or Leray spikes will be found. An alternate approach to intermittency, based on fluctuation of time averages of higher moments of vorticity, has recently been developed in [19]. Since higher moments are bounded by higher derivatives, and Gevrey norms "contain" all derivatives, these approaches may be related. It is also noteworthy that the suggested scenario leading to intermittency in [19] involves fluctuations in higher moments of vorticity. In our case, fluctuation in Gevrey norms along the solution curves of (1) leads to inverse energy cascade. Unlike in two dimensions, in three-dimensional setting, inverse energy cascade is an intermittent phenomenon.

The paper is organized as follows. In Sect. 2, we establish our notation and setting, while in Sect. 3, we recall preliminary estimates and results from [16] that we need. In Sects. 4 and 5, we study properties of the domain $\mathcal{D}_{\mathbf{u}}$ and of the solution to (1). In Sect. 6, we establish the previously mentioned blow-up result and in Sect. 7, we provide a connection between boundary features of $\mathcal{D}_{\mathbf{u}}$ and intermittency. In Sect. 8, we derive an integral formula for the solution of (1), and in Sect. 9, we derive an estimate of the analyticity radius for the entire interval of existence $(0, t_\infty)$ of the solution $\mathbf{u}(\cdot)$. Finally, in Sect. 10, we provide a connection between the behavior of $\mathbf{u}(\cdot)$ along the solution curves of (1) and a possible inverse energy cascade for the three-dimensional

Navier–Stokes equations. We tried to make the presentation as self-contained as possible.

2 Preliminaries

First, we define the functional spaces for our study (for details we refer to [33] or [10]). Let $\Omega = [0, L]^3$ and denote by $L^2(\Omega)$ the Hilbert space of all $L-$periodic functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ that are square integrable on Ω with respect to the Lebesgue measure. We denote by H the set of all real-valued, divergence-free, periodic functions $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ with period L that are in $L^2(\Omega)$ and have mean zero (i.e., $\int_\Omega \mathbf{u} = 0$). The space H is a Hilbert space with the inner product inherited from $L^2(\Omega)$ and is the closure (in $L^2(\Omega)$) of trigonometric polynomials $P(x)$ of the form

$$P(x) = \sum_{|k|\le N, k\in\mathbb{Z}^3} a_k e^{\frac{2\pi\iota}{L} k\cdot x}, a_k \in \mathbb{C}^3, k \cdot a_k = 0, a_0 = 0, a_{-k} = \bar{a}_k.$$

Here, $k \cdot x$ denotes the usual dot product, that is, $k \cdot x = k_1x_1 + k_2x_2 + k_3x_3$. Denote by $\widetilde{\mathbb{Z}} = \mathbb{Z}\setminus\{0\}$. Since each $\mathbf{u} \in H$ is a periodic function belonging to $L^2(\Omega)$ with range included in $\mathbb{R}^3$ and is divergence free with zero space average, it admits a Fourier expansion of the form

$$\mathbf{u}(x) = \sum_{k\in\widetilde{\mathbb{Z}}^3} \mathbf{u}_k e^{\frac{2\pi\iota}{L} k\cdot x}, k \cdot \mathbf{u}_k = 0, \mathbf{u}_{-k} = \bar{\mathbf{u}}_k,$$

where $\mathbf{u}_k = \hat{\mathbf{u}}(k) = (\frac{1}{L})^3 \int_\Omega \mathbf{u}(x) e^{-\frac{2\pi\iota}{L} k\cdot x}\, \mathrm{d}x.$

For $\mathbf{u} \in H$, we will denote by $|\mathbf{u}|$ its L^2 norm, that is, $|\mathbf{u}|^2 = \int_\Omega |\mathbf{u}(x)|^2\, \mathrm{d}x = \left(\frac{L}{2\pi}\right)^3 \sum |\mathbf{u}_k|^2$, the last relation being Parseval's equality. We will also denote by $(\mathbf{u}, \mathbf{v}) := \int_\Omega \mathbf{u}(x) \cdot \bar{\mathbf{v}}(x)\, \mathrm{d}x$ the corresponding L^2 inner product. In addition to denoting the norm in H, by an abuse of notation, $|\cdot|$ will also denote the absolute value of a real number as well as the standard Euclidean norm in $\mathbb{R}^3$ or $\mathbb{C}^3$; precisely which usage is meant will be clear from context. The space H is a closed subspace of $L^2(\Omega)$. The orthogonal projection on H is denoted by $\mathbb{P}$ and is known as the Helmholtz–Leray projection operator. In terms of the Fourier coefficients, it is given by

$$(\mathbb{P}\mathbf{u})_k = \mathbf{u}_k - \frac{k}{|k|^2} (k \cdot \mathbf{u}_k).$$

Additionally, for any $N \in \mathbb{N}$, we will denote by P_N the Galerkin projection on H defined by

$$(P_N\mathbf{u})(x) = \sum_{k\in\widetilde{\mathbb{Z}}^3, |k|\le N} \mathbf{u}_k e^{\frac{2\pi\iota}{L} k\cdot x}.$$

Note that the projections P_N are increasing ($P_N \le P_{N+1}$) and $P_N \to I$ strongly.

We also introduce the space $V \subset H$, a dense subspace, defined by the property

$$V = \left\{ \mathbf{u} \in H : \|\mathbf{u}\|^2 := \sum_{k\in\widetilde{\mathbb{Z}}^3} \frac{4\pi^2}{L^2} |k|^2 |\mathbf{u}_k|^2 < \infty \right\}.$$

V is also a Hilbert space with norm $\|\cdot\|$. The inner product in V is

$$((\mathbf{u},\mathbf{v})) := \frac{4\pi^2}{L^2}\sum_{k\in\widetilde{\mathbb{Z}}^3}|k|^2\mathbf{u}_k\cdot\bar{\mathbf{v}}_k = \sum_{i=1}^{3}(\partial_i\mathbf{u},\partial_i\mathbf{v}),$$

where ∂_i denotes the partial derivative with respect to the space variable x_i, $i=1,2,3$. It is well known that by Poincaré inequality, the norm $\|\mathbf{u}\|$ is equivalent to the standard $\mathbb{H}^1$ norm.

Let Δ denote the Laplacian. It is easy to see that the operator Δ maps the space $D(\Delta)\cap H$ onto H. Henceforth, we will also denote $A=(-\Delta)|D(\Delta)\cap H$. The operator A is self-adjoint and positive, and its domain comprises precisely of $\mathbf{u}\in H$ such that $|A\mathbf{u}|^2 := \sum|k|^4|\mathbf{u}_k|^2 < \infty$. In terms of Fourier coefficients, the operator A can be expressed as

$$(A\mathbf{u})_k = \left(\frac{2\pi}{L}\right)^2|k|^2\mathbf{u}_k,\ k\in\widetilde{\mathbb{Z}}^3.$$

Clearly, the space V is in the domain of the operator $A^{1/2}$, and moreover, $\|\mathbf{u}\|=|A^{1/2}\mathbf{u}|$ for $\mathbf{u}\in V$. Henceforth, we will denote by

$$\kappa_0 = \frac{2\pi}{L}$$

the lowest eigenvalue of the operator $A^{1/2}$. Note also that the projections P_N commute with (any power of) A.

Remark 1 We recall that for any (densely defined) self-adjoint operator S on H that commutes with P_N, and satisfies the condition $|SP_N\mathbf{v}|\le C$ for some $\mathbf{v}\in H$ and for all $N\in\mathbb{N}$, we have $\mathbf{v}\in D(S)$ and $SP_N\mathbf{v}$ converges (in H) to $S\mathbf{v}$.

3 Specific preliminaries

We consider the Navier–Stokes equations of a viscous incompressible fluid in $\mathbb{R}^3$ with L-periodic boundary condition in the space variables, and with potential force. The unknown function is the vector-valued velocity

$$\mathbf{u}(t)=\mathbf{u}(\mathbf{x},t)=(u_1(\mathbf{x},t),u_2(\mathbf{x},t),u_3(\mathbf{x},t)),\ \mathbf{x}\in\mathbb{R}^3,\ t\ge 0,$$

which is also divergence free (i.e., $\nabla\cdot\mathbf{u}=0$).

When projected on the divergence-free vector fields, the equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t)+\nu A\mathbf{u}(t)+B(\mathbf{u}(t),\mathbf{u}(t))=0,\ \nabla\cdot\mathbf{u}(t)=0,\ \mathbf{u}(0)=\mathbf{u}_0\in V. \tag{2}$$

Here,

$$B(\mathbf{u},\mathbf{v})=\mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{v} \text{ and } A\mathbf{u}=(-\Delta)\mathbf{u},\ (\mathbb{P}=\text{Leray projector}).$$

We will now introduce the Gevrey norms (see [FT]). For $\alpha\ge 0$ and $\mathbf{u}\in D(e^{\alpha A^{1/2}})$ and $\mathbf{v}\in D(A^{1/2}e^{\alpha A^{1/2}})$, denote

$$|\mathbf{u}|_\alpha = |e^{\alpha A^{1/2}}\mathbf{u}| \text{ and } \|\mathbf{v}\|_\alpha = |A^{1/2}e^{\alpha A^{1/2}}\mathbf{v}|. \tag{3}$$

We will need the following inequality (see [FT]):

$$|(B(\mathbf{u},\mathbf{v}),A^{2\beta}e^{2\alpha A^{1/2}}\mathbf{w})|\le c_0|A^{1/2}\mathbf{u}|_\alpha|A^{\beta+\frac{1}{4}}\mathbf{v}|_\alpha|A^{\beta+\frac{1}{2}}\mathbf{w}|_\alpha,\ \alpha\ge 0,\beta=0,1/4. \tag{4}$$

In (4) above, and henceforth in the paper, we will denote by $c_0, c_1, \ldots$ various absolute constants that are nondimensional and do not depend on the parameters in our problem like ν, L, δ etc. When $\beta = \frac{1}{2}$, by using Agmon's inequality, one obtains the following estimate concerning the bilinear map B (see Lemma 2.1 in [FT]):

$$|(B(\mathbf{u}, \mathbf{v}), e^{2\alpha A^{1/2}} A\mathbf{w})| \leq c_0 \|\mathbf{u}\|_\alpha^{1/2} |A\mathbf{u}|_\alpha^{1/2} \|\mathbf{v}\|_\alpha |A\mathbf{w}|_\alpha. \tag{5}$$

This in particular implies that B can be extended as a (bounded) bilinear map $B : D(Ae^{\alpha A^{1/2}}) \times D(A^{1/2} e^{\alpha A^{1/2}}) \to D(e^{\alpha A^{1/2}})$. Also, using Parseval equality and the fact that $\sup_{x>0} x^m e^{-\eta x} < \infty$, for all $\mathbf{u} \in H$ and $m > 0$, $\eta > 0$, we have

$$|A^{m/2} e^{-\eta A^{1/2}} \mathbf{u}| \leq e^{-m} m^m \frac{1}{\eta^m} |\mathbf{u}|. \tag{6}$$

We will need the following lemma.

Lemma 3.1 *Let* $\alpha_1 \geq \alpha_2 \geq 0$ *and* $|A^{3/4}\mathbf{u}|_{\alpha_1} < \infty$. *Then*

$$\begin{aligned}\max\{(1 - e^{-2(\alpha_1-\alpha_2)\kappa_0}) \|\mathbf{u}\|_{\alpha_1}^2, 2\kappa_0(\alpha_1 - \alpha_2)\|\mathbf{u}\|_{\alpha_2}^2\} &\leq \|\mathbf{u}\|_{\alpha_1}^2 - \|\mathbf{u}\|_{\alpha_2}^2 \\ &\leq 2\,(\alpha_1 - \alpha_2)\,|A^{3/4}\mathbf{u}|_{\alpha_1}^2. \end{aligned}\tag{7}$$

Proof Let $\widetilde{\mathbb{Z}}^3 = \mathbb{Z}^3 \setminus \{0\}$ and note that

$$\begin{aligned}\|\mathbf{u}\|_{\alpha_1}^2 - \|\mathbf{u}\|_{\alpha_2}^2 &= \sum_{k\in\widetilde{\mathbb{Z}}^3} \left(e^{2\alpha_1\kappa_0|k|} - e^{2\alpha_2\kappa_0|k|}\right) \kappa_0^2 |k|^2 |\mathbf{u}_k|^2 \\ &= \sum_{k\in\widetilde{\mathbb{Z}}^3} \left(1 - e^{-2(\alpha_1-\alpha_2)\kappa_0|k|}\right) e^{2\alpha_1\kappa_0|k|} \kappa_0^2 |k|^2 |\mathbf{u}_k|^2. \end{aligned}\tag{8}$$

Note now that for all $k \in \widetilde{\mathbb{Z}}^3$, we also have

$$1 - e^{-2(\alpha_1-\alpha_2)\kappa_0} \leq 1 - e^{-2(\alpha_1-\alpha_2)\kappa_0|k|} \leq 2(\alpha_1 - \alpha_2)\kappa_0|k|, \tag{9}$$

where the first inequality in the line above follows from the fact that $|k| \geq 1$, while the second follows from the elementary fact $1 - e^{-x} \leq x$ $(x \geq 0)$. Inserting these estimates in (8) and using Poincaré inequality, we obtain

$$(1 - e^{-2(\alpha_1-\alpha_2)\kappa_0}) \|\mathbf{u}\|_{\alpha_1}^2 \leq \|\mathbf{u}\|_{\alpha_1}^2 - \|\mathbf{u}\|_{\alpha_2}^2 \leq 2\,(\alpha_1 - \alpha_2)\,|A^{3/4}\mathbf{u}|_{\alpha_1}^2.$$

On the other hand, we also have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \|\mathbf{u}\|_\alpha^2 = 2|A^{3/4}\mathbf{u}|_\alpha^2 \geq 2\kappa_0 \|\mathbf{u}\|_\alpha^2.$$

Integrating this inequality (with respect to α) between α_2 to α_1 and using the fact that for all $\beta \in [\alpha_2, \alpha_1]$, we have $\|\mathbf{u}\|_\beta^2 \geq \|\mathbf{u}\|_{\alpha_2}^2$, we obtain

$$\|\mathbf{u}\|_{\alpha_1}^2 - \|\mathbf{u}\|_{\alpha_2}^2 \geq 2\kappa_0(\alpha_1 - \alpha_2)\|\mathbf{u}\|_{\alpha_2}^2.$$

This finishes the proof. □

Recall that our initial data $\mathbf{u}_0$ in (2) belongs to V. We will assume that a regular solution of (2) exists up to time $0 < t_\infty \leq \infty$. This means for all $T < t_\infty$, there exists a unique Leray–Hopf weak solution of (2) on $[0, T]$; moreover, it satisfies $\sup_{0\leq t\leq T} \|u(t)\| < \infty$. Furthermore, since the force in (2) is potential, for any weak solution $\mathbf{u}$ we know from Leray's result that $\lim_{t\to\infty} \|\mathbf{u}(t)\| = 0$. This implies that $\lim_{t\nearrow t_\infty} \|u(t)\| = \infty$ if and only if $t_\infty < \infty$.

We recall the following result analogous to [16] with a sketch of the proof. The existence time (10) below is slightly different from [16]. The one presented here can also be found in [11]. It should be mentioned that all computations leading to the relevant estimates in the proof of this result are formal. As usual, the proof of the computations and the estimates can be made rigorous by considering a Galerkin approximation based on the eigenfunctions of A (see [10]).

Proposition 3.2 *Let $\alpha_0 \geq 0$, $\gamma \geq 0$ and $0 \leq t_0 \leq t$. If $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$, then*

$$\sup_{s\in[t_0,\,t]} \|\mathbf{u}(s)\|^2_{\alpha_0+\gamma s} < \infty \text{ if } t < t_0 + \frac{\nu}{\gamma^2}\ln\left(1+\frac{\nu^2\gamma^2}{c_1\|\mathbf{u}(t_0)\|^4_{\gamma t_0}}\right), \text{ where } c_1 = \frac{27c_0^4}{16}. \tag{10}$$

Moreover, in case the smallness conditions $\|\mathbf{u}_0\|_{\alpha_0} < \frac{\nu\kappa_0^{1/2}}{4c_0}$ and $\gamma < \frac{\nu\kappa_0}{2}$ hold, we have

$$\|\mathbf{u}(t)\|^2_{\alpha_0+\gamma t} \leq \|\mathbf{u}_0\|^2_{\alpha_0} e^{-\frac{1}{2}\nu\kappa_0^2 t} \text{ for all } t > 0. \tag{11}$$

Proof For notational simplicity, we will only prove this result for $\alpha_0 = 0$. Taking the inner product (in H) of (2) with $e^{2\gamma t A^{1/2}}\mathbf{u}(\cdot)$ and using (5), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{\gamma t} + \nu|A\mathbf{u}|^2_{\gamma t} &\leq c_0\|\mathbf{u}\|^{3/2}_{\gamma t}|A\mathbf{u}|^{3/2}_{\gamma t} + \gamma\|\mathbf{u}\|_{\gamma t}|A\mathbf{u}|_{\gamma t} \\ &\leq \frac{27c_0^4}{32\nu^3}\|\mathbf{u}\|^6_{\gamma t} + \frac{\gamma^2}{2\nu}\|\mathbf{u}\|^2_{\gamma t} + \nu|A\mathbf{u}|^2_{\gamma t}.\end{aligned} \tag{12}$$

Letting $\xi(t) = \|\mathbf{u}(t)\|^2_{\gamma t}$, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi \leq \frac{c_1}{\nu^3}\xi^3 + \frac{\gamma^2}{\nu}\xi.$$

This immediately yields

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}\left(\frac{1}{\xi} - \frac{(c_1/\nu^3)\xi}{(c_1/\nu^3)\xi^2 + (\gamma^2/\nu)}\right) \leq \frac{\gamma^2}{\nu}.$$

Integrating this differential inequality between t_0 to t, and rearranging we obtain

$$\xi^2\left(1 - \Xi_0^2 e^{\frac{2\gamma^2}{\nu}(t-t_0)}\right) \leq \frac{\nu^2\gamma^2}{c_1}\Xi_0^2 e^{2\frac{\gamma^2}{\nu}(t-t_0)},\quad \Xi_0 := \left(\frac{\frac{c_1}{\nu^3}\xi_0^2}{\frac{c_1}{\nu^3}\xi_0^2 + \frac{\gamma^2}{\nu}}\right)^{1/2}.$$

This yields (10).

We will now prove (11). Note that from (12), using Poincaré inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{\gamma t} + |A\mathbf{u}|^2_{\gamma t}(\nu - \frac{c_0}{\kappa_0^{1/2}}\|\mathbf{u}\|_{\gamma t} - \frac{\gamma}{\kappa_0}) \leq 0. \tag{13}$$

If the smallness condition holds at $t = 0$, then the same is true on an interval $[0, \epsilon]$ for some $\epsilon > 0$. From (13), this means that $\|\mathbf{u}(t)\|_{\gamma t}$ is nonincreasing on $[0, \epsilon]$ and therefore $\|\mathbf{u}(\epsilon)\|_{\gamma\epsilon} \leq \|\mathbf{u}_0\| < \frac{\nu\kappa_0^{1/2}}{4c_0}$. By repeating this argument, for all $t > 0$, we have $\|\mathbf{u}(t)\|_{\gamma t} \leq \|\mathbf{u}_0\| < \frac{\nu\kappa_0^{1/2}}{4c_0}$. From (13), we now immediately obtain (11). □

We will need some elementary facts which are collected in the following lemma.

Lemma 3.3 *Let c_1 be as in (10). For $0 \le t_0 < \infty$, define*

$$\tau(\gamma; t_0, \alpha_0) = \begin{cases} \frac{\nu}{3\gamma^2} \ln\left(1 + \frac{\nu^2\gamma^2}{c_1 \|\mathbf{u}(t_0)\|_{\alpha_0}^4}\right) & \text{for } 0 < \gamma < \infty \\ \frac{\nu^3}{3c_1 \|\mathbf{u}(t_0)\|_{\alpha_0}^4} & \text{for } \gamma = 0 \\ 0 & \text{for } \gamma = \infty. \end{cases}$$

The parametric curve

$$\{(t, \alpha) \in \mathbb{R}^2 : t = t_0 + \tau(\gamma; t_0, \alpha_0), \alpha = \alpha_0 + \gamma\tau(\gamma; t_0, \alpha_0), \gamma \in (0, \infty)\}$$

can be represented by a function $\alpha = \ell(t; t_0, \alpha_0)$, $t_0 < t < t_0 + \tau(0; t_0, \alpha_0)$ which has the following properties:

(i) *The function $\ell(\cdot; t_0, \alpha_0)$ is a continuous function on the compact set $[t_0, t_0+\tau(0; t_0, \alpha_0)]$ and satisfies $\ell(t_0; t_0, \alpha_0) = \ell(t_0 + \tau(0; t_0, \alpha_0); t_0, \alpha_0) = \alpha_0$.*
(ii) *$\ell(\cdot; t_0, \alpha_0)$ is continuously differentiable on $(t_0, t_0 + \tau(0; t_0, \alpha_0))$ and concave, and has vertical tangent lines at $t = t_0$ and $t = t_0 + \tau(0; t_0, \alpha_0)$. More precisely,*

$$\lim_{t \searrow t_0^+} \ell'(t; t_0, \alpha_0) = \infty, \quad \lim_{t \nearrow (t_0+\tau(0;t_0,\alpha_0))^-} \ell'(t; t_0, \alpha_0) = -\infty.$$

(iii) *Let $\zeta_0 > 0$ be the unique number satisfying the equation $2\zeta_0 - (1 + \zeta_0)\ln(1 + \zeta_0) = 0$, i.e., $\zeta_0 \approx 3.9216$. The function $\ell(\cdot; t_0, \alpha_0)$ has a unique maximum at t_{max} with the maximum value $\ell(t_{max}; t_0, \alpha_0)$ given by*

$$t_{max} = t_0 + \frac{2}{(1 + \zeta_0)} \frac{\nu^3}{3c_1 \|\mathbf{u}(t_0)\|_{\alpha_0}^4}, \quad \ell(t_{max}; t_0, \alpha_0) = \alpha_0 + \frac{2}{3\sqrt{c_1}(1 + \zeta_0)} \frac{\nu^2}{\|\mathbf{u}(t_0)\|_{\alpha_0}^2}.$$

4 An auxiliary ODE for the analyticity radius

In this paper, our main tool in the study of the radius of analyticity of a regular solution $\mathbf{u}(t)$ is the ODE (1). One of our early objectives is to find a domain $\mathcal{D}$, adjacent to the maximal interval of regularity $(0, t_\infty)$ of the solution $\mathbf{u}(\cdot)$, where this ODE is defined and its right hand side is a continuous function (jointly in both variables) on $\mathcal{D}$ and locally Lipschitz continuous in the variable α. The solution of (1) will be denoted by $\alpha_\delta(t; t_0, \alpha_0)$. In case δ is fixed, for notational simplicity, we will omit it and write $\alpha(t; t_0, \alpha_0)$ instead of $\alpha_\delta(t; t_0, \alpha_0)$. If δ, t_0, α_0 are all fixed in a discussion, then we will simply write $\alpha(t)$ instead of $\alpha_\delta(t; t_0, \alpha_0)$. We will start with a lemma that we will need later.

Lemma 4.1 *Let $I \subset [0, t_\infty)$ be an interval such that*

$$M_\beta(I) = \sup\{\|e^{\beta A^{1/2}}\mathbf{u}(t)\| : t \in I\} < \infty. \tag{14}$$

Denoting $G(t, \alpha) = A^{1/2}e^{\alpha A^{1/2}}\mathbf{u}(t)$ and $f(t, \alpha) = \|e^{\alpha A^{1/2}}\mathbf{u}(t)\|^2$, we have the following:

(i) *$G(t, \alpha) \in \mathcal{D}(A^\gamma)$ for all $\gamma > 0$ and $(t, \alpha) \in I \times [0, \beta)$.*
(ii) *The functions $G(t, \alpha)$ and $f(t, \alpha)$ are jointly continuous in both variables on $I \times [0, \beta)$ and locally Lipschitz in α.*

Proof For (i) we simply note that $G(t, \alpha)$ belongs to $\mathcal{D}(e^{\epsilon A^{1/2}})$ where $\epsilon = \beta - \alpha$. We will focus on (ii). We will first prove the continuity of G in strong topology in H jointly in both variables. Let $(t_n, \alpha_n) \to (t', \alpha') \in I \times [0, \beta)$. Since $t \to \mathbf{u}(t)$, $t \in I$ is strongly continuous in both H and V, we must have $\lim_{n\to\infty} A^{1/2}\mathbf{u}(t_n) = A^{1/2}\mathbf{u}(t')$. Due to (14), $\sup_n |G(t_n,\alpha_n)| < \infty$ and $|A^{1/2}e^{\alpha' A^{1/2}}\mathbf{u}(t')| < \infty$. Therefore, $G(t_n,\alpha_n)$ is weakly compact in H. On the other hand, it is easy to see that $G(t,\alpha)$ is weakly continuous on $I\times[0,\beta)$ with values in H. This implies that any weak limit is unique, and in particular, $w - \lim_{n\to\infty} G(t_n,\alpha_n) = A^{1/2}e^{\alpha' A^{1/2}}\mathbf{u}(t')$. We will show that this convergence is in fact strong in H. From (14), it follows that with $\epsilon = \dfrac{\beta-\alpha'}{3}$, $G(t_n,\alpha_n)$ belongs to $\mathcal{D}(e^{\epsilon A^{1/2}})$ for n large enough, and moreover, $\lim\sup_n |e^{\epsilon A^{1/2}}G(t_n,\alpha_n)| < \infty$. Since the operator $e^{-\epsilon A^{1/2}}$ is a compact operator on H, $G(t_n,\alpha_n)$ converges strongly in H to $A^{1/2}e^{\alpha' A^{1/2}}\mathbf{u}(t')$.

We will now show the local Lipschitz property. Let $t\in I$, $\alpha_1\le\alpha_2\le\alpha<\beta$. Using the notation $\hat{\mathbf{u}}(k) = \mathbf{u}_k$ and recalling that $\kappa_0 = \frac{2\pi}{L}$ denotes the lowest eigenvalue of $A^{1/2}$, we have

$$\|e^{\alpha_2 A^{1/2}}\mathbf{u}(t) - e^{\alpha_1 A^{1/2}}\mathbf{u}(t)\|^2 = \sum_{k\in\mathbb{Z}^3}\left(e^{\alpha_2\kappa_0|k|} - e^{\alpha_1\kappa_0|k|}\right)^2 \kappa_0^2|k|^2|\mathbf{u}_k|^2$$

$$= \sum_{k\in\mathbb{Z}^3}\left(1-e^{-(\alpha_2-\alpha_1)\kappa_0|k|}\right)^2 e^{2\alpha_1\kappa_0|k|}\kappa_0^2|k|^2|\mathbf{u}_k|^2 \le 4(\alpha_2-\alpha_1)^2\sum_{k\in\mathbb{Z}^3} e^{2\alpha_1\kappa_0|k|}\kappa_0^4|k|^4|\mathbf{u}_k|^2$$

$$\le 4(\alpha_2-\alpha_1)^2\sum_{k\in\mathbb{Z}^3} e^{-2(\beta-\alpha)\kappa_0|k|}e^{2\beta\kappa_0|k|}\kappa_0^4|k|^4|\mathbf{u}_k|^2 \le \frac{4}{e^2}\frac{M_\beta^2(I)}{(\beta-\alpha)^2}(\alpha_2-\alpha_1)^2.$$

Here, the inequality in the second line above follows from (9) while the second inequality in the last line above follows by noting

$$\sup_{k\in\mathbb{Z}^3}\kappa_0^2|k|^2e^{-2(\beta-\alpha)\kappa_0|k|} \le \frac{1}{(\beta-\alpha)^2e^2}.$$

This shows that G is locally Lipschitz in α.

The statements concerning f follow immediately from the corresponding ones established for G. □

Thus far, it is not clear whether or not the Eq. (1) has any solution. We will introduce a set $\mathcal{D} = \mathcal{D}_{\mathbf{u}}$ in the plane where this equation can be solved uniquely and which will play a central role in our study.

Definition 4.2 Denote $\mathbb{R}_+^2 = \{(t,\alpha) : t>0, \alpha\ge 0\}$ and let

$$\begin{aligned}\mathcal{D} :&= \{(t,\alpha)\in\mathbb{R}_+^2 :\ \exists\epsilon>0 \text{ with } \sup_{s\in[t-\epsilon,t+\epsilon]}\|\mathbf{u}(s)\|_{\alpha+\epsilon}<\infty\}\\ &= \{(t,\alpha)\in\mathbb{R}_+^2 :\ \exists\epsilon>0 \text{ with } \sup_{s\in[t-\epsilon,t]}\|\mathbf{u}(s)\|_{\alpha+\epsilon}<\infty\}.\end{aligned}\tag{15}$$

The second equality in the definition of $\mathcal{D}$ in (15) follows immediately from (10). The fact that $\mathcal{D}$ is not empty and abuts $(0,t_\infty)$ is established in the theorem below in which we will also collect some elementary properties of $\mathcal{D}$. We will need the following notation to proceed. For $(t_0,\alpha_0)\in\mathbb{R}_+^2$ such that $\|\mathbf{u}(t_0)\|_{\alpha_0}<\infty$, and with the functions τ and ℓ as defined in Lemma 3.3, let

$$\Gamma_{(t_0,\alpha_0)} = \{(t,\alpha)\in\mathbb{R}_+^2 : t\in(t_0, t_0+\frac{\nu^3}{3c_1\|\mathbf{u}(t_0)\|_{\alpha_0}^4}), \alpha_0<\alpha<\ell(t;t_0,\alpha_0)\}.\tag{16}$$

It will also be convenient to denote the interior $\mathcal{D}^o$ (in the topology of the whole space $\mathbb{R}^2$) of $\mathcal{D}$ and the "upper boundary" $\partial^+\mathcal{D}$ of $\mathcal{D}$ by

$$\mathcal{D}^o = \{(t,\alpha) \in \mathcal{D} : \alpha > 0\} \text{ and } \partial^+\mathcal{D} = \overline{\mathcal{D}} \setminus (\mathcal{D} \cup \{(s,0) : 0 \le s \le t_\infty\}), \tag{17}$$

where the closure $\overline{\mathcal{D}}$ is taken in $\mathbb{R}^2$.

Theorem 4.3 *The set $\mathcal{D}$ has the following properties stated below.*

(i) *$\mathcal{D}$ is the maximal open subset of $\mathbb{R}^2_+$ on which the function $f(t,\alpha)$ is defined and continuous. Moreover, f is also a locally Lipscitz on $\mathcal{D}$.*
(ii) *If $(t_0,\alpha_0) \in \mathcal{D}$, then there exists an $\epsilon > 0$ such that the rectangle $\{(t,\alpha) : t \in [t_0-\epsilon, t_0+\epsilon], \alpha \in [0,\alpha_0+\epsilon]\} \subset \mathcal{D}$. In particular, this means that $\mathcal{D}$ is simply connected.*
(iii) *If (t_0,α_0) is such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$, then $(t_0,\alpha_0) \in \overline{\mathcal{D}}$ and*

$$\Gamma_{(t_0,\alpha_0)} \subset \mathcal{D}^0\,,\ \widetilde{\Gamma}_{(t_0,\alpha_0)} := \Gamma_{(t_0,\alpha_0)} \cup \{(t,\alpha_0) : t \in (t_0, t_0 + \frac{\nu^3}{3c_1\|\mathbf{u}(t_0)\|^4_{\alpha_0}})\} \subset \mathcal{D}.$$

(iv) *The set $\{(t,0) : t \in (0,t_\infty)\} \subset \mathcal{D}$.*
(v) *If $\gamma : (t_0,T) \to (0,\infty)$ is a continuous function such that for all $T_1 < T$*

$$\sup_{(t_0,T_1)} \|\mathbf{u}(t)\|_{\gamma(t)} < \infty,$$

then the set $\{(t,\gamma') : t_0 < t < T, 0 \le \gamma' < \gamma(t)\} \subset \mathcal{D}$.
(vi) *Assume that $T := \frac{32c_0^2|\mathbf{u}_0|^2}{\nu^3\kappa_0} < t_\infty$ and let $0 < \gamma < \frac{\nu\kappa_0}{2}$. In this case, the domain $\mathcal{D}$ contains the triangular set $\{(t,\alpha) : t > T, 0 \le \alpha < \gamma(t-T)\}$.*

Proof Clearly, $\mathcal{D}^o$ is open in $\mathbb{R}^2$ and $\mathcal{D}$ is open in $\mathbb{R}^2_+$. We will now prove maximality. Let $\mathcal{D}'$ be the maximal domain in $\mathbb{R}^2_+$ on which f is continuous. If $(t_0,\alpha_0) \in \mathcal{D}'$, there exists an $\epsilon > 0$ such that the square $R = \{|t-t_0| \le \epsilon, |\alpha-\alpha_0| \le \epsilon\} \subset \mathcal{D}'$. In this case, f, being continuous, is bounded in the square which in particular implies $\sup_{s\in[t_0-\epsilon,t_0+\epsilon]} \|\mathbf{u}(s)\|_{\alpha_0+\epsilon} < \infty$. Consequently, $(t_0,\alpha_0) \in \mathcal{D}$, that is, $\mathcal{D}' \subset \mathcal{D}$. Conversely, Lemma 4.1 implies that f is continuous on $\mathcal{D}$. The fact that f is locally Lipschitz on $\mathcal{D}$ also follows immediately from Lemma 4.1.

Note that the first part of (ii) follows from Lemma 4.1. The second part follows from the first part by noting any point in $(t_0,\alpha_0) \in \mathcal{D}$ can be connected within $\mathcal{D}$ to $(t_0,0)$ by a vertical line.

We will now prove (iii). We will take $(t_0,\alpha_0) = (0,0)$ for notational simplicity; the general case is similar. Denote $\ell(t) = \ell(t;0,0)$. Let $(t',\alpha') \in \Gamma_{(0,0)}$. Then, $\alpha' < \ell(t';0)$, and moreover, from the parametric representation, there exists γ' such that $t' = \tau(\gamma')$, $\ell(t';0) = \gamma' t'$. From the definition of $\tau(\gamma')$ and (10), it follows that $\sup_{s\in[0,1.2t']} \|\mathbf{u}(s)\|_{s\gamma'} \le K < \infty$. Since $\alpha' < \gamma' t'$, this means that there exists $\epsilon > 0$ such that the closed rectangle $\{(t,\alpha) : |t-t'| \le \epsilon, |\alpha-\alpha'| \le \epsilon\}$ is contained in the open triangular region $\{(t,\alpha) : t \in (0,1.2t'), 0 < \alpha < \gamma' t\}$. Therefore, $\sup_{t\in[t'-\epsilon,t'+\epsilon]} \|\mathbf{u}(t)\|_{\alpha'+\epsilon} \le K$. This implies by (15) that $(t',\alpha') \in \mathcal{D}$. Thus, $\Gamma_{(t_0,\alpha_0)} \subset \mathcal{D}^o$. The fact that $\widetilde{\Gamma}_{(t_0,\alpha_0)} \subset \mathcal{D}$ now follows easily from part (ii).

Part (iv) follows from part (iii) by noting that

$$\{(t,0) : t \in (0,t_\infty)\} \subset \bigcup_{t\in(0,t_\infty)} \widetilde{\Gamma}_{(t,0)}.$$

To prove part (v), let $t_0 < t' < T$ and $\gamma' < \gamma(t')$ and let $\epsilon = \frac{\gamma(t')-\gamma'}{2}$. By continuity of $\gamma(\cdot)$, there exists $\delta > 0$ such that $(t'-\delta, t'+\delta) \subset (t_0, T)$, and for all $t \in (t'-\delta, t'+\delta)$, we have $\gamma(t) > \gamma' + \epsilon$. Consequently,

$$\sup_{t\in(t'-\delta,t'+\delta)} \|\mathbf{u}(t)\|_{\gamma'+\epsilon} \le \sup_{t\in(t'-\delta,t'+\delta)} \|\mathbf{u}(t)\|_{\gamma(t)} < \infty.$$

From Lemma 4.1, it follows that $(t', h') \in \mathcal{D}$.

Finally, we will prove part (vi). From Leray's energy inequality, for all $t_0 < t_\infty$, we have

$$\frac{1}{t_0}\int_0^{t_0} \|\mathbf{u}(t)\|^2 \mathrm{d}t \le \frac{|\mathbf{u}_0|^2}{\nu t_0}.$$

This implies that there exists $t_0 \le T$ such that $\|\mathbf{u}(t_0)\|^2 \le \frac{\nu^2\kappa_0}{16c_0^2}$, where T is as in part (vi). Invoking (11) now completes the proof. □

The theorem above shows that $\mathcal{D}$ contains a neighborhood of $(0, t_\infty)$ (in the topology of $\mathbb{R}^2_+$, where $\mathbb{R}^2_+$ is as in Definition 4.2). In the proposition below, we make this more precise.

Proposition 4.4 *There exists a continuously differentiable, strictly increasing function* $\theta : [\frac{2}{(1+\zeta_0)}\frac{\nu^3}{3c_1\|\mathbf{u}_0\|^4}, t_\infty) \to [0, t_\infty)$ *satisfying*

$$t = \theta(t) + \frac{2}{(1+\zeta_0)}\frac{\nu^3}{3c_1\|\mathbf{u}(\theta(t))\|^4} \ \textit{and} \ \lim_{t\nearrow t_\infty}\theta(t) = t_\infty, \tag{18}$$

where c_1 *is as in* (10) *and* ζ_0 *as in Lemma* 3.3. *Moreover, the set*

$$\{(t,\alpha) \in \mathbb{R}^2_+ : 0 \le \alpha < \frac{2\nu^2}{3\sqrt{c_1}(1+\zeta_0)\|\mathbf{u}(\theta(t))\|^2}, 0 < t < t_\infty\} \subset \mathcal{D}.$$

Proof Consider the function

$$\omega(t) = t + \frac{2}{(1+\zeta_0)}\frac{\nu^3}{3c_1\|\mathbf{u}(t)\|^4},$$

where ζ_0 is as defined in Lemma 3.3). Note that $\omega(0) = \frac{2}{(1+\zeta_0)}\frac{\nu^3}{3c_1\|\mathbf{u}_0\|^4}$ and $\lim_{t\nearrow t_\infty}\omega(t) = t_\infty$. The last equality clearly holds if $t_\infty = \infty$ since $\omega(t) \ge t$. On the other hand, if $t_\infty < \infty$, then it is well known that $\lim_{t\nearrow t_\infty}\|\mathbf{u}(t)\| = \infty$, and therefore, the claimed equality holds in this case as well. Now, a standard argument as in the proof of Proposition 3.2 (with $\gamma = 0$) yields

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2}{\|\mathbf{u}(t)\|^6} \le \frac{c_1}{8\nu^3}, \tag{19}$$

where c_1 is as in (10). Using (19), we immediately obtain that ω is continuously differentiable and $\omega'(t) \ge \frac{5}{6}$. Thus $\theta = \omega^{-1}$ satisfies the properties asserted in the proposition. From part (iii) of Theorem 4.3 with $\alpha_0 = 0$, we have $\Gamma_{(\theta(t),0)} \subset \mathcal{D}$. On the other hand, from part (iii) of Lemma 3.3, any point (t,α) with $0 \le \alpha \le \frac{2}{\sqrt{3}(1+\zeta_0)}(\nu\tau(0;\theta(t),0))^{1/2}$ belongs to $\Gamma_{(\theta(t),0)}$. □

Remark 2 The function $\theta(t)$ in the above theorem was first introduced in [6].

Since the function $f(t,\alpha)$ is locally Lipschitz in $\mathcal{D}$, we state below the classical Cauchy–Lipschitz result concerning the (local) existence of a solution to (1) in $\mathcal{D}$ [22].

Theorem 4.5 *Consider the ordinary differential equation* (1) *in $\mathcal{D}$. The following hold:*

(i) *If $(t_0,\alpha_0) \in \mathcal{D}^o$, then given $\delta \geq 0$, there exists $\epsilon > 0$ such that an unique solution to* (1) *exists on the interval $(t_0-\epsilon, t_0+\epsilon)$ with $(t,\alpha(t;t_0,\alpha_0))$ belonging to $\mathcal{D}$.*

(ii) *For $(t_0,\alpha_0) \in \mathcal{D}^o$, let $(T_m(t_0,\alpha_0), T_M(t_0,\alpha_0))$ be the maximal interval of existence of a solution and denote*

$$\lim_{t \nearrow t_0+T_M(t_0,\alpha_0)} \alpha(t;t_0,\alpha_0) := \alpha_M,\ \alpha_m := \lim_{t \searrow T_m(t_0,\alpha_0)} \alpha(t;t_0,\alpha_0).$$

Then, $\alpha_M \in (\alpha_0,\infty]$ and $\alpha_m \in [0,\alpha_0)$ and the points $(T_M(t_0,\alpha_0),\alpha_M)$ and $(t_0 - T_m(t_0,\alpha_0),\alpha_m)$ belong to the boundary of $\mathcal{D}$, where the point $(T_M(t_0,\alpha_0),\infty)$ is interpreted to be on the boundary of $\mathcal{D}$ if $\alpha_M = \infty$.

(iii) *Let (t_0,α_0) be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$. The point (t_0,α_0) belongs to $\overline{\mathcal{D}}$, the closure of $\mathcal{D}$, and moreover given $\delta \geq 0$, there exists $\epsilon > 0$ such that a solution to* (1) *exists on the interval $[t_0, t_0+\epsilon)$ with $(t,\alpha(t)) \in \mathcal{D}$ for all $t \in (t_0,t_0+\epsilon)$ and $\sup_{t\in[t_0,t_0+\epsilon)} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty$.*

(iv) *The set*

$$\{(t,\beta) : t_0 < t < T_M(t_0,\alpha_0), 0 \leq \beta \leq \alpha(t;t_0,\alpha_0,\delta)\} \subset \mathcal{D}^o.$$

Proof Since (t_0,α_0) and δ is fixed here, we will write $\alpha(t) = \alpha_\delta(t;t_0,\alpha_0)$. Due to Lemma 4.1, part (i) and (ii) follow from standard existence, uniqueness theory for ordinary differential equation (see, for instance [22]).

We will now prove (iii). Due to part (iii) of Theorem 4.3, the set $\Gamma_{(t_0,\alpha_0)} \subset \mathcal{D}$ and consequently, (t_0,α_0) belongs $\overline{\mathcal{D}}$. Now note that $\frac{\delta}{\nu}\|\mathbf{u}(t)\|_\alpha^2$ is a locally Lipschitz, continuous function in the open domain $\Gamma_{(t_0,\alpha_0)}$ defined in (16). Furthermore, recall that the function ℓ defined in Lemma 3.3 has the property that $\frac{\mathrm{d}}{\mathrm{d}t}\ell(t;t_0,\alpha_0)|_{t=t_0} = \infty$. Consequently, the vector field in the equation $\frac{\mathrm{d}}{\mathrm{d}t}\alpha = \frac{\delta}{\nu}\|\mathbf{u}(t)\|_{\alpha(t)}^2$ at $t=t_0$ points to the interior of the domain $\Gamma_{(t_0,\alpha_0)}$ for all $\delta \geq 0$ and moreover, $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$. Invoking standard local existence theory for the ODE completes the proof.

Part (iv) follows by noting that for each $t \in (t_0, T_M(t_0,\alpha_0))$, the norm $\|\mathbf{u}(t)\|_{\alpha(t)} < \infty$. This implies, by part (iii) of Theorem 4.3, that $\Gamma_{(t,\alpha(t))} \subset \mathcal{D}$. Consequently, the claim follows. □

5 More on the auxiliary ODE

We will present some more geometric properties of $\mathcal{D}$ in this section. We begin by establishing some estimates that will be necessary for our study.

Proposition 5.1 *Let (t_0,α_0) be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$, c_1 be as defined in* (10) *and $\alpha(t)(:= \alpha_\delta(t;t_0,\alpha_0))$ be a solution of* (1).

(i) *Let* $T > 0$ *be such that* $t_0 < T < t_0 + \frac{\nu^3}{2(c_1+\delta^2)\|u(t_0)\|_{\alpha_0}^4}$ *and assume that for all* t *satisfying* $t_0 < t < T$*, the points* $(t, \alpha(t)) \in \mathcal{D}$*. In this case, we have the estimate*

$$\|u(t)\|_{\alpha(t)}^2 \leq \frac{\|u(t_0)\|_{\alpha_0}^2}{\sqrt{1 - \frac{2(c_1+\delta^2)}{\nu^3}\|u(t_0)\|_{\alpha_0}^4 (t - t_0)}} \tag{20}$$

holds. Additionally, we also have

$$\int_{t_0}^{t} |Au(\tau)|_{\alpha(\tau)}^2 \, d\tau < \infty \text{ for all } t < T_M(t_0, \alpha_0). \tag{21}$$

(ii) *Let* $t_1 > t_0$ *be such that* $(t, \alpha(t)) \in \mathcal{D}$ *for all* $t_0 < t < t_1$*. Assume moreover that* $\alpha(t_1; t_0, \alpha_0) < \alpha_0 + \frac{\delta\nu^2}{(c_1+\delta^2)\|u(t_0)\|_{\alpha_0}^2}$*. Then we have the estimates*

$$\|u(t)\|_{\alpha(t)}^2 \leq \frac{\|u(t_0)\|_{\alpha_0}^2}{\sqrt{1 - \frac{(c_1+\delta^2)}{\delta\nu^2}\|u(t_0)\|_{\alpha_0}^2 (\alpha(t) - \alpha_0)}}, \; t_0 < t < t_1. \tag{22}$$

Proof Due to the assumption in part (i), $\alpha(t) \in \mathcal{D}$ for any $t_0 < t < T$ and by Lemma 4.1, the map $s \to e^{\alpha(s)A^{1/2}}u(s)$ defines a continuous function from the interval (t_0, T) into H. Consequently, for any compact subinterval $[t_1, t_2] \subset (t_0, T)$, there exists $\epsilon > 0$ such that $\sup_{t\in[t_1,t_2]} |\mathbf{u}(t)|_{\alpha(t)+\epsilon} < \infty$. Thus from (6), for any $[t_1, t_2] \subset (t_0, T)$ and $\beta > 0$, we have

$$\mathbf{u}(t) \in D(A^\beta e^{\alpha(t)A^{1/2}}), \; t \in [t_1, t_2] \text{ and } \sup_{t\in[t_1,t_2]} |A^\beta \mathbf{u}(t)|_{\alpha(t)} < \infty. \tag{23}$$

For any $N \in \mathbb{N}$, taking the inner product (in H) of (2) with $P_N e^{2\alpha(\cdot)A^{1/2}} Au(\cdot)$ we obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}|P_N A^{1/2}u(t)|_\alpha^2 + \nu|P_N Au|_\alpha^2 \\ &= -(B(u,u), P_N e^{2\alpha A^{1/2}}Au) + \dot{\alpha}(u, P_N e^{2\alpha(t)A^{1/2}}A^{\frac{3}{2}}u(t)) \\ &= -(e^{\alpha A^{1/2}}B(u,u), P_N e^{\alpha A^{1/2}}Au) + \tfrac{\delta}{\nu}\|u\|_\alpha^2 |P_N A^{\frac{3}{4}}u|_\alpha^2. \end{aligned} \tag{24}$$

The last equality is obtained by using the fact [see (5)] that the bilinear operator B extends as a continuous map $B : D(Ae^{\alpha A^{1/2}}) \times D(A^{1/2}e^{\alpha A^{1/2}}) \to D(e^{\alpha A^{1/2}})$. We would first like to establish this balance Eq. (24) without the projections P_N. Note that by (5), we have

$$|(e^{\alpha A^{1/2}}B(u,u), P_N e^{\alpha A^{1/2}}Au)| \leq c_0\|u\|_\alpha^{3/2}|Au|_\alpha^{3/2}.$$

In view of (23), for any t_1, t_2 with $t_0 < t_1 < t_2 < T$, we can integrate (24) and apply the dominated convergence theorem to get the relation

$$\begin{aligned} &\frac{1}{2}\left(\|\mathbf{u}(t_2)\|_{\alpha(t_2)}^2 - \|\mathbf{u}(t_1)\|_{\alpha(t_1)}^2\right) + \nu\int_{t_1}^{t_2} |A\mathbf{u}(\tau)|_{\alpha(\tau)}^2 \, \mathrm{d}\tau \\ &= \int_{t_1}^{t_2} -(e^{\alpha A^{1/2}}B(u,u), e^{\alpha A^{1/2}}Au) + \frac{\delta}{\nu}\|u\|_\alpha^2 |A^{\frac{3}{4}}u|_\alpha^2 \, \mathrm{d}\tau. \end{aligned} \tag{25}$$

Consequently, using (5) on the nonlinear term followed by Young's inequality, we obtain

$$\frac{1}{2}\left|\left(\|\mathbf{u}(t_2)\|^2_{\alpha(t_2)}-\|\mathbf{u}(t_1)\|^2_{\alpha(t_1)}\right)\right|+\nu\int_{t_1}^{t_2}|A\mathbf{u}(\tau)|^2_{\alpha(\tau)}\,\mathrm{d}\tau$$
$$\le \int_{t_1}^{t_2} c_0\|u\|_\alpha^{3/2}|Au|_\alpha^{3/2}+\frac{\delta}{\nu}\|u\|^3_\alpha|Au|_\alpha\mathrm{d}\tau$$
$$\le \int_{t_1}^{t_2}\frac{1}{\nu^3}(c_1+\delta^2)\|u\|^6_\alpha+\nu\int_{t_1}^{t_2}|A\mathbf{u}(\tau)|^2_{\alpha(\tau)}\,\mathrm{d}\tau$$

This immediately shows that the function $\|\mathbf{u}(t)\|^2_{\alpha(t)}$ is absolutely continuous and thus differentiable $a.e.$ on (t_0,T). We can now apply the fundamental theorem of calculus and rewrite (25) in integral form as

$$\frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|\mathbf{u}(\tau)\|^2_{\alpha(\tau)}\,\mathrm{d}\tau+\nu\int_{t_1}^{t_2}|A\mathbf{u}(\tau)|^2_{\alpha(\tau)}\,\mathrm{d}\tau$$
$$=\int_{t_1}^{t_2}\left(-(e^{\alpha A^{1/2}}B(u,u),e^{\alpha A^{1/2}}Au)+\frac{\delta}{\nu}\|u\|^2_\alpha|A^{\frac{3}{4}}u|^2_\alpha\right)\mathrm{d}\tau.$$

Since $t_1<t_2$ in (t_0,T) is arbitrary, from this equality we obtain the relation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^\beta u(t)|^2_\alpha+\nu|A^{\beta+\frac{1}{2}}u|^2_\alpha$$
$$=-(e^{\alpha A^{1/2}}B(u,u),e^{\alpha A^{1/2}}A^{2\beta}u)+\frac{\delta}{\nu}\|u\|^2_\alpha|A^{\beta+\frac{1}{4}}u|^2_\alpha, \tag{26}$$

where $\beta=1/2$. For later application, we note that a similar proof (but using (4) instead) works for $\beta=0,1/4$ as well.

From (26) with $\beta=1/2$ and proceeding as we did before by applying (5) followed by Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_\alpha\le\frac{1}{\nu^3}(c_1+\delta^2)\|u\|^6_\alpha. \tag{27}$$

Now integrating (27) between t_0 and t, we get (20).

On the other hand, if we introduce the definition of $\dot{\alpha}$ from (1) on the right hand side of (27) and then integrate, we obtain (22).

Using Young's inequality with different constants in (26) and proceeding exactly as in the previous case, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_\alpha+\frac{\nu}{4}|Au|^2_\alpha\le\left(\frac{27c_0^4}{\nu^3}+\frac{\delta^2}{2\nu^3}\right)\|u\|^6_\alpha. \tag{28}$$

By integrating both sides of the resulting inequality, we infer (21). □

In Theorem 4.5, we have used standard ODE theory to obtain existence, uniqueness and stability of the ODE (1) for initial points in $\mathcal{D}$. We have also established that for certain points (t_0,α_0) belonging to the boundary of $\mathcal{D}$, namely those for which $\|\mathbf{u}(t_0)\|_{\alpha_0}<\infty$, we can solve (1) locally on a maximal interval of the form $[t_0,T_M(t_0,\alpha_0))$. However, since we do not in general have local Lipschitz property at points on the boundary of $\mathcal{D}$, we cannot conclude from the standard ODE theory that this solution is unique. Moreover, provided $\lim_{t\nearrow T_M(t_0,\alpha_0)}\alpha(t)=\alpha_1<\infty$, from the standard ODE theory, we only know that $(T_M(t_0,\alpha_0),\alpha_1)$ belongs to the boundary of $\mathcal{D}$. However, due to the special structure of (1), more information can be obtained. This is established in Theorem 5.3 below. We need the following lemma.

Lemma 5.2 *Let $\alpha(t) = \alpha_\delta(t; t_0, \alpha_0)$ be a solution of (1) such that $\alpha(t) \in \mathcal{D}$ for $t \in (t_0, T)$. Assume moreover that $\liminf_{t \nearrow T} \|\mathbf{u}(t)\|^2_{\alpha(t)} < \infty$ and denote $\alpha_1 = \lim_{t \nearrow T} \alpha(t)$. Then $\alpha_1 < \infty$, the point $(T, \alpha_1) \in \mathcal{D}$ and the vector $e^{\alpha(t)A^{1/2}}\mathbf{u}(t)$ converges to $e^{\alpha_1 A^{1/2}}\mathbf{u}(T)$ in V as $t \nearrow T$.*

Proof We first remark that the hypothesis implies that $\limsup_{t \nearrow T} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty$. Indeed, due to the assumption, there exists a sequence of points $t_n \nearrow T$ such that

$$\|\mathbf{u}(t)\|^2_{\alpha(t_n)} < \infty \text{ for all } n. \tag{29}$$

Due to the uniqueness established in Theorem 4.5, for any $t \in (t_n, T)$, we have $\alpha(t; t_n, \alpha_n) = \alpha(t; t_0, \alpha_0)$, where $\alpha_n = \alpha(t_n; t_0, \alpha_0)$. By solving for α starting from (t_n, α_n) and using (20) (where (t_0, α_0) is successively replaced with (t_n, α_n)) and (29), it follows immediately that

$$\limsup_{t \to T} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty. \tag{30}$$

Due to (30), by choosing t_0 close to T, we can without loss of generality assume that $a := \sup_{t \in [t_0, T)} \|\mathbf{u}(t)\|^2_{\alpha(t)} < \infty$. Consequently, from (1), $\alpha_1 < \infty$. Let $\gamma_1 = \frac{2\delta}{\nu} a$ and $t' \in (t_0, T)$ be a point such that $T - t' \leq \frac{\nu}{2\gamma_1^2} \ln\left(1 + \frac{\nu^2\gamma_1^2}{c_1 A}\right)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\alpha \leq \frac{\delta}{\nu} a = \frac{\gamma_1}{2}$ for all $t \in (t', T)$, we must have $\alpha_1 \leq (T - t')\frac{\gamma_1}{2} + \alpha'$, where $\alpha' = \alpha(t'; t_0, \alpha_0)$. On the other hand, by Proposition 3.2,

$$\sup_{t \in [t', t' + \frac{\nu}{4\gamma_1^2} \ln(1 + \frac{\nu^2\gamma_1^2}{c_1 A}))} \|\mathbf{u}(t)\|^2_{\alpha' + (t - t')\gamma_1} < \infty.$$

Since $\alpha_1 < \alpha' + (T - t')\gamma_1$, from part (v) of Theorem 4.3 (with $\gamma(t) = (t - t')\gamma$), it follows that $(T, \alpha_1) \in \mathcal{D}$. The assertion concerning strong convergence now follows from the definition of $\mathcal{D}$. □

Theorem 5.3 *If $(t_0, \alpha_0) \in \mathbb{R}^2_+$ be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$, then the following hold.*

- (i) *The solution $\alpha_\delta(t; t_0, \alpha_0)$ of (1) is unique on the interval $[t_0, T_M(t_0, \alpha_0))$, where $[t_0, T_M(t_0, \alpha_0))$ denotes the maximal interval of existence.*
- (ii) *Let $\alpha_M = \lim_{t \nearrow T_M(t_0, \alpha_0)} \alpha(t)$ and assume that $\alpha_M < \infty$. In this case, with $\partial^+\mathcal{D}$ as defined in (17), the point $(T_M(t_0, \alpha_0), \alpha_M) \in \partial^+\mathcal{D}$ and $\alpha(t; t_0, \alpha_0) \in \mathcal{D}$ for all $t \in (t_0, T_M(t_0, \alpha_0))$. Moreover, $\lim_{t \nearrow T_M(t_0, \alpha_0)} \|\mathbf{u}(t)\|_{\alpha(t)} = \infty$.*

Proof Let us first prove part (i). Recall that since $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$, the set $\Gamma_{(t_0, \alpha_0)} \subset \mathcal{D}$. In view of the result in part (i) of Theorem 4.5, it suffices to prove local uniqueness at (t_0, α_0). Without loss of generality, we will prove it for $t_0 = 0$, $\alpha_0 = 0$. Assume now we have two solutions α_1 and α_2 such that $\alpha_1(t), \alpha_2(t) \in \mathcal{D}$ for all $t \in (0, T)$ and $\sup_{t \in [0, T)} \max\left\{\|\mathbf{u}\|_{\alpha_1}, \|\mathbf{u}\|_{\alpha_2}\right\} < \infty$. Integrating the inequality in (28), it follows that

$$\int_0^T |A\mathbf{u}(t)|^2_{\alpha_i}\, dt < \infty, \quad i = 1, 2. \tag{31}$$

We now make the following observation. If there exists a $t' \in (0, T)$ such that $\alpha_1(t') > \alpha_2(t')$, this implies that $\alpha_1(t) > \alpha_2(t)$ for all $t \in [t', T]$ (this follows by comparing the derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\alpha_i$ on $[t', T]$). So, without loss of generality, we may assume $\alpha_1 \geq \alpha_2$ on $[0, T]$. Using the Poincaré inequality it follows from (21) that $\int_0^T |A^{3/4}\mathbf{u}|_{\alpha_i}^2\, dt < \infty, i = 1, 2$. From (7), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1 - \alpha_2) \le 2\frac{\delta}{\nu}(\alpha_1 - \alpha_2)|A^{3/4}\mathbf{u}|_{\alpha_1}^2.$$

This in turn implies

$$0 \le \alpha_1(t) - \alpha_2(t) \le e^{2\frac{\delta}{\nu}\int\limits_0^T |A^{3/4}\mathbf{u}(t)|_{\alpha_1}^2\, dt}(\alpha_1 - \alpha_2)(0) = 0.$$

Part (ii) follows immediately from Lemma 5.2. □

Proposition 5.1 and Theorem 5.3 immediately yield the following corollary.

Corollary 5.4 *We have the relations:*

$$t_0 + \frac{\nu^3}{2(c_1 + \delta^2)\|u(t_0)\|_{\alpha_0}^4} \le T_M(t_0, \alpha_0) \text{ and} \tag{32}$$

$$\lim_{t \nearrow T_M(t_0,\alpha_0)} \alpha(t; t_0, \alpha_0) \ge \alpha_0 + \frac{\delta\nu^2}{(c_1 + \delta^2)\|u(t_0)\|_{\alpha_0}^2}. \tag{33}$$

The corollary below is a straightforward extension of an argument due to Prodi. It follows from Corollary 5.4, more specifically from (32) and (33), and the fact that

$$\alpha(s; t_0, \alpha_0) = \alpha(s; t, \alpha(t; t_0, \alpha_0)), t_0 < t < s < T_M(t_0, \alpha_0).$$

Corollary 5.5 *Denote* $\alpha(T_M(t_0, \alpha_0); t_0, \alpha_0) = \lim_{t\nearrow T_M(t_0,\alpha_0)} \alpha(t; t_0, \alpha_0)$. *For all* $t_0 < t < T_M(t_0, \alpha_0)$, *we have*

$$\|\mathbf{u}(t)\|_{\alpha(t;t_0,\alpha_0)}^2 \ge \max\left\{\frac{\nu^{\frac{3}{2}}}{\sqrt{2(c_1+\delta^2)}}\frac{1}{\sqrt{T_M(t_0,\alpha_0) - t}},\right.$$
$$\left.\frac{\delta\nu^2}{(c_1+\delta^2)}\frac{1}{\alpha(T_M(t_0,\alpha_0); t_0, \alpha_0) - \alpha(t; t_0, \alpha_0)}\right\}. \tag{34}$$

We will now prove a stability result concerning the auxiliary ODE.

Theorem 5.6 *Let* $(t_0^{(n)}, \alpha_0^{(n)}, \delta^{(n)})$ *be a sequence such that* $(t_0^{(n)}, \alpha_0^{(n)}, \delta^{(n)}) \to (t_0, \alpha_0, \delta)$ *and* $\sup_{n\ge 0} \|\mathbf{u}(t_0^{(n)})\|_{\alpha_0^{(n)}} < \infty$. *Then,* $\liminf_{n\to\infty} T_M(t_0^{(n)}, \alpha_0^{(n)}) \ge T_M(t_0, \alpha_0)$ *and for any* $\epsilon > 0$ *and* $T < T_M(t_0, \alpha_0, \delta)$, *we have*

$$\lim_{n\to\infty} \sup_{[t_0+\epsilon, T]} |\alpha_{\delta^{(n)}}(t; t_0^{(n)}, \alpha_0^{(n)}) - \alpha_\delta(t; t_0, \alpha_0)| = 0. \tag{35}$$

Additionally, ϵ *can be taken to be zero if* $t_0^{(n)} \le t_0$ *for* n *large.*

Proof We will denote $\alpha_{\delta^{(n)}}(t; t_0^{(n)}, \alpha_0^{(n)}) = \alpha_n(t)$ and $\alpha_\delta(t; t_0, \alpha_0) = \alpha(t)$. It will be enough to give the proof in case $t_0^{(n)} \ge t_0$, the other case being similar. Let

$$M := 2\max\left\{\sup_{n\ge 0} \|\mathbf{u}(t_0^{(n)})\|_{\alpha_0^{(n)}}, \sup_{t\in[t_0,T]} \|\mathbf{u}(t)\|_{\alpha(t)}\right\},$$

and note that $M < \infty$. Let moreover $T_1 = \dfrac{\nu^3}{8(c_1 + (2\sup_n(\delta^{(n)}))^2)M^4} > 0$ and fix $\epsilon < T_1$. Take n sufficiently large so that $t_0^{(n)} < t_0 + \epsilon$. From (20), for each n we get,

$$\max\{\|\mathbf{u}(t)\|_{\alpha(t)}, \sup_{[t_0, t_0+2T_1]} \|\mathbf{u}(t)\|_{\alpha_n(t)}\} \leq 2\|\mathbf{u}(t_0^{(n)})\|_{\alpha_0^{(n)}} \leq M_1.$$

Due to Corollary 5.4, we have $T_M(t_0^{(n)}, \alpha_0^{(n)}) > t_0 + T_1$. Moreover, from (28)

$$\begin{aligned} &\max\left\{\int_{t_0}^{t_0+T_1} |A\mathbf{u}(t)|^2_{\alpha_n(t)}, \int_{t_0}^{t_0+T_1} |A\mathbf{u}(t)|^2_{\alpha_n(t)}\right\} \\ &\quad \leq \frac{4}{\nu}\left(\|\mathbf{u}(t_0^{(n)})\|^2_{\alpha_0^{(n)}} + M_1^3 T_1 (\frac{54c_0^4 + (\delta^{(n)})^2}{2\nu^3})\right) < \infty. \end{aligned} \tag{36}$$

We claim that the paths $\alpha_n(\cdot)$ and $\alpha(\cdot)$ meet at most at one point. Indeed, if they meet at a point t', then if $\delta^{(n)} > \delta$, then for all $t > t'$, we must have $\alpha_n(t) \geq \alpha(t)$. Assume now that $\alpha_n(\cdot) > \alpha(\cdot)$. From (7), it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\alpha_n - \alpha| = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha_n - \alpha) &\leq \tfrac{\delta^{(n)}}{\nu}(\|\mathbf{u}\|^2_{\alpha_n} - \|\mathbf{u}\|^2_{\alpha}) + \tfrac{|\delta^{(n)} - \delta|}{\nu}\|\mathbf{u}\|^2_{\alpha} \\ &\leq 2\tfrac{\delta^{(n)}}{\nu}|\alpha_n - \alpha||A^{3/4}\mathbf{u}|^2_{\alpha_n} + \tfrac{|\delta^{(n)} - \delta|}{\nu}\|\mathbf{u}\|^2_{\alpha}. \end{aligned}$$

Similarly, in case $\alpha_n(\cdot) < \alpha(\cdot)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\alpha_n - \alpha| \leq 2\frac{\delta}{\nu}|\alpha_n - \alpha||A^{3/4}\mathbf{u}|^2_{\alpha} + \frac{|\delta^{(n)} - \delta|}{\nu}\|\mathbf{u}\|^2_{\alpha_n}.$$

Integrating this relation between $t_0^{(n)}$ to $t \leq t_0 + T_1$ and using $\alpha_n(t_0^{(n)}) = \alpha_0^{(n)}$, we obtain

$$\begin{aligned} 0 &\leq |\alpha_n(t) - \alpha(t)| \\ &\leq e^{2\frac{\delta}{\nu}\int_{t_0}^{t_0+T_1} \max\{|A^{3/4}\mathbf{u}(t)|^2_{\alpha}, |A^{3/4}\mathbf{u}(t)|^2_{\alpha_n}\}\,dt} \\ &\quad \times \left(\frac{|\delta^{(n)} - \delta|}{\nu}\max\{\alpha_n(t), \alpha(t)\} + |\alpha_0^{(n)} - \alpha(t_0^{(n)}; t_0, \alpha_0)|\right). \end{aligned}$$

By continuity of $\alpha(t)$, we have $|\alpha_0^{(n)} - \alpha(t_0^{(n)}; t_0, \alpha_0)| \to 0$. Due to the uniform bound in (36), which by Poincaré inequality also implies that $\alpha_n(t) = \int_{t_0}^{t} \|\mathbf{u}(s)\|^2_{\alpha_n(s)}\,ds$ is uniformly bounded, the claimed convergence in (35) holds on the interval $[t_0 + \epsilon, t_0 + T_1]$. In case $t_0 + T_1 > T$, the proof is complete.

Assume now that $t_0 + T_1 < T < T_M(t_0, \alpha_0)$. Since $\sup_{t \in [t_0, t_0+T_1)} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty$, by Proposition 5.2, the point $(t_0 + T_1, \alpha(t_0 + T_1)) \in \mathcal{D}$. Then due to the convergence of $\alpha_n(\cdot)$ just established, $\|\mathbf{u}(t_0 + T_1)\|_{\alpha_n(t_0+T_1)}$ converges to $\|\mathbf{u}(t_0 + T_1)\|_{\alpha(t_0+T_1)}$. We now repeat the previous step replacing $t_0^{(n)}$ by $t_0 + T_1$ for all n, $\alpha_0^{(n)}$ with $\alpha_n(t_0 + T_1)$ and leaving $\delta^{(n)}$ unchanged. This process continues, and during each application, we increase the interval of convergence by the fixed amount T_1. So, after finitely many steps, we are in the situation when $t_0 + T_1 > T$. This finishes the proof. □

6 Blowup and monotonicity of certain Gevrey norms

Until now, in all results concerning analyticity, only lower estimates for the analyticity radius of the solution $\mathbf{u}(t)$ were obtained. However, here we will show that if δ is chosen large enough, the norm $\|\mathbf{u}(t)\|_{\alpha(t;t_0)}$ blows up at or before any $T \in (t_0, t_\infty)$. Due to Theorem 5.3, this means that given any $T \in (0, t_\infty)$, we are guaranteed to reach a boundary point of the domain $\mathcal{D}$ in the interval $(0, T)$. Additionally, we obtain that along our solution curve, $|\mathbf{u}(t)|_{\alpha(t)}$ is monotonically increasing provided δ is large enough. This leads to an unusual regularity test for the solution $\mathbf{u}(t)$ (see Corollary 6.3 and subsequent Remark 5). These properties play an useful role in our discussion concerning energy cascade in Sect. 10.

Theorem 6.1 *Let (t_0, α_0) be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$. Assume that*

$$\delta > \max\{\frac{4\nu c_0}{\kappa_0^{1/2}|\mathbf{u}(t_0)|_{\alpha_0}}, \frac{4\nu^2}{\kappa_0|\mathbf{u}(t_0)|_{\alpha_0}^2}\}. \tag{37}$$

In this case, for all t such that $t_0 < t < T_M(t_0, \alpha_0)$, $|\mathbf{u}(t)|_{\alpha(t)}$ is (strictly) increasing in t and we have the estimates

$$|\mathbf{u}(t)|_{\alpha(t)} \geq |\mathbf{u}(t_0)|_{\alpha_0} e^{\kappa_0(\alpha-\alpha_0)} \text{ and } |\mathbf{u}(t)|_{\alpha(t)}^2 \geq \frac{|\mathbf{u}(t_0)|_{\alpha_0}^2}{1 - \frac{\delta}{\nu}|\mathbf{u}(t_0)|_{\alpha_0}^2 \kappa_0^3 (t - t_0)}. \tag{38}$$

Moreover, $\|\mathbf{u}(t)\|_{\alpha(t)}$ blows up at or before $t_0 + \frac{\nu}{|\mathbf{u}(t_0)|_{\alpha_0}^2 \delta\kappa_0^3}$, i.e.,

$$T_M(t_0, \alpha_0) \leq t_0 + \frac{\nu}{|\mathbf{u}(t_0)|_{\alpha_0}^2 \delta\kappa_0^3}. \tag{39}$$

Proof From (26) and (4) (with $\beta = 0$) we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|_\alpha^2 &\geq -\nu|A^{1/2}\mathbf{u}|_\alpha^2 - c_0|A^{1/2}\mathbf{u}|_\alpha^2|A^{1/4}\mathbf{u}|_\alpha + \frac{\delta}{\nu}|A^{1/2}\mathbf{u}|_\alpha^2|A^{1/4}\mathbf{u}|^2 \\ &= |A^{1/4}\mathbf{u}|_\alpha^2|A^{1/2}\mathbf{u}|_\alpha^2\left(\frac{\delta}{\nu} - \frac{c_0}{|A^{1/4}\mathbf{u}|_\alpha} - \frac{\nu}{|A^{1/4}\mathbf{u}|_\alpha^2}\right). \end{aligned} \tag{40}$$

Note now that Poincaré inequality and the choice of δ as in (37) imply

$$\frac{\delta}{\nu} - \frac{c_0}{|A^{1/4}\mathbf{u}(t_0)|_{\alpha_0}} - \frac{\nu}{|A^{1/4}\mathbf{u}(t_0)|_{\alpha_0}^2} \geq \frac{\delta}{\nu} - \frac{c_0}{\kappa_0^{1/2}|\mathbf{u}(t_0)|_{\alpha_0}} - \frac{\nu}{\kappa_0|\mathbf{u}(t_0)|_{\alpha_0}^2} \geq \frac{\delta}{2\nu}.$$

Thus, from (40), it follows by continuity $|\mathbf{u}|_\alpha$ is increasing on an interval $[t_0, t_0 + \eta]$. At $t = t_0 + \eta$ we must have

$$\begin{aligned} \frac{\delta}{\nu} - \frac{c_0}{|A^{1/4}\mathbf{u}(t)|_{\alpha(t)}} - \frac{\nu}{|A^{1/4}\mathbf{u}(t)|_{\alpha(t)}^2} &\geq \frac{\delta}{\nu} - \frac{c_0}{\kappa_0^{1/2}|\mathbf{u}(t)|_{\alpha(t)}} - \frac{\nu}{\kappa_0|\mathbf{u}(t)|_{\alpha(t)}^2} \\ &> \frac{\delta}{\nu} - \frac{c_0}{\kappa_0^{1/2}|\mathbf{u}(t_0)|_{\alpha_0}} - \frac{\nu}{\kappa_0|\mathbf{u}(t_0)|_{\alpha_0}^2} \geq \frac{\delta}{2\nu}. \end{aligned} \tag{41}$$

Consequently, by (40), $|\mathbf{u}|_\alpha$ must be strictly increasing at $t_0 + \eta$ and the estimate (41) holds at $t = t_0 + \eta$. This then implies that (41) holds for all $t \in (t_0, T_M(t_0, \alpha_0))$.

Inserting the estimate (41) in (40) and applying Poincaré inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|_\alpha^2 \geq \kappa_0\frac{\delta}{\nu}|\mathbf{u}|_\alpha^2\|\mathbf{u}\|_\alpha^2 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|_\alpha^2 \geq \kappa_0^3\frac{\delta}{\nu}|\mathbf{u}|_\alpha^4.$$

Also, integrating both these relations [and recalling the definition of $\alpha(t)$ in (1)], we get (38). The assertion concerning the blowup follows immediately from the second inequality in (38). □

Remark 3 From Proposition 5.1 and Theorem 6.1, provided δ satisfies (37), we get

$$T_M(t_0,\alpha_0) \geq t_0 + \frac{\nu^3}{2(c_1+\delta^2)\|\mathbf{u}(t_0)\|_{\alpha_0}^4} \text{ and } T_M(t_0,\alpha_0) \leq t_0 + \frac{\nu}{\delta\kappa_0^3|\mathbf{u}(t_0)|_{\alpha_0}^2}.$$

To see that these results are consistent, we need to show

$$\frac{\nu^3}{2(c_1+\delta^2)\|\mathbf{u}(t_0)\|_{\alpha_0}^4} \leq \frac{\nu}{\delta\kappa_0^3|\mathbf{u}(t_0)|_{\alpha_0}^2}.$$

Note that $\frac{\delta}{(c_1+\delta^2)} \leq \frac{1}{\delta}$ and by (37), $\delta \geq \delta_0 := \frac{2\nu^2}{\kappa_0|\mathbf{u}(t_0)|_{\alpha_0}^2}$. The inequality above now immediately follows from the Poincaré inequality.

Due to Theorem 6.1, blowup of $\|\mathbf{u}\|_\alpha$ is ensured provided δ is sufficiently large. From Theorem 5.3, we know that the point on the curve α where the blowup of $\|\mathbf{u}\|_\alpha$ occurs, necessarily belongs to the boundary of $\mathcal{D}$. We provide a more refined analysis of this point in the proposition below.

Proposition 6.2 *Let* $(t_0,\alpha_0) \in \overline{\mathcal{D}}$ *with* $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$ *and* δ *be such that it satisfies* (37). *Denote* $\alpha(t) = \alpha(t;t_0,\alpha_0)$ *and* $T := T_M(t_0,\alpha_0)$. *Assume that*

$$\alpha(T) := \lim_{t\nearrow T} \alpha(t;t_0,\alpha_0) < \infty \text{ and } |\mathbf{u}(T)|_{\alpha(T)} < \infty. \tag{42}$$

Then, in fact we have

$$\sup_{t\in[t_0,T)} |\mathbf{u}(t)|_{\alpha(t;t_0,\alpha_0)} \leq |\mathbf{u}(T)|_{\alpha(T)} \text{ and } e^{\alpha(t)A^{1/2}}\mathbf{u}(t) \to e^{\alpha(T)A^{1/2}}\mathbf{u}(T) \text{ in } H. \tag{43}$$

Proof Clearly, in view of Theorem 4.3, without loss of generality, we can (and will) assume that $\alpha_0 > 0$ and $(t_0,\alpha_0) \in \mathcal{D}$. Choose a sequence $\{\alpha_0^{(n)}\}$ such that $(t_0,\alpha_0^{(n)}) \in \mathcal{D}$, $\alpha_0^{(n)} \to \alpha_0$ and $\alpha_0^{(n)} < \alpha_0$ for all n. Since the point $(t_0,\alpha_0^{(n)}) \to (t_0,\alpha_0) \in \mathcal{D}$, we have $|\mathbf{u}(t_0)|_{\alpha_0^{(n)}} \to |\mathbf{u}(t_0)|_{\alpha_0}$. Thus, we may (and therefore will) also assume that (37) holds with α_0 replaced by $\alpha_0^{(n)}$ for all n. Denoting $\alpha_n(t) = \alpha(t;t_0,\alpha_0^{(n)})$, Theorem 6.1, now implies that for each n, the norm $|\mathbf{u}(t)|_{\alpha_n(t)}$ is increasing for $t \in (t_0, T_M(t_0,\alpha_0^{(n)}))$.

Note first that due to Theorem 7.2, we must have $\alpha(T) = \beta_c(T)$ and the point $(T,\alpha(T)) \in \partial\mathcal{D}$. Due to Corollary 7.3, we also have $T_M(t_0,\alpha_0^{(n)}) > T$. Consequently, for each fixed n and $t_0 < t < T$, we have

$$\alpha_n(t) < \alpha(t),\ (t,\alpha_n(t)), (t,\alpha(t)) \in \mathcal{D} \text{ and } |\mathbf{u}(t)|_{\alpha_n(t)} \leq |\mathbf{u}(t)|_{\alpha_n(T)}. \tag{44}$$

In the previous line, the first relation follows from Corollary 7.3, the second from part (ii) of Theorem 5.3 and the third from part (ii) of Theorem 6.1. By Corollary 7.3, we also have $\alpha_n(T) < \alpha(T)$ for each n. Therefore, from the third relation in (44) and assumption (42), we immediately obtain

$$|\mathbf{u}(t)|_{\alpha_n(t)} \leq |\mathbf{u}(t)|_{\alpha_n(T)} \leq |\mathbf{u}(t)|_{\alpha(T)} < \infty. \tag{45}$$

Now by Theorem 5.6, for each fixed $t \in (t_0,T)$, we have $\lim_{n\to\infty}\alpha_n(t) = \alpha(t)$. Due to (44), the points $(t,\alpha_n(t)), (t,\alpha(t)) \in \mathcal{D}$ and therefore, $|\mathbf{u}(t)|_{\alpha_n(t)} \to |\mathbf{u}(t)|_{\alpha(t)}$. Letting $n\to\infty$ and using the uniform bound in (45), for each fixed $t \in (t_0,T)$ we obtain

$$|\mathbf{u}(t)|_{\alpha(t)} \leq |\mathbf{u}(T)|_{\alpha(T)} \text{ for } t_0 < t < T. \tag{46}$$

On the other hand, due to the weak continuity of the map $t \to \mathbf{u}(t)$, and the fact that $\alpha(t) \nearrow \alpha(T)$ as $t \nearrow T$, it follows that $e^{\alpha(t)A^{1/2}}\mathbf{u}(t) \to e^{\alpha(T)A^{1/2}}\mathbf{u}(T)$ weakly. From (46), we also obtain $\limsup_{t\nearrow T} |e^{\alpha(t)A^{1/2}}\mathbf{u}(t)| \le |e^{\alpha(T)A^{1/2}}\mathbf{u}(T)|$. The strong convergence of $e^{\alpha(t)A^{1/2}}\mathbf{u}(t)$ to $e^{\alpha(T)A^{1/2}}\mathbf{u}(T)$ now follows by recalling the following general fact: if $v_n \to v$ weakly in a Hilbert space H and $\limsup_{n\to\infty} \|v_n\| \le \|v\|$, then $v_n \to v$ strongly. □

Remark 4 The following result is a generalization of a classical result [18] and provides an indication of what happens in case δ and the initial data are small. The result in [18] corresponds to $\delta = 0, \alpha_0 = 0$ in our setting below. We omit the proof since it is also classical.

Assume that

$$|A^{1/4}\mathbf{u}(t_0)|_{\alpha_0}^2 \le \frac{\nu^2}{16c_0^2} \text{ and } 0 \le \delta \le 4c_0^2.$$

In this case, we have $T_M(t_0, \alpha_0) = \infty$, and moreover, for all $t \ge t_0$, we have

$$\|\mathbf{u}\|_{\alpha(t;t_0,\alpha_0)}^2 \le \|\mathbf{u}(t_0)\|_{\alpha_0}^2 e^{-\kappa_0^2\nu(t-t_0)} \text{ and } \alpha(t;t_0,\alpha_0) \le \alpha_0 + \frac{\delta}{\nu^2\kappa_0^2}\|\mathbf{u}(t_0)\|_{\alpha_0}^2. \tag{47}$$

More generally, if

$$|A^{1/4}\mathbf{u}(t_0)|_{\alpha_0}^2 \le \frac{\nu^2}{16c_0^2} \text{ and } 0 \le \delta \le \frac{\nu^2}{4|A^{1/4}\mathbf{u}(t_0)|_{\alpha_0}^2}, \tag{48}$$

then we have (47) for all $t \ge t_0$.

The proof of Theorem 6.1 also yields the following corollary, which, in turn, yields a regularity criterion for the 3D Navier–Stokes equations (see Remark 5 below).

Corollary 6.3 *Let $\delta \ge 64c_0^2$ and $(t_0, \alpha_0) \in \overline{\mathcal{D}}$ be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$. Assume that $t_\infty < \infty$. Then, for all t satisfying $t_0 < t < T_M(t_0, \alpha_0)$, the quantity $|\mathbf{u}(t)|_{\alpha(t)}$ is (strictly) increasing in t and we have the estimates* (38) *and* (39).

Proof Due to the result in [18] (see Remark 4 with $\alpha_0 = 0, \delta = 0$), it follows that

$$|A^{1/4}\mathbf{u}(t)| \ge \frac{\nu^2}{16c_0^2} \quad \text{for all} \quad 0 \le t < t_\infty, \text{ if } t_\infty < \infty. \tag{49}$$

Note that this implies

$$\frac{\delta}{\nu} - \frac{c_0}{|A^{1/4}\mathbf{u}(t)|_{\alpha(t)}} - \frac{\nu}{|A^{1/4}\mathbf{u}(t)|_{\alpha(t)}^2} \ge \frac{\delta}{\nu} - \frac{c_0}{|A^{1/4}\mathbf{u}(t)|} - \frac{\nu}{|A^{1/4}\mathbf{u}(t)|} \ge \frac{\delta}{2\nu},$$

where the last inequality in the line above follows from (49) and the assumption $\delta \ge 16c_0^2$. □

Remark 5 Regularity criteria for the 3D Navier–Stokes equations have a rich history. Usually, they involve boundedness or time integrability of certain norms of the solution or regularity of vortex alignment direction (see [1,8,9,31,32] and references there in). Note that Corollary 6.3 also provides a regularity test for the solution $\mathbf{u}(t)$ but of a completely different type, namely,

If $\delta \ge 64c_0^2$ and $|\mathbf{u}(t)|_{\alpha(t)}$ is not increasing for some interval $[0, t_1]$ where $t_1 < T_M(0,0)$, then the solution $\mathbf{u}$ must be globally regular.

The authors would like to thank the reviewer for the following observation.

Remark 6 As pointed out in the introduction, in certain cases, the analyticity radius of the solution may in fact be infinite for all positive times. The blowup in Theorem 6.1 in those cases mean that the curve $\alpha(\cdot)$ goes to infinity as t approaches the blow-up point. One may view this as the solution curve reaching the boundary point (T_M, ∞) of the domain $\mathcal{D}$. In fact, if one considers the linear Stokes' equation (in which case the nonlinearity is absent), Theorem 6.1 still applies. It is well known that solutions to the Stokes' equation are entire for all $t > 0$. The blowing up of $|\mathbf{u}|_{\alpha(\cdot)}$ in this case means $\alpha(\cdot)$ approaches infinity in a finite interval of time.

7 Topological structure of the domain $\mathcal{D}$

One of our goals in this paper is to connect the topology of the domain $\mathcal{D}$, particularly its boundary, to turbulence. In this section, we establish additional topological properties of the domain $\mathcal{D}$. These facts will be useful in exploring this connection in more detail in Sect. 10; see also Remark 8 below. We will start with the following proposition.

Proposition 7.1 *Let* (t_0, α_0) *be such that* $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$. *Denote* $t_1 = T_M(t_0, \alpha_0)$ *and assume that* $\lim_{t \nearrow t_1} \alpha(t) = \alpha_1 < \infty$. *Then, for all* $t_0 \le t < t_1$ *and* $\beta < \alpha(t; t_0, \alpha_0)$, *we have the estimate*

$$\sup_{t \le \tau \le t_1} \|\mathbf{u}(\tau)\|_\beta \le \|\mathbf{u}(t)\|_\beta \exp\left\{ c_3 \sqrt{\frac{2(c_1 + \delta^2)}{\delta^3}} \left(\frac{\alpha_1 - \alpha(t; t_0, \alpha_0)}{\alpha(t; t_0, \alpha_0) - \beta} \right)^{3/2} \right\}, \tag{50}$$

where $c_3 = c_0\sqrt{(1.5)^3 e^{-3}}$ *and* c_1 *as in* (10).

Proof We first note that from part (i) of Theorem 4.3, the set $\{(\tau, \beta) : t_0 < t < t_1, 0 < \beta < \alpha(t; t_0, \alpha_0)\} \subset \mathcal{D}$, and moreover, by the uniqueness of solutions (Theorem 5.3), for all $t_0 \le t \le \tau < t_1$, we must have

$$\alpha(\tau; t, \alpha(t; t_0, \alpha_0)) = \alpha(\tau; t_0, \alpha_0) \text{ and } \alpha_1 = \alpha(t; t_0, \alpha_0) + \frac{\delta}{\nu} \int_t^{t_1} \|\mathbf{u}(\tau)\|^2_{\alpha(\tau; t_0, \alpha_0)} \, d\tau, \tag{51}$$

Subsequently in this proof, denote $\widetilde{\alpha} := \alpha(t; t_0, \alpha_0)$. From (2), for $t_0 \le t \le \tau < t_1$ we obtain

$$\begin{aligned} \tfrac{1}{2} \frac{d}{d\tau} \|\mathbf{u}(\tau)\|^2_\beta + \nu |A\mathbf{u}(\tau)|^2_\beta &= -(B(\mathbf{u}(\tau), \mathbf{u}(\tau)), e^{2\beta A^{1/2}} A\mathbf{u}(\tau)) \\ &\le c_0 \|\mathbf{u}(\tau)\|^2_\beta |A^{5/4}\mathbf{u}(\tau)|_\beta \le \tfrac{c_3}{(\widetilde{\alpha} - \beta)^{3/2}} \|\mathbf{u}(\tau)\|^2_\beta \|\mathbf{u}(\tau)\|_{\widetilde{\alpha}}, \end{aligned}$$

where $c_3 = c_0\sqrt{(1.5)^3 e^{-3}}$. The first inequality in the above line follows from (5) and interpolation while the second follows from the observation that

$$\begin{aligned} |A^{5/4}\mathbf{u}(\tau)|_\beta &= |e^{-(\widetilde{\alpha} - \beta)A^{1/2}} A^{3/4} A^{1/2} e^{\widetilde{\alpha} A^{1/2}} \mathbf{u}|_{\widetilde{\alpha}} \\ &\le \|e^{-(\widetilde{\alpha} - \beta)A^{1/2}} A^{3/4}\|_{op} \|\mathbf{u}\|_{\widetilde{\alpha}} \le \frac{\sqrt{(1.5)^3 e^{-3}}}{(\widetilde{\alpha} - \beta)^{3/2}} \|\mathbf{u}\|_{\widetilde{\alpha}}, \end{aligned}$$

where $\|\cdot\|_{op}$ denotes the operator norm on the space V. Noting that $\widetilde{\alpha} \le \alpha(\tau; t_0, \alpha_0)$ and applying Gronwall's inequality, we obtain for any $t \le \tilde{t}_1 \le t_1$

$$\begin{aligned}
\|\mathbf{u}(\tilde{t}_1)\|_\beta^2 &\le \|\mathbf{u}(t)\|_\beta^2 e^{\frac{2c_3}{(\widetilde{\alpha}-\beta)^{3/2}} \int_t^{t_1} \|\mathbf{u}(\tau)\|_{\alpha(\tau;t_0,\alpha_0)}\, \mathrm{d}\tau} \\
&\le \|\mathbf{u}(t)\|_\beta^2\, e^{\frac{2c_3}{(\widetilde{\alpha}-\beta)^{3/2}} (t_1-t)^{1/2} (\int_t^{t_1} \|\mathbf{u}(\tau)\|^2_{\alpha(\tau;t_0,\alpha_0)}\, \mathrm{d}\tau)^{1/2}} \\
&= \|\mathbf{u}(t)\|_\beta^2\, e^{\frac{2c_3}{(\widetilde{\alpha}-\beta)^{3/2}} (t_1-t)^{1/2} (\frac{\nu}{\delta})^{1/2} (\alpha_1-\widetilde{\alpha})^{1/2}}.
\end{aligned} \tag{52}$$

On the other hand, since $\lim_{\tau \nearrow t_1} \|\mathbf{u}(\tau)\|_{\alpha(\tau;t,\widetilde{\alpha})} = \lim_{s \nearrow t_1} \|\mathbf{u}(s)\|_{\alpha(s;t_0,\alpha_0)} = \infty$, from (20) we have the estimate

$$\|\mathbf{u}(\tau)\|^2_{\alpha(\tau;t,\widetilde{\alpha})} \ge \frac{\nu^{3/2}}{\sqrt{2(c_1+\delta^2)}} \frac{1}{(t_1-\tau)^{1/2}}.$$

Integrating this inequality (in τ) between t to t_1 and using (1), we readily obtain

$$\frac{(t_1-t)^{1/2}}{(\alpha_1-\widetilde{\alpha})} \le \sqrt{\frac{c_1+\delta^2}{2\delta^2\nu}}.$$

Inserting this estimate in (52) finishes the proof. □

Some topological properties of $\mathcal{D}$ are listed in Theorem 4.3. Here, we list several others mostly concerning its boundary. For $t \in [0, t_\infty)$, define

$$\beta_c(t) = \sup\{\alpha \ge 0 : \|\mathbf{u}(t)\|_\alpha < \infty\} \text{ and } \alpha_c(t) = \sup\{\alpha \ge 0 : \{t\}\times[0,\alpha) \subset \mathcal{D}\}. \tag{53}$$

In view of Theorem 4.3 (more specifically, part (ii)), the functions α_c and β_c are well defined and due to Proposition 4.4, $\alpha_c(t) > 0$ for $0 < t < t_\infty$. Furthermore,

$$\beta_c(t) \ge \alpha_c(t),\ \{(t,\alpha) : \alpha < \alpha_c(t)\} \subset \mathcal{D} \text{ and } \{(t,\alpha) : \alpha \ge \alpha_c(t)\} \subset \mathcal{D}^c, \tag{54}$$

where $\mathcal{D}^c$ denotes the complement of $\mathcal{D}$. This means that the domain $\mathcal{D}$ lies strictly under the curve $(t, \alpha_c(t))$, $0 < t < t_\infty$ and, moreover, the points $(t, \alpha_c(t)) \in \partial^+\mathcal{D}$ for $0 < t < t_\infty$. We will also need the following definitions. Let

$$\begin{aligned}
&\overline{\alpha}_c(t) = \limsup_{s\to t} \alpha_c(s),\ \underline{\alpha}_c(t) := \liminf_{s\to t} \alpha_c(s), \\
&\overline{\beta}_c(t) = \limsup_{s\to t} \beta_c(s) \text{ and } \underline{\beta}_c(t) := \liminf_{s\to t} \beta_c(s).
\end{aligned} \tag{55}$$

Theorem 7.2 *The functions introduced in* (53) *and* (55) *have the following supplementary properties:*

(i) *α_c is lower semi-continuous and β_c is lower semi-continuous from the right, that is, we have*

$$\underline{\alpha}_c(t_0) \ge \alpha_c(t_0) \textit{ and } \liminf_{t\to t_0^+} \beta_c(t) \ge \beta_c(t_0).$$

Moreover,

$$\limsup_{t\to t_0^+} \alpha_c(t) = \limsup_{t\to t_0^+} \beta_c(t),\ \limsup_{t\to t_0^-} \alpha_c(t) = \limsup_{t\to t_0^-} \beta_c(t) \textit{ and } \overline{\alpha}_c(t_0) = \overline{\beta}_c(t_0). \tag{56}$$

(ii) *For* $0 < t_0 < t_\infty$*, we have*

$$I_{t_0} := \{\, (t_0, \alpha) : \alpha \in [\alpha_c(t_0), \overline{\alpha}_c(t_0)] \,\} = \partial^+\mathcal{D} \cap \{(t_0, \alpha) : \alpha > 0\},$$

where $\partial^+\mathcal{D}$ *is as defined in* (17)*. Moreover,* $I_{t_0} = (t_0, \alpha_c(t_0))$ *precisely when* t_0 *is a continuity point of* $\alpha_c(t_0)$*. In this case, it is also a continuity point of* $\beta_c(t_0)$*, and we have*

$$\beta_c(t_0) = \alpha_c(t_0) = \underline{\beta}_c(t_0) = \underline{\alpha}_c(t_0) = \overline{\beta}_c(t_0) = \overline{\alpha}_c(t_0).$$

(iii) *Let* $\mathcal{T}$ *be the set of all* $t_0 \in (0, t_\infty)$ *such that* I_{t_0} *is a singleton point. Then,* $\mathcal{T}$ *is a dense* G_δ *subset of* $(0, t_\infty)$*.*

(iv) *Let* $T = T_M(t_0, \alpha_0)$ *and denote* $\alpha(t) = \alpha(t; t_0, \alpha_0)$*,* $t_0 < t < T$*. If*

$$\alpha(T) := \lim_{t \nearrow T} \alpha(t; t_0, \alpha_0) < \infty,$$

then the set

$$\{(T, \beta) : 0 \le \beta < \alpha(T)\} \subset \mathcal{D}, \alpha_c(T) = \alpha(T) \text{ and } (T, \alpha(T)) \in \partial^+\mathcal{D}.$$

(v) *Let* $\alpha_c(T) < \beta_c(T)$*, where we use the notation in part (iv). Then*

$$\lim_{t \nearrow T} \|\mathbf{u}(t)\|_{\alpha(t)} = \infty \text{ and } \|\mathbf{u}(T)\|_{\alpha(T)} < \infty.$$

Proof We will first prove that α_c is lower semi-continuous. Let $\alpha < \alpha_c(t_0)$. By (54), the point $(t_0, \alpha) \in \mathcal{D}$ and there exists $\tau > 0$ such that $[t_0 - \tau, t_0 + \tau] \times [\alpha, \alpha + \tau] \subset \mathcal{D}$. Consequently, $\alpha_c(t) \ge \alpha + \tau$ for $\tau \in [t_0 - \tau, t_0 + \tau]$. Since α is arbitrary, $\liminf_{t \to t_0} \alpha_c(t) \ge \alpha_c(t_0)$.

To prove lower semi-continuity from the right of β_c, we note that if $\alpha < \beta_c(t_0)$, then $\Gamma_{(t_0,\alpha)}$ [defined in (16)] is contained in $\mathcal{D}$. Thus, there exists $\tau = \tau(\|\mathbf{u}(t_0)\|_\alpha) > 0$ such that for all $t_0 < t < t_0 + \tau$, we have $\beta_c(t) \ge \alpha$, which in turn implies that $\limsup_{t \to t_0^+} \beta_c(t) \ge \alpha$. Since $\alpha < \beta_c(t_0)$ is arbitrary, the claim follows.

To prove (56), let $(t_n, \beta_c(t_n))$ be a sequence converging to $(t_0, \overline{\beta}_c(t_0))$. Note that $\Gamma_{(t_n, \beta_c(t_n) - \frac{1}{n})} \subset \mathcal{D}$ and there exists a sequence of points $t_n' > t_n, t_n' - t_n < \frac{1}{n}$ and $\alpha_c(t_n') > \beta_c(t_n) - \frac{1}{n}$. Since $\beta_c(t) \ge \alpha_c(t)$ for all $t \in (0, t_\infty)$, this implies (56).

Next, we will prove (ii). If $(t_n, \alpha_n) \in \mathcal{D}$ and $(t_n, \alpha_n) \to (t_0, \alpha)$, then $\alpha_c(t_n) \ge \alpha_n$, and consequently, $\overline{\alpha}_c(t_0) \ge \limsup_{n \to \infty} \alpha_c(t_n) \ge \alpha$. This shows that if $(t_0, \alpha) \in \partial^+\mathcal{D}$, then $\alpha \le \overline{\alpha}_c(t_0)$. Obviously, $(t_0, \alpha_c(t_0)) \in \partial\mathcal{D}$. In view of (54), to finish the proof of the first part of (ii), we need to show that $\{t_0\} \times (\alpha_c(t_0), \overline{\alpha}_c(t_0)] \subset \partial^+\mathcal{D}$. Let $\alpha_c(t_0) < \alpha \le \overline{\alpha}_c(t_0)$ and choose a sequence of points t_n such that $(t_n, \alpha_c(t_n)) \to (t_0, \overline{\alpha}_c(t_0))$. Due to the second relation in (54), for each n, the set $\{(t_n, \widetilde{\alpha}), \widetilde{\alpha} < \alpha_c(t_n)\} \subset \mathcal{D}$. Therefore, we can choose a sequence α_n with $(t_n, \alpha_n) \in \mathcal{D}$ and $(t_n, \alpha_n) \to (t_0, \alpha)$. To prove the second part of part (ii), we simply note that due to part (i) and the first inequality of (54), we have the obvious inequalities

$$\alpha_c(t_0) \le \underline{\alpha}_c(t_0) \le \underline{\beta}_c(t_0) \text{ and } \alpha_c(t_0) \le \beta_c(t_0) \le \overline{\beta}_c(t_0) = \overline{\alpha}_c(t_0).$$

We will now prove part (iii). In view of part (ii), any $t_0 \in \mathcal{T}$ is a continuity point of α_c. Recall that the function α_c is nonnegative lower semi-continuous. This implies that the function $\hat{\beta}(t) = \frac{\alpha_c(t)}{1+\alpha_c(t)}$ is a bounded, lower semi-continuous function on the Baire space $(0, t_\infty)$. It is consequently a pointwise limit of a sequence (in fact, increasing) of continuous functions. Due to Baire's theorem (see Baire's Theorem 2 on page 12 in [35]), the set of continuity points of $\hat{\beta}$ is a dense G_δ subset of $(0, t_\infty)$. The same then holds for α_c provided we adopt the convention that t_0 is also a continuity point of α_c in case $\alpha_c(t_0) = \infty$ and $\lim_{t\to t_0} \alpha_c(t) = \infty$.

To prove of part (iv), let $\beta < \alpha(T_M)$ and $t_1 \in (t_0, T_M)$ be such that $\beta_1 := \alpha(t_1) > \beta$. From (50), it follows that $\sup_{\tau\in[t_1,T_M]} \|\mathbf{u}(\tau)\|_{\beta_1} < \infty$ which, in view of (15), implies part (iv). □

Theorem 7.2 has the following immediate corollary.

Corollary 7.3 *Let* $\alpha_0^{(1)} < \alpha_0, t_0^{(1)} < t_0$ *with* $\max\{\|\mathbf{u}(t_0)\|_{\alpha_0}, \|\mathbf{u}(t_0^{(1)})\|_{\alpha_0}\} < \infty$. *Moreover, assume that*

$$\max\{\alpha(T_M(t_0,\alpha_0); t_0, \alpha_0), \alpha(T_M(t_0^{(1)},\alpha_0), t_0^{(1)}, \alpha_0)\} < \infty.$$

Then, $T_M(t_0,\alpha_0) < T_M(t_0,\alpha_0^{(1)})$ *and* $T_M(t_0^{(1)},\alpha_0) < T_M(t_0,\alpha_0)$. *In particular,* $T_M(t_0, 0)$ *is a strictly increasing function of* t_0 *for all* $t_0 \in [0, t_\infty)$.

Proof We will first prove the relation $T_M(t_0,\alpha_0) < T_M(t_0,\alpha_0^{(1)})$. Note that since $\alpha_0^{(1)} < \alpha_0$ and

$$\|\mathbf{u}(t_0)\|_{\alpha_0^{(1)}} < \|\mathbf{u}(t_0)\|_{\alpha_0} < \infty.$$

This inequality, together with the first inequality in (7), implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t; t_0, \alpha_0) \geq \frac{\mathrm{d}}{\mathrm{d}t}\alpha(t; t_0, \alpha_0^{(1)});$$

first by continuity in a neighborhood of t_0 and consequently for all $t_0 \leq t < T_M(t_0,\alpha_0)$. Thus, we have

$$T_M(t_0,\alpha_0) \leq T_M(t_0,\alpha_0^{(1)}) \text{ and } \alpha(t; t_0, \alpha_0) - \alpha(t; t_0, \alpha_0^{(1)}) \geq \alpha_0 - \alpha_0^{(1)} > 0.$$

This, however, means that

$$\alpha(T_M(t_0,\alpha_0); t_0, \alpha_0) - \alpha(T_M(t_0,\alpha_0); t_0, \alpha_0^{(1)}) \geq \alpha_0 - \alpha_0^{(1)} > 0.$$

By part (iv) of Theorem 7.2, we must have $(T_M(t_0,\alpha_0), \alpha(T_M(t_0,\alpha_0); t_0, \alpha_0^{(1)})) \in \mathcal{D}$. Thus, by (part (ii) of) Theorem 5.3, the solution $\alpha(t; t_0, \alpha_0^{(1)})$ can be extended beyond $T_M(t_0,\alpha_0)$, i.e., $T_M(t_0,\alpha_0) < T_M(t_0,\alpha_0^{(1)})$.

Concerning the second assertion, there is nothing to prove in case $T_M(t_0^{(1)},\alpha_0) \leq t_0$. Otherwise, we note that $\alpha_1 := \alpha(t_0; t_0^{(1)}, \alpha_0) > \alpha_0$. By uniqueness, for all $t_0 < t$, we have $\alpha(t; t_0^{(1)}, \alpha_0) = \alpha(t; t_0, \alpha_1)$. Now we apply the previous part to arrive at the desired conclusion. □

Remark 7 Due to Theorem 7.2 (more specifically, part (ii)), if $\alpha_c(\tau_0) < \overline{\alpha}_c(\tau_0)$ then the set I_{τ_0} is a proper interval (i.e., I_{τ_0} is not a singleton point) belonging to $\partial^+\mathcal{D}$ and in this case, we will call I_{τ_0} a " downward spike" in the domain $\mathcal{D}$. We will also call a downward spike I_{τ_0} a "sharp downward spike" if

$$\underline{\alpha}_c(\tau_0) = \liminf_{t\to\tau_0} \alpha_c(t) > \alpha_c(\tau_0).$$

The above terminologies are motivated by the fact that α_c is a lower semi-continuous function. Note that in case I_{τ_0} is a sharp downward spike, then for any $\beta_0 \in [\alpha_c(\tau_0), \underline{\alpha}_c(\tau_0))$, there exists a $\delta > 0$ such that the set

$$\{(t,\beta) : t \neq \tau_0, \tau_0 - \delta \leq t \leq \tau_0 + \delta, 0 \leq \beta \leq \beta_0\} \subset \mathcal{D}.$$

Note that by part (ii) of Theorem 7.2, $\beta_c(\tau_0) > \alpha_c(\tau_0)$ is possible only in case I_{τ_0} is a downward spike. The following result concerns the occurrence of downward spikes in general, as well as sharp downward spikes, and spikes for which $\beta_c(\tau_0) > \alpha_c(\tau_0)$.

Theorem 7.4 *Let $\ell_1, \ell_2 > 0$ and define the sets*

$$X_{\ell_1,\ell_2} = \{t \in [0, t_\infty) : \alpha_c(t) \leq \ell_1 \textit{ and } \overline{\alpha}_c(t) - \alpha_c(t) \geq \ell_2\},$$
$$X = \{t \in (0, t_\infty) : \underline{\alpha}_c(t) > \alpha_c(t)\} \textit{ and } Y = \{t \in (0, t_\infty) : \beta_c(t) > \alpha_c(t)\}.$$

The set X_{ℓ_1,ℓ_2} is closed and nowhere dense while the set X, which is precisely the set of all (time points of) sharp downward spikes, and the set Y are countable.

Proof The lower semi-continuity of α_c and the definition of the set X_{ℓ_1,ℓ_2} show that it is closed. Let $t_0 \in X_{\ell_1,\ell_2}$. Then, t_0 is a point of discontinuity of the lower semi-continuous function α_c. As in proof of part (iii) of Theorem 7.2, the set of such points is nowhere dense.

To show that X is countable, for $\ell_1, \ell_2 > 0$, let

$$\underline{X}_{\ell_1,\ell_2} = \{t \in [0, t_\infty) : \alpha_c(t) \leq \ell_1 \text{ and } \underline{\alpha}_c(t) - \alpha_c(t) \geq \ell_2\}.$$

Let $t_0 \in \underline{X}_{\ell_1,\ell_2}$ and $\epsilon = \dfrac{\ell_2}{3}$ and note that there exists $\delta > 0$ such that for all $t \in [t_0 - \delta, t_0 + \delta]$, $t \neq t_0$, we have $\alpha_c(t) > \ell_1 + \ell_2 - \epsilon$. This implies that the set

$$\{(t, \beta) : t \in [t_0 - \delta, t_0 + \delta], \beta \in [0, \ell_1 + \ell_2 - \epsilon]\} \subset \mathcal{D},$$

and consequently

$$\alpha_c(t) \geq \ell_1 + \ell_2 - \epsilon > \ell_1 \text{ for all } t_0 - \delta < t < t_0 + \delta, t \neq t_0.$$

Thus, given $t_0 \in \underline{X}_{\ell_1,\ell_2}$, there exists $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \cap \underline{X}_{\ell_1,\ell_2} = \{t_0\}$, that is, $\underline{X}_{\ell_1,\ell_2}$ consists of isolated points and is therefore countable. Taking union over all positive rational numbers, we conclude that X is countable.

Note that if $\beta < \beta_c(t)$, then the set $\Gamma_{(t,\beta)}$, defined in (16), is contained in $\mathcal{D}$. This implies that $\liminf_{s\to t^+} \alpha_c(s) \geq \beta_c(t)$. Thus, in order to show that Y is countable, it will be enough to show that the set $\{t \in (0, t_\infty) : \liminf_{s\to t^+} \alpha_c(s) > \alpha_c(t)\}$ is countable. To that end, for $\ell_1, \ell_2 > 0$, define

$$Y_{\ell_1,\ell_2} = \{t \in (0, t_\infty) : \alpha_c(t) \leq \ell_1 \text{ and } \liminf_{s\to t^+} \alpha_c(s) - \alpha_c(t) \geq \ell_2\}.$$

Let $t_0 \in Y_{\ell_1,\ell_2}$ and $\epsilon = \dfrac{\ell_2}{3}$ and note that $\|\mathbf{u}(t_0)\|_\beta < \infty$ and $\Gamma_{9t_0,\beta)} \subset \mathcal{D}$, where $\beta = \ell_1 + \ell_2 - \epsilon$. Consequently, there exists maximal $\delta_{t_0} \in (0, \infty]$ such that the set

$$\{(t, \beta) : t_0 < t < t_0 + \delta_{t_0}, \beta = \ell_1 + \ell_2 - \epsilon\} \subset \mathcal{D}$$

and either the point $(t_0 + \delta_{t_0}, \ell_1 + \ell_2 - \epsilon) \in \partial^+\mathcal{D}$ or $t_0 + \delta_0 = t_\infty$. Clearly, the open interval $(t_0, t_0 + \delta_{t_0})$ does not belong to Y_{ℓ_1,ℓ_2} since for $t \in (t_0, t_0 + \delta_{t_0})$, $\alpha_c(t) \geq \beta > \alpha_c(t_0)$. Pick a rational number $r(t_0)$ belonging to the open interval $(t_0, t_0 + \delta_{t_0})$. Since for $t_0, t_0' \in Y_{\ell_1,\ell_2}$, the intervals $(t_0, t_0 + \delta_{t_0})$ and $(t_0', t_0' + \delta_{t_0'})$ are disjoint, this defines an injective map into the rationals. Thus, the set Y_{ℓ_1,ℓ_2} is countable. Taking union over all rationals $\ell_1, \ell_2 > 0$ finishes the proof of the lemma. □

Remark 8 The time events described in Theorem 7.4 may turn out to be related to intermittent events in the dynamics of the flow represented by the solution $\mathbf{u}(t)$. If this turns out to be true, then it would be interesting to see if ℓ_1 and ℓ_2 play a role similar to the micro- and macro-length scales in the Kolmogorov theory. See also Theorem 10.5 for a connection between these downward spikes and energy cascades. An alternate approach to intermittency, based on time averages of higher moments of the vorticity, is presented in [19]. Since higher moments can be bounded by higher order derivatives and the Gevrey norm "contains" all higher derivatives, there may be a connection between these approaches.

Definition 7.5 We say that a point $(T, \alpha_c(T)) \in \partial^+\mathcal{D}$ is "reachable" if there exists a $(t_0, \alpha_0) \in \mathcal{D}$, $\delta > 0$ and a solution curve $\alpha(t; t_0, \alpha_0)$ of (1) such that

$$T_M(t_0, \alpha_0) = T \text{ and } \alpha(T; t_0, \alpha_0) := \lim_{t \nearrow T} \alpha(t; t_0, \alpha_0) = \alpha_c(T).$$

Remark 9 By part (iv) of Theorem 7.2, if I_{τ_0} is a downward spike and for a solution curve $\alpha(\cdot; t_0, \alpha_0)$ we have $T_M(t_0, \alpha_0) = \tau_0$ and $\alpha(\tau_0; t_0, \alpha_0) < \infty$, then $\alpha(\tau_0; t_0, \alpha_0) = \alpha_c(t_0)$ and the last equality in the above line automatically holds. This in particular implies that if a solution curve $\alpha(\cdot)$ reaches a downward spike I_{τ_0} of "finite length" (i.e., $\overline{\alpha}_c(\tau_0) < \infty$), it only does so at its lowest endpoint $\alpha_c(t_0)$.

If $t_1 < t_2 < t_\infty$, it follows from Theorem 6.1 that there exists a solution $\alpha(t; t_1, 0)$ which blows up in the open interval (t_1, t_2), that is, $T_M(t_1, 0) \le t_2$. Thus, if there exists an interval (t_1, t_2) such that $\alpha_c(t) < \infty$ for all $t \in (t_1, t_2)$, then $\{t \in (t_1, t_2) : (t, \alpha_c(t)) \text{ is reachable}\}$ is dense subset of (t_1, t_2).

In the proposition below, we establish some sufficient conditions for points on the boundary $\partial^+\mathcal{D}$ to be reachable. In the proof of this proposition, we will need to consider a family of solutions curves of (1) where the parameter δ changes. Thus, instead of omitting δ from the notation, we will explicitly write $\alpha_\delta(t; t_0, \alpha_0)$ for a solution of (1).

Proposition 7.6 *Let $T < t_\infty$, $\alpha_c(T) < \infty$.*

(i) *Assume that there exists a continuous curve $\gamma : [t_0, T] \to \mathbb{R}^+$, with $(t, \gamma(t)) \in \mathcal{D}$ for $t \in (t_0, T)$ and satisfying the conditions*

$$\gamma(T) = \alpha_c(T),\ \frac{d}{dt}\gamma(t) \text{ exists with } \frac{d}{dt}\gamma(t) \le M < \infty \text{ for } t \in (t_0, T). \tag{57}$$

In this case, the boundary point $(T, \alpha_c(T))$ is reachable via a solution curve $\alpha_\delta(t; t_0, 0)$, and moreover, the corresponding δ can be chosen to satisfy (37).

(ii) *If*

$$\limsup_{s \to T^-} \frac{\alpha_c(T) - \alpha_c(s)}{T - s} \le M < \infty, \tag{58}$$

then there exists a γ satisfying condition in part (i) and, consequently, the conclusions as in part (i) hold.

(iii) *Let $\alpha_{\delta'}(\cdot; t_0', \alpha_0')$ satisfy $T_M(t_0', \alpha_0') = T$ and $\alpha_{\delta'}(T; t_0', \alpha_0') = \infty$. Then, $\overline{\alpha}_c(\tau_0) = \infty$ and the boundary point $(T, \alpha_c(T))$ is reachable via a solution $\alpha_\delta(\cdot; t_0, 0)$ where δ satisfies (37).*

(iv) *Assume that the point $(T, \alpha_c(T))$ is reachable. Then, there exists a δ satisfying (37) such that the solution $\alpha_\delta(\cdot; t_0, 0)$ reaches $(T, \alpha_c(T))$.*

Proof For notational simplicity, for any $\delta > 0$, we will denote by $\alpha_\delta(t)$ the solution $\alpha_\delta(t; t_0, 0)$; similarly, the corresponding existence time $T_M(t_0, 0, \delta)$ will be denoted by T_δ. We will first prove part (i). Let

$$M_1 = \inf_{s \in [t_0, T]} |\mathbf{u}(s)| \text{ and } \delta_0 = \max\left\{ \frac{4\nu c_0}{\kappa_0^{1/2} M_1}, \frac{4\nu^2}{\kappa_0 M_1^2}, \frac{2M\nu}{\kappa_0^2 M_1^2} \right\}. \tag{59}$$

Due to (32), by choosing t_0 sufficiently close to T, we can without loss of generality, assume that $T_{\delta_0} > T$. Now define

$$\Upsilon = \{\delta \geq \delta_0 : T_\delta > T\} \quad \text{and} \quad \Upsilon_{sol} := \{\alpha_\delta(\cdot) : \delta \in \Upsilon\}.$$

Clearly,the set Υ is nonempty since $\delta_0 \in \Upsilon$ and, for all $\delta \in \Upsilon$, the condition (37) holds. Moreover, since $T_\delta > T$ for all $\delta \in \Upsilon$, due to Theorem 4.5 part (iv), we have

$$\alpha_\delta(T) < \alpha_c(T), \quad (\delta \in \Upsilon). \tag{60}$$

We now observe that

$$\alpha_\delta(t) \leq \gamma(t) \quad \text{for all} \quad \delta \in \Upsilon, t \in [t_0, T). \tag{61}$$

Assume to the contrary that there exists $\delta \in \Upsilon$, $t_1 \in (t_0, T)$ such that $\alpha_\delta(t_1) > \gamma(t_1)$. Due to (59), for all $\delta \in \Upsilon$ and for all $t \in (t_0, T)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_\delta(t) - \gamma(t)) \geq (\frac{\delta}{\nu}\|\mathbf{u}(t)\|^2 - M) \geq M.$$

This in particular means that $\alpha_\delta(t) - \gamma(t) > \alpha_\delta(t_1) - \gamma(t_1)$ for all $t \in (t_1, T)$. Consequently, we must have $\alpha_\delta(T) > \gamma(T) = \alpha_c(T)$ which contradicts (60).

Due to the blow-up result (39), there exists $\delta' > \delta_0$ such that $T_{\delta'} < T$. Since $\alpha_\delta(t)$ is increasing in δ for each fixed t, this implies $T_\delta \leq T_{\delta'} < T$ for all $\delta \geq \delta'$. In other words, if $\delta \in \Upsilon$, then $\delta < \delta'$. It follows that the set Υ is bounded above, and we will denote $\delta_{\text{sup}} = \sup\{\delta : \delta \in \Upsilon\}$. We claim that

$$T_{\delta_{\text{sup}}} = T \text{ and } \lim_{t \nearrow T} \alpha_{\delta_{\text{sup}}}(t) = \alpha_c(T).$$

To see this, choose a (strictly) increasing sequence $\delta_n \nearrow \delta_{\text{sup}}$, $\delta_n \in \Upsilon$ and denote $\alpha_n(\cdot) = \alpha_{\delta_n}(\cdot)$. For each fixed t, the sequence $\alpha_n(t)$ is an increasing sequence and thus $\alpha(t) := \lim_{n\to\infty} \alpha_n(t)$ exists, and moreover, due to (61), $\alpha(t) \in \mathcal{D}$. Furthermore, for all $\delta \in \Upsilon$, since $\alpha_\delta(T) = \lim_{t\to T^-} \alpha_\delta(t) \leq \alpha_c(T) < \infty$, $\alpha_\delta(\cdot)$ is a continuous function on the compact set $[t_0, T]$. Applying Dini's theorem, we conclude that $\alpha_n(\cdot)$ converge uniformly to the continuous function $\alpha(\cdot)$ on the compact interval $[t_0, T]$.

For a fixed $t \in (t_0, T)$, since $\alpha(t) \in \mathcal{D}$, there exists a square

$$\mathcal{R} = \{(s, \beta) : |s - t| \leq \epsilon, |\beta - \alpha(t)| \leq \epsilon\} \subset \mathcal{D}.$$

The uniform convergence of $\alpha_n(\cdot)$ to $\alpha(\cdot)$ now implies that $\frac{\delta_n}{\nu}\|\mathbf{u}(\cdot)\|^2_{\alpha_n(\cdot)}$ converge uniformly to $\frac{\delta}{\nu}\|\mathbf{u}(\cdot)\|^2_{\alpha(\cdot)}$ on the set $[t - \epsilon, t + \epsilon]$. This readily implies that $\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)$ exists for all $t \in (t - \epsilon, t + \epsilon)$ and $\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s) = \frac{\delta}{\nu}\|\mathbf{u}(s)\|^2_{\alpha(s)}$. Since $t \in (t_0, T)$ is arbitrary, this implies that $\alpha(\cdot)$ is the unique solution $\alpha_{\delta_{\text{sup}}}(\cdot)$, and moreover, we must have $T_{\delta_{\text{sup}}} \geq T$. Note now that if $T_{\delta_{\text{sup}}} > T$, then by part (iv) of Theorem 7.2, we have $\alpha_{\delta_{\text{sup}}}(T) < \alpha_c(T)$, and by the stability result, namely Theorem 5.6, there exists $\delta > \delta_{\text{sup}}$ such that $T_\delta > T$. This contradicts the definition of δ_{sup}. Thus, $T_{\delta_{\text{sup}}} = T$ and $\alpha_{\delta_{\text{sup}}}(T) = \alpha_c(T)$.

To prove part (ii), note that if (58) holds, then we can take $\gamma(t)$ to be the straight line passing through the point $(T, \alpha_c(T))$ and having slope $2M$.

In order to prove part (iii), observe that since $\alpha_{\delta'}(T; t_0', \alpha_0') = \infty$, there exists $t_1 < T$ such that $\alpha_{\delta'}(t_1; t_0', \alpha_0') = \alpha_c(T) < \infty$. Since the region under the curve $\alpha_{\delta'}(\cdot; t_0', \alpha_0')$ belongs to $\mathcal{D}$, we can take the function $\gamma(t)$ to be the constant function $\gamma(t) = \alpha_c(T), t_1 \leq t \leq T$. Clearly, $\gamma(\cdot)$ satisfies (57) (in fact, with $M = 0$), and we can apply part (i) to obtain the desired conclusion.

We will now prove (iv). Let $\alpha_{\delta'}(t; t_0', \alpha_0')$ be a curve reaching $(T, \alpha_c(T))$. Note that without loss of generality (by choosing a larger t_0' if necessary), we may assume that $\alpha_0' > 0$. Let δ_0 be as defined in (59) with $M = 0$. Choose $t_0 > t_0'$ sufficiently close to T so that $T_{\delta_0} > T$ and define the set Υ as in part (i). Note now that if there exists $\delta \in \Upsilon$ and $t_1 > t_0$ such that $\alpha_\delta(t_1) > \alpha_{\delta'}(t_1; t_0', \alpha_0')$, then since $t_0 > t_0'$ and $\alpha_\delta(t_0) = 0 < \alpha_0'$, it must be the case that $\delta > \delta'$. This then implies that $\alpha_\delta(T) > \alpha_{\delta'}(T; t_0', \alpha_0') = \alpha_c(T)$ contradicting (60). Consequently, we must have

$$\alpha_\delta(t) \leq \alpha_{\delta'}(t; t_0', \alpha_0') \text{ for all } t \in (t_0, T), \delta \in \Upsilon.$$

Due to this condition and the fact that $\alpha_{\delta'}(t; t_0', \alpha_0') \in \mathcal{D}$ for all $t \in (t_0', T)$, setting $\gamma(t) = \alpha_{\delta'}(t; t_0', \alpha_0')$, we conclude that (61) holds. We can now essentially repeat the proof of part (i) to obtain the conclusion of part (iv). □

Definition 7.7 We say that a boundary point $(T, \alpha_c(T))$ of $\mathcal{D}$ with $T < t_\infty$ is a "Leray boundary point" if $\alpha_c(T) < \infty$, $(T, \alpha_c(T))$ is reachable (see Definition 7.5) and $\|\mathbf{u}(T)\|_{\alpha_c(T)} < \infty$. On the other hand, we say that I_T is a "Leray spike" if $(T, \alpha_c(T))$ is a Leray boundary point and $\beta_c(T) > \alpha_c(T)$.

Remark 10 Recall that according to Leray, a weak solution $\mathbf{u}(t)$ of the Navier–Stokes equation may lose regularity as one approaches point T from the left, but might regain regularity at the point T. More precisely, this means that $\lim_{t \nearrow T^-} \|\mathbf{u}(t)\| = \infty$ while $\|\mathbf{u}(T)\| < \infty$. Such a scenario is believed to be connected to the turbulence of the flow. The definition of Leray boundary points and Leray spikes given above are motivated by the Leray blow-up scenario as we clarify below.

(i) If I_T is a Leray spike, then it is indeed a downward spike since $\beta_c(T) > \alpha_c(T)$. Moreover, due to Theorem 7.4, the set of all Leray spikes is countable.

(ii) If $(T, \alpha_c(T))$ is a Leray boundary point and $\alpha(\cdot)$ is a solution curve that reaches it starting from a point $(t_0, \alpha_0) \in \mathcal{D}$, then $T = T_M(t_0, \alpha_0)$, $\alpha_c(T) = \alpha(T)$ and $e^{\alpha(t)A^{1/2}}\mathbf{u}(t) \in V$ for all $t \in [t_0, T)$. Additionally,

$$\sup_{t \in [t_0, T_1]} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty, \lim_{t \nearrow T^-} \|\mathbf{u}(t)\|_{\alpha(t)} = \infty \text{ and } \|\mathbf{u}(T)\|_{\alpha(T)} < \infty,$$

where $t_0 < T_1 < T$. This precisely mimics the Leray blow-up scenario in Gevrey classes.

(iii) Due to Proposition 7.6, all sharp downward spikes are reachable. Consequently, a sharp downward spike I_T with $\beta_c(T) > \alpha_c(T)$ is a Leray spike.

(iv) Some connections between Leray boundary points and Leray spikes and turbulence are discussed in Sect. 10. A more detailed analysis of the behavior of these downward spikes will be presented in [2].

8 An integral formula and applications

In the sequel, as before, we will write $\alpha(t, s) = \alpha(t; s, 0)$ and $\alpha(t)$ for $\alpha(t, 0)$. If the second variable in $\alpha(t, s)$, $t \geq s$ is not zero, it will be explicitly written.

Theorem 8.1 *Let (t_0, α_0) be such that $\|\mathbf{u}(t_0)\|_{\alpha_0} < \infty$ and $0 \leq t_0 \leq t < T_M(t_0, \alpha_0)$. In this case, we have*

$$\alpha(t; t_0, \alpha_0) = \alpha_0 + \frac{\delta}{\nu} \int_{t_0}^{t} ds\, \|u(s)\|^2\, e^{\frac{2\delta}{\nu} \int_s^t |A^{3/4}u(\tau)|^2_{\alpha(\tau; s, \alpha_0)} d\tau}. \tag{62}$$

In particular, this implies that the right-hand side of (62) *is finite for all* $t < T_M(t_0, \alpha_0)$*, and moreover,* (62) *holds.*

Proof From (1), it follows that

$$\alpha(t; t_0, \alpha_0) = \frac{\delta}{\nu} \int_{t_0}^{t} \|e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau)\|^2 \mathrm{d}\tau, \quad t_0 \le t < T_M(t_0, \alpha_0). \tag{63}$$

Note that due to (21), for all $t_0 \le t < T_M(t_0, \alpha_0)$ we have

$$\int_{t_0}^{t} |A^{3/4} e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau)|^2 \mathrm{d}\tau \le \lambda_0^{-1/2} \int_{t_0}^{t} |A e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau)|^2 \mathrm{d}\tau < \infty,$$

where λ_0 denotes the smallest eigenvalue of A. We may therefore differentiate the integral in (63) with respect to t_0 and obtain

$$\begin{aligned} &\frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0} \\ &= -\frac{\delta}{\nu} \|u(t_0)\|_{\alpha_0}^2 + 2\frac{\delta}{\nu} \int_{t_0}^{t} (A e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau), A^{1/2} e^{\alpha(\tau, t_0) A^{1/2}} u(\tau)) \frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0} \mathrm{d}\tau \\ &= -\frac{\delta}{\nu} \|u(t_0)\|_{\alpha_0}^2 + 2\frac{\delta}{\nu} \int_{t_0}^{t} |A^{3/4} e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau)|^2 \frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0} \mathrm{d}\tau. \end{aligned} \tag{64}$$

Now by differentiating (64) with respect to t, we also have

$$\frac{\partial}{\partial t} \left(\frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0} \right) = 2\frac{\delta}{\nu} |A^{3/4} e^{\alpha(t; t_0, \alpha_0) A^{1/2}} u(t)|^2 \frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0}.$$

Furthermore, $\frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0}|_{t_0 = t} = -\frac{\delta}{\nu} \|u(t_0)\|_{\alpha_0}^2$, and consequently,

$$\frac{\partial \alpha(t; t_0, \alpha_0)}{\partial t_0} = -\frac{\delta}{\nu} \|u(t_0)\|^2 e^{\frac{2\delta}{\nu} \int_{t_0}^{t} |A^{3/4} e^{\alpha(\tau; t_0, \alpha_0) A^{1/2}} u(\tau)|^2 \mathrm{d}\tau}. \tag{65}$$

Integrating this equation with respect to t_0 yields (62). □

The integral equation (62) immediately yields some lower bounds on α. Under some circumstances, it improves to the linear lower estimate of [FT]. Note also that all these estimates are valid for $t < T_M(t_0, \alpha_0)$, which on the other hand obeys the inequality (32).

Recall that we denoted by $\kappa_0 = \frac{2\pi}{L}$ the first eigenvalue of $A^{1/2}$. Due to Remark 4, we will assume that

$$\|u(t)\|^2 \geq \frac{\kappa_0 \nu^2}{4c_0^2}, t \in [0, T]. \tag{66}$$

Otherwise, if $\|u(t')\| < \frac{\kappa_0 \nu^2}{4c_0^2}$ then we have exponential decrease of $\|\mathbf{u}(t)\|$ for all $t \geq t'$.

Proposition 8.2 *For* $t < T_M(t_0, \alpha_0)$

$$\alpha(t, t_0) \geq \alpha_0 + \frac{1}{2\kappa_0} \left(e^{\frac{2\delta}{\nu} \int_{t_0}^{t} \|\mathbf{u}(\tau)\|^2 \, d\tau} - 1 \right) = \alpha_0 + \frac{1}{2} \left(e^{\frac{2\delta}{\nu}(|\mathbf{u}(t_0)|^2 - |\mathbf{u}(t)|^2)} - 1 \right). \tag{67}$$

In case (66) *holds, then for all* $t < T_M(t_0, \alpha_0)$

$$\alpha(t; t_0, \alpha_0) \geq \alpha_0 + \frac{1}{2\kappa_0} \left(e^{\frac{\delta \kappa_0 \nu}{2c_0^2}(t - t_0)} - 1 \right). \tag{68}$$

Additionally, if $t_\infty < \infty$*, then for an adequate dimensionless absolute constant* c_3*, for all* $t < T_M(t_0, \alpha_0)$ *we have the estimate*

$$\alpha(t; t_0, \alpha_0) \geq \alpha_0 + \frac{1}{2\kappa_0} e^{\frac{c_3 \delta \sqrt{\nu}(t - t_0)}{\sqrt{t_\infty}}}. \tag{69}$$

Proof Inequality (67) follows from (62) by observing

$$\alpha(t, t_0) \geq \alpha_0 + \frac{\delta}{\nu} \int_{t_0}^{t} \mathrm{d}s \, \|\mathbf{u}(s)\|^2 e^{\frac{2\kappa_0 \delta}{\nu} \int_s^t \|\mathbf{u}(\tau)\|^2 \, \mathrm{d}\tau}$$

and recognizing that the integrand is an exact derivative.

The subsequent estimate (68) for α is obtained from (66). Finally, (69) follows by replacing the estimate (66) by the estimate (34), with $t_0 = 0$, $\delta = 0$, $\alpha_0 = 0$ and $T_M(t_0, \alpha_0) = t_\infty$. □

9 Global boundedness of $|u|_\beta$

In this section, we will set $\alpha_0 = 0$. We will also assume that $t_\infty < \infty$. We will need the following lemma and the subsequent proposition to proceed.

From Theorem 4.5, recall that $T_M(t_0) = T_M(t_0, 0)$ is the maximal interval of existence of (1) with $\alpha_0 = 0$.

Lemma 9.1 *There exists a nondecreasing function* ψ *from* $[0, t_\infty)$ *onto* $[0, t_\infty)$ *satisfying* $\psi(t) < t < T_M(\psi(t))$ *for each* $t \in [0, t_\infty)$*. Moreover, there exists* $t_1 \in (0, t_\infty)$ *such that for all* $t \geq t_1$ *, with* c_1 *be as in* (10)*, we have*

$$\beta(t) := \alpha(t, \psi(t)) = \frac{\epsilon \nu^2}{\|u(\psi(t))\|^2}, \text{ where } \epsilon = \frac{\delta}{16(c_1 + \delta^2)}. \tag{70}$$

Moreover, $\dot{\psi}(t)$ *exists for all* $t \in (t_1, t_\infty)$ *with*

$$0 < \dot{\psi}(t) \leq 4 \text{ and } \|u(t)\|^2_{\beta(t)} \leq 2\|u(\psi(t))\|^2. \tag{71}$$

Proof Recall that $\alpha(t_0; t_0) = 0$ and due to (33) and moreover, $\|\mathbf{u}(t)\|_{\alpha(t, t_0)} < \infty$ if $\alpha(t, t_0) < \dfrac{\delta \nu^2}{(c_1 + \delta^2)\|\mathbf{u}(t_0)\|^2}$. On the other hand, loss of regularity occurs at $t_\infty < \infty$ by our assumption. Together with the monotonicity of $\alpha(\cdot, t_0)$, this implies that for each $0 \le t_0 < t_\infty$, there exists an unique point $\phi(t_0)$ with $t_0 < \phi(t_0) < T_M(t_0)$ such that

$$\alpha(\phi(t_0), t_0) = \frac{\epsilon\nu^2}{\|u(t_0)\|^2}.$$

By differentiating this expression, we get

$$\phi'(t_0) = \frac{-\frac{\epsilon\nu^2}{\|\mathbf{u}(t_0)\|^4}\frac{\mathrm{d}}{\mathrm{d}\tau}\|\mathbf{u}(\tau)\|^2|_{\tau=t_0} - \frac{\partial\alpha(\phi(t_0),t_0)}{\partial t_0}}{\frac{\partial\alpha(\phi(t_0),t_0)}{\partial t}|_{t=\phi(t_0)}}. \tag{72}$$

A standard computation as in the derivation of (27) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 \le \frac{c_2}{\nu^3}\|\mathbf{u}\|^6 \text{ where } c_2 = \frac{27}{128}c_0^4.$$

On the other hand, (65) and (22) immediately yield

$$\frac{\partial\alpha(\phi(t_0), t_0)}{\partial t_0} \le -\frac{\delta}{\nu}\|\mathbf{u}(t_0)\|^2 \text{ and } \|\mathbf{u}(t)\|^2_{\alpha(\phi(t_0),t_0)} \le 2\|\mathbf{u}(t_0)\|^2. \tag{73}$$

Recalling that $\frac{\partial\alpha(\phi(t_0),t_0)}{\partial t}|_{t=\phi(t_0)} = \frac{\delta}{\nu}\|\mathbf{u}(\phi(t_0))\|^2_{\alpha(\phi(t_0),t_0))}$ and inserting the three estimates obtained above in (72), we get

$$\phi'(t_0) \ge \frac{-\frac{c_2\epsilon}{\nu}\|\mathbf{u}(t_0)\|^2 + \frac{\delta}{\nu}\|\mathbf{u}(t_0)\|^2}{2\frac{\delta}{\nu}\|\mathbf{u}(t_0)\|^2} \ge \frac{1}{4}.$$

In the very last inequality, we used the definitions of ϵ, c_1 and c_2.

This implies that $\phi(t)$ is strictly increasing. Moreover, since $t < \phi(t) < t_\infty$, clearly, $\lim_{t\to t_\infty^-} \phi(t) = t_\infty$, that is, ϕ maps $[0, t_\infty)$ monotonically onto the interval $[\phi(0), t_\infty)$. Consequently, we may define

$$\psi(t) = \begin{cases} 0 & \text{if } t \le t_1 := \phi(0) \\ \phi^{-1}(t) & \text{if } t \in (t_1, t_\infty] \end{cases}$$

Clearly, ψ is strictly increasing on the interval (t_1, t_∞), and therefore, $\psi' > 0$. The upper estimate on ψ' follows from the lower estimate obtained above on $\phi'(t_0)$. In view of (73), the proof of (71) is now complete. □

We will need the following proposition to proceed.

Proposition 9.2 *Let* $\alpha > 0$. *We have the estimate*

$$|(B(\mathbf{u}, \mathbf{u}), e^{2\alpha A^{1/2}}\mathbf{u})| \le c_4\alpha^{1/2}\|\mathbf{u}\|^3_\alpha, \ c_4 = 2c_0\frac{1}{\sqrt{e}}. \tag{74}$$

Proof Let $|\mathbf{u}|_m = |A^{m/2}\mathbf{u}|$ and $m_i \ge 0, i = 1, 2, 3;\ m_1 + m_2 + m_3 \ge 3/2$ such that at least two of $\{m_1, m_2, m_3\}$ are strictly positive. Then (see [CF]),

$$|(B(\mathbf{u}, \mathbf{v}), \mathbf{w})| \le c_0|\mathbf{u}|_{m_1}|\mathbf{v}|_{m_2+1}|\mathbf{w}|_{m_3}. \tag{75}$$

Let $\mathbf{u}, \mathbf{v} \in H$, $0 \le \beta \le \alpha$ and assume $e^{\alpha A^{1/2}}\mathbf{u}, e^{\alpha A^{1/2}}\mathbf{v} \in H$. We have

$$
\begin{aligned}
|(B(\mathbf{u},\mathbf{v}), e^{2\beta A^{1/2}}A^{1/2}\mathbf{v})| &\le (\kappa_0)^3 \sum_{h+k+l=0} |\mathbf{u}_k||h||\mathbf{v}_h||l|e^{2\beta|l|}|\mathbf{v}_l| \\
&\le (\kappa_0)^3 \sum_{h+k+l=0} e^{\beta|k|}|\mathbf{u}_k||h|e^{\beta|h|}|\mathbf{v}_h||l|e^{\beta|l|}|\mathbf{v}_l| \qquad (76) \\
&\le c_0|A^{5/8}e^{\beta A^{1/2}}\mathbf{u}||A^{5/8}e^{\beta A^{1/2}}\mathbf{v}||A^{1/2}e^{\beta A^{1/2}}\mathbf{v}| \qquad (77) \\
&= c_0|A^{1/8}e^{-(\alpha-\beta)A^{1/2}}A^{1/2}\mathbf{u}||A^{1/8}e^{-(\alpha-\beta)A^{1/2}}A^{1/2}\mathbf{v}||e^{-(\alpha-\beta)A^{1/2}}A^{1/2}\mathbf{v}| \\
&= c_0\frac{1}{2\sqrt{e}}\frac{1}{(\alpha-\beta)^{1/2}}\|\mathbf{u}\|_\alpha\|\mathbf{v}\|_\alpha^2 \qquad (78)
\end{aligned}
$$

where to obtain the inequality in (76), we used $-l = h + k$ and the triangle inequality and to obtain the inequality in (77), we proceed as in the proof of Lemma 2.1 in [FT] and use (75) with $m_1 = 5/4, m_2 = 1/4, m_3 = 0$. To obtain (78), we used (6).

Using the orthogonality $(B(\mathbf{u},\mathbf{u}),\mathbf{u}) = 0$ and the fact that

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha}(B(\mathbf{u},\mathbf{u}), e^{2\alpha A^{1/2}}\mathbf{u}) = 2(B(\mathbf{u},\mathbf{u}), e^{2\alpha A^{1/2}}A^{1/2}\mathbf{u})
$$

we have

$$
|(B(\mathbf{u},\mathbf{u}), e^{2\alpha A^{1/2}}\mathbf{u})| = 2\left|\int_0^\alpha (B(\mathbf{u},\mathbf{u}), e^{2\beta A^{1/2}}A^{1/2}\mathbf{u})\,\mathrm{d}\beta\right|
$$

$$
\begin{aligned}
&\le 2\int_0^\alpha |(B(\mathbf{u},\mathbf{u}), e^{2\beta A^{1/2}}A^{1/2}\mathbf{u})|\,\mathrm{d}\beta \le 2c_0\frac{1}{2\sqrt{e}}\|\mathbf{v}\|_\alpha^3\int_0^\alpha \frac{1}{(\alpha-\beta)^{1/2}}\mathrm{d}\beta \qquad (79) \\
&= 2c_0\frac{1}{\sqrt{e}}\alpha^{1/2}\|\mathbf{v}\|_\alpha^3,
\end{aligned}
$$

where to obtain (79), we used (78). □

We will now prove the main theorem of this section.

Theorem 9.3 *Let $t_\infty < \infty$ and β and ψ be as in Lemma 9.1. We have the estimates*

$$
\sup_{t\in(0,t_\infty)} |u(t)|_{\beta(t)} \le |\mathbf{u}_0| + \left(\frac{\delta}{8(c_1+\delta^2)}\right)^{3/2} \nu^{5/4}t_\infty^{1/4} < \infty \text{ and} \qquad (80)
$$

$$
\frac{1}{2\kappa_0}\ln\left(\frac{1}{1-\frac{2c_3\delta\kappa_0\nu^{1/2}(t-\psi(t))}{\sqrt{t_\infty-\psi(t)}}}\right) \le \beta(t) \le \frac{\delta\nu^{1/2}\sqrt{t_\infty-\psi(t)}}{16c_3(c_1+\delta^2)} \qquad (81)
$$

Proof If $t_\infty < \infty$, then (34) holds (with $t_0 = 0, \delta = 0, \alpha_0 = 0$ and $T_M(t_0,\alpha_0) = t_\infty$). Due to (70), this implies the second inequality in (81).

For a fixed t, denote $\gamma(s) := \alpha(s,\psi(t))$, $\psi(t) \le s \le t$. By taking inner product with $e^{2\gamma(s)A^{1/2}}\mathbf{u}$ in (2) we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|u(s)|_{\gamma(s)}^2 + \nu\|u(s)\|_{\gamma(s)}^2 &\le c_4\gamma^{1/2}(s)\|u(s)\|_{\gamma(s)}^3 + \dot\gamma(s)|A^{1/4}u(s)|_{\gamma(s)}^2 \\
&\le c_4\frac{\epsilon^{1/2}\nu}{\|u(\psi(t))\|}\|u(s)\|_{\gamma(s)}^3 + \dot\gamma(s)|A^{1/4}u(s)|_{\gamma(s)}^2 \\
&\le (2\epsilon)^{1/2}c_4\nu\|u(s)\|_{\gamma(s)}^2 + \dot\gamma(s)|A^{1/4}u(s)|_{\gamma(s)}^2. \qquad (82)
\end{aligned}
$$

In the above chain of inequalities, the first one is obtained using (74), the second follows by noting $\gamma(s) \le \gamma(t) = \beta(t)$ and the definition of β in (70) while the third inequality follows from (22). Using Cauchy–Schwartz inequality, the second inequality in (71), the fact that $\gamma(s) \le \beta(t)$ and the definition of β in (70), we have

$$\|A^{1/4}u(s)\|^2_{\gamma(s)} = |(e^{\gamma(s)A^{1/2}}u(s), e^{\gamma(s)A^{1/2}}A^{1/2}u(s))| \le |u(s)|_{\gamma(s)}\|u(s)\|_{\gamma(s)} \le \frac{(2\epsilon)^{1/2}\nu}{\gamma^{1/2}(s)}.$$

Using this and noting that $\frac{\mathrm{d}}{\mathrm{d}s}|u(s)|^2_{\gamma(s)} = |u(s)|_{\gamma(s)}\frac{\mathrm{d}}{\mathrm{d}s}|u(s)|_{\gamma(s)}$, from (82), we get

$$\frac{\mathrm{d}}{\mathrm{d}s}|u(s)|_{\gamma(s)} + \frac{\nu}{2}\frac{\|u(s)\|^2_{\gamma(s)}}{|u(s)|_{\gamma(s)}} \le \dot\gamma(s)\frac{1}{\sqrt{\gamma(s)}}(2\epsilon)^{1/2}\nu.$$

Integrating this inequality between $\psi(t)$ to t and recalling $\gamma(\psi(t)) = \alpha(\psi(t), \psi(t)) = 0$, one obtains

$$|u(t)|_{\beta(t)} + \frac{\nu}{2}\int_{\psi(t)}^{t}\frac{\|u(s)\|^2_{\gamma(s)}}{|u(s)|_{\gamma(s)}}\,ds \le |u(\psi(t))| + 2(2\epsilon)^{1/2}\nu\sqrt{\beta(t)}. \tag{83}$$

Inserting the second estimate from (81) in (83) and noting $|\mathbf{u}(s)| \le |\mathbf{u}_0|$, (80) follows.

We will now prove the first estimate in (81). From (65), using Poincaré inequality and (1), we have

$$\frac{\partial\alpha(t,\tau)}{\partial\tau} \le -\frac{\delta}{\nu}\|u(\tau)\|^2 e^{2\kappa_0\alpha(t,\tau)}$$

which implies

$$-\frac{\partial}{\partial\tau}(e^{-2\kappa_0\alpha(t,\tau)}) \le -\frac{2\delta\kappa_0}{\nu}\|u(\tau)\|^2.$$

Integrating this inequality (in τ) between $\psi(t)$ to t and using (34), we get

$$1 - e^{-2\kappa_0\alpha(t,\psi(t))} \ge \frac{2\delta\kappa_0}{\nu}\int_{\psi(t)}^{t}\|u(\tau)\|^2\,\mathrm{d}\tau \ge \frac{2\delta\kappa_0}{\nu}c_3\nu^{3/2}\int_{\psi(t)}^{t}\frac{1}{\sqrt{t_\infty - \tau}}\,\mathrm{d}\tau$$
$$\ge 2c_3\delta\kappa_0\nu^{1/2}\frac{t-\psi(t)}{\sqrt{t_\infty - \psi(t)}}$$

To obtain the last inequality, we have also used the fact that $\psi(t) < t$. This inequality is equivalent to the first inequality in (81). □

Remark 11 Note that the bound on the right-hand side of (80) involves only $|\mathbf{u}_0|$ (and t_∞) but not $\|\mathbf{u}_0\|$.

10 Cascade of energy and Gevrey norms

In this section, we discuss in more detail some physical effects connected to the oscillations of the norm $\|\mathbf{u}(t)\|^2_{\alpha(t)}$ (Theorem 10.4) and to the occurrence of Leray boundary points and Leray spikes on the domain $\mathcal{D}$ (Theorem 10.5). We will show that the two above-mentioned cases are connected to inverse energy cascades. We have seen in Sect. 7 that the occurrence of Leray boundary points and Leray spikes and consequently the inverse energy cascade associated with them are intermittent events. We need the following preparatory lemmas and propositions to proceed.

Lemma 10.1 *Let* $\{r_i\}_{i=1}^N, \{\eta_i\}_{i=1}^N$, $1 \le N \le \infty$ *be two sequences such that*

$$0 < \eta_{j-1} < \eta_j, \ \sum_{i=1}^{n} \eta_i r_i < 0 \text{ for all } n \le N, 2 \le j \le N. \tag{84}$$

Then for all $m \in \{1, 2, \ldots, N\}$, *we must have* $\sum_{i=1}^m r_i < 0$.

Proof Clearly, the lemma holds for $m = 1$. Assume by induction that it holds for $m = m_0$, i.e., $\sum_{i=0}^m r_i < 0$ for all $m \le m_0$. We now want to show that $\sum_{i=0}^{m_0+1} r_i < 0$. Assume on the contrary that $\sum_{i=0}^{m_0+1} r_i \ge 0$. Due to the induction assumption, this then implies

$$\sum_{i=k}^{m_0+1} r_i \ge -\sum_{i=1}^{k-1} r_i > 0 \text{ for each } 2 \le k \le m_0 + 1. \tag{85}$$

Due to (84), we have by (85) with $k = m_0 + 1$ that

$$\eta_1 r_1 + \cdots + \eta_{m_0}(r_{m_0} + r_{m_0+1}) < \eta_1 r_1 + \cdots + \eta_{m_0} r_{m_0} + \eta_{m_0+1} r_{m_0+1} < 0.$$

Now by (85) with $k = m_0$ and since $0 < \eta_{m_0-1} < \eta_{m_0}$ we have

$$\eta_1 r_1 + \cdots + \eta_{m_0-1}(r_{m_0-1} + r_{m_0} + r_{m_0+1}) < \eta_1 r_1 + \cdots ++ \eta_{m_0-1} r_{m_0-1} + \eta_{m_0}(r_{m_0} + r_{m_0+1}) < 0.$$

Continuing in this way, we get $\eta_1 \left(\sum_{i=1}^{m_0+1} r_i\right) < 0$ which in turn implies $\sum_{i=1}^{m_0+1} r_i < 0$. This leads to a contradiction. □

Proposition 10.2 *Let* $\gamma_1 < \gamma_2$ *and* $\mathbf{v}, \mathbf{w} \in \mathcal{H}$ *be such that*

$$|\mathbf{v}|_{\gamma_1} < |\mathbf{w}|_{\gamma_2} \text{ and } \|\mathbf{w}\|_{\gamma_2} \le \|\mathbf{v}\|_{\gamma_1} < \infty. \tag{86}$$

Let $\Lambda_1 < \Lambda_2 < \cdots$ *be the distinct eigenvalues of* A *and let* R_i *denoting the orthogonal projection onto the eigen space of* Λ_i, $i \ge 1$. *Then, there exists* $m \ge 1$ *such that for all* $l \ge 0$,

$$\sum_{n=m}^{m+l} e^{2(\gamma_2-\gamma_1)\Lambda_n^{1/2}} |R_n \mathbf{w}|^2 < \sum_{n=m}^{m+l} |R_n \mathbf{v}|^2. \tag{87}$$

Proof Let $p_n = |R_n \mathbf{w}|_{\gamma_2}^2 = e^{2\gamma_2 \Lambda_n^{1/2}} |R_n \mathbf{w}|^2$, $q_n = |R_n \mathbf{v}|_{\gamma_1}^2 = e^{2\gamma_1 \Lambda_n^{1/2}} |R_n \mathbf{v}|^2$ and note that

$$\|\mathbf{w}\|_{\gamma_2}^2 = \sum \Lambda_n p_n, \ |\mathbf{w}|_{\gamma_2}^2 = \sum_n p_n, \ \|\mathbf{v}\|_{\gamma_1}^2 = \sum \Lambda_n q_n, \ |\mathbf{v}|_{\gamma_1}^2 = \sum_n q_n. \tag{88}$$

If f_n, g_n are positive and f_n nondecreasing, Abel's (summation by parts) formula yields

$$\sum_{n=0}^{\infty} f_n g_n = f_0 \sum_{n=0}^{\infty} g_n + \sum_{j=1}^{\infty} (f_j - f_{j-1}) \left(\sum_{n=j}^{\infty} g_n\right). \tag{89}$$

Denoting $Q_n = I - \sum_{j=1}^{n-1} R_j$, note that $\sum_{n=j}^{\infty} p_n = |Q_j \mathbf{w}|_{\gamma_2}^2$, $\sum_{n=j}^{\infty} q_n = |Q_j \mathbf{v}|_{\gamma_1}^2$. Applying (89) to (88), we now obtain

$$\|\mathbf{w}\|_{\gamma_2}^2 = \Lambda_0 |\mathbf{w}|_{\gamma_2}^2 + \sum_{j=1}^{\infty} \left(\sum_{n=j}^{\infty} p_n\right)(\Lambda_j - \Lambda_{j-1}) = \Lambda_0 |\mathbf{w}|_{\gamma_2}^2 + \sum_{j=1}^{\infty} |Q_j \mathbf{w}|_{\gamma_2}^2 \left(\Lambda_j - \Lambda_{j-1}\right) \text{ and}$$

$$\|\mathbf{v}\|_{\gamma_1}^2 = \Lambda_0 |\mathbf{v}|_{\gamma_1}^2 + \sum_{j=1}^{\infty} \left(\sum_{n=j}^{\infty} q_n\right)(\Lambda_j - \Lambda_{j-1}) = \Lambda_0 |\mathbf{v}|_{\gamma_1}^2 + \sum_{j=1}^{\infty} |Q_j \mathbf{v}|_{\gamma_1}^2 \left(\Lambda_j - \Lambda_{j-1}\right).$$

Subtracting one from the other, we obtain

$$\begin{aligned}&\sum_{j=1}^{\infty}\left(|Q_j\mathbf{w}|_{\gamma_2}^2-|Q_j\mathbf{v}|_{\gamma_1}^2\right)(\Lambda_j-\Lambda_{j-1})\\&=-\Lambda_0\left(|\mathbf{w}|_{\gamma_2}^2-|\mathbf{v}|_{\gamma_1}^2\right)-\left(\|\mathbf{v}\|_{\gamma_1}^2-\|\mathbf{w}\|_{\gamma_2}^2\right)<0,\end{aligned} \tag{90}$$

where the last inequality follows from (86). On the other hand, the finiteness condition in (86) implies that $|Q_n\mathbf{w}|_{\gamma_2}, |Q_n\mathbf{v}|_{\gamma_1}\to 0$ as $n\to\infty$. Consequently, from (90), it follows that there exists a largest $m<\infty$ such that

$$\min_{n\geq 1}\left(|Q_n\mathbf{w}|_{\gamma_2}^2-|Q_n\mathbf{v}|_{\gamma_1}^2\right)=\left(|Q_m\mathbf{w}|_{\gamma_2}^2-|Q_m\mathbf{v}|_{\gamma_1}^2\right)=\mu<0.$$

By definition of μ and m, this implies that

$$\left(|Q_{m+l+1}\mathbf{w}|_{\gamma_2}^2-|Q_{m+l+1}\mathbf{v}|_{\gamma_1}^2\right)>\mu \text{ for all } l\geq 0. \tag{91}$$

By subtracting, this in turn yields

$$\begin{aligned}&\sum_{n=m}^{m+l}e^{2\gamma_2\Lambda_n^{1/2}}|R_n\mathbf{w}|^2-\sum_{n=m}^{m+l}e^{2\gamma_1\Lambda_n^{1/2}}|R_n\mathbf{v}|^2\\&=|(Q_m-Q_{m+l+1})\mathbf{w}|_{\gamma_2}^2-|(Q_m-Q_{m+l+1})\mathbf{v}|_{\gamma_1}^2\\&=(|Q_m\mathbf{w}|_{\gamma_2}^2-|Q_{m+l+1}\mathbf{w}|_{\gamma_2}^2)-(|Q_m\mathbf{v}|_{\gamma_1}^2-|Q_{m+l+1}\mathbf{v}|_{\gamma_1}^2)\\&=(|Q_m\mathbf{w}|_{\gamma_2}^2-|Q_m\mathbf{v}|_{\gamma_1}^2)-(|Q_{m+l+1}\mathbf{w}|_{\gamma_2}^2-|Q_{m+l+1}\mathbf{v}|_{\gamma_1}^2)<0,\end{aligned}$$

where the very last inequality follows from the definition of μ above and (91). In other words, for all $l\geq 0$, we have (84) with

$$\eta_n=e^{2\gamma_1\Lambda_n^{1/2}} \text{ and } r_n=e^{2(\gamma_2-\gamma_1)\Lambda_n^{1/2}}|R_n\mathbf{w}|^2-|R_n\mathbf{v}|^2.$$

The conclusion now follows from the Lemma 10.1. □

The following proposition will be useful later.

Proposition 10.3 *Let $\alpha(t):=\alpha(t;t_0,\alpha_0)$, $t_0<t<T_M(t_0,\alpha_0)$ be a solution of (1) with δ large as in (37). Denote $T=T_M(t_0,\alpha_0)$ and assume that $\alpha_c(T)<\beta_c(T)\leq\infty$. Then, there exists a sequence $\{t_j\}_{j=1}^{\infty}\subset\mathbb{R}_+$ with $t_j\nearrow T$ and $\{m_j\}_{j=1}^{\infty}\subset\mathbb{N}$, such that (87) holds with*

$$\mathbf{v}=\mathbf{u}(t_j),\ \gamma_1=\alpha(t_j),\ m=m_j,\ j=1,2,\ldots,$$

and

$$\mathbf{w}=\mathbf{u}(T),\ \gamma_2=\begin{cases}\beta_c(T), & \text{if } \beta_c(T)<\infty,\\ \beta,\ \alpha_c(T)<\beta<\infty & \text{if } \beta_c(T)=\infty.\end{cases}$$

In case $\beta_c(T)<\infty$, we have $\gamma_2-\gamma_1\geq\beta_c(T)-\alpha_c(T)$ and $m_j\to\infty$. On the other hand, if $\liminf_{j\to\infty} m_j<\infty$, then $\mathbf{u}(T)$ is a trigonometric polynomial and $\beta_c(T)=\infty$.

Proof Note that since $\alpha_c(T)<\beta_c(T)$, I_T is a downward spike. By Theorem 7.2, $\alpha(T):=\lim_{t\nearrow T}\alpha(t)=\alpha_c(T)$ and moreover, $T<t_\infty$ (otherwise by Theorem 7.2, $(t_\infty,0)\in\mathcal{D}$ while the blow-up point t_∞ cannot belong to $\mathcal{D}$).

Fix an arbitrary number β with $\alpha_c(T)<\beta<\beta_c(T)$ and note that $M:=\|\mathbf{u}(T)\|_\beta<\infty$. By Theorem (5.3), $\lim_{t\nearrow T}\|\mathbf{u}(t)\|_{\alpha(t)}=\infty$. Noting that the points $(t,\alpha(t))\in\mathcal{D}$, this immediately implies that I_T is a Leray spike. Choose a sequence of points t_j, $j \in \mathbb{N}$ such that $\|\mathbf{u}(t_j)\|_{\alpha(t_j)} \geq 2M$ and $t_j \nearrow T$.

Note now that with the notation

$$\gamma_1 = \alpha(t_j), \gamma_2 = \beta, \mathbf{u} = \mathbf{u}(t_j), \mathbf{w} = \mathbf{u}(T), \tag{92}$$

by Proposition 6.2, we must have

$$M \geq |\mathbf{u}(T)|_\beta \geq |\mathbf{u}(T)|_{\alpha(T)} \geq |\mathbf{u}(t_m)|_{\alpha(t_m)}. \tag{93}$$

Due to the fact that $\|\mathbf{u}(t_j)\|_{\alpha(t_j)} \geq 2M \geq \|\mathbf{u}(T)\|_\beta$ and (93), we can now apply Proposition 10.2 to conclude that (87) hold. Note that since $\alpha(\cdot)$ is increasing, we have $\gamma_1 \leq \alpha_c(T)$. In case $\beta_c(T) < \infty$, let $\beta \to \beta_c(T)$ to conclude that $\gamma_2 - \gamma_1 \geq \beta_c(T) - \alpha_c(T)$.

Assume now $\liminf_{j\to\infty} m_j < \infty$. Since $m_j \in \mathbb{N}$ for all j, passing through a subsequence, we may without loss of generality assume that $m_j = m_0$ for all j and for some $m_0 \in \mathbb{N}$. Let $l > m_0$ and note that by weak continuity of the map $t \to u(t)$, $\sum_{n=m_0}^{l} |R_n\mathbf{u}(t_j)| \to \sum_{n=m_0}^{l} |R_n\mathbf{u}(T)|$ as $t_j \nearrow T$. Note that for all j, we have $\gamma_2 - \gamma_1 \geq \beta - \alpha_c(T) > 0$, where $\beta = \beta_c(T)$ if $\beta_c(T) < \infty$ and $\beta > \alpha_c(T)$ is arbitrary if $\beta_c(T) = \infty$. Due to (87), we now conclude that $\sum_{n=m_0}^{l} |R_n\mathbf{u}(T)| = 0$, for all $l > m_0$, that is, $\mathbf{u}(T)$ is a trigonometric polynomial. In this case, $\beta_c(T) = \infty$. □

The following theorems, which are the main results of this section, provide the possible connections to physical phenomena of our earlier results concerning blowup and monotonicity properties of Gevrey norms presented in Sect. 6. More precisely, Theorem 10.4 shows that an inverse energy cascade occurs if the norm $\|\mathbf{u}(t)\|_{\alpha(t)}$ oscillates (for large δ), while Theorem 10.5 asserts that the same holds in case there are Leray boundary points and/or Leray spikes on the boundary of $\mathcal{D}$.

Theorem 10.4 *Let $\alpha(t) := \alpha(t; t_0, \alpha_0)$ be such that $|\mathbf{u}(t)|_{\alpha(t)}$ is strictly monotonically increasing in the interval $[t_1, T)$, where $T = T_M(t_0, \alpha_0)$. Assume that there exists a point $t_2 \in (t_1, T)$ such that $\|\mathbf{u}(t_2)\|_{\alpha(t_2)} \leq \|\mathbf{u}(t_1)\|_{\alpha(t_1)}$. In this case, there exists $t_3 \in (t_2, T)$ and an adequate $m \in \mathbb{N}$ such that the inequality*

$$\sum_{n=m}^{l} |R_n\mathbf{u}(t_1)|^2 > e^{2(\alpha(t_3)-\alpha(t_1))\Lambda_m^{1/2}} \sum_{n=m}^{l} |R_n\mathbf{u}(t_3)|^2 \tag{94}$$

holds for all $l = m, m+1, \ldots, \infty$. Moreover, we also have the estimate

$$\alpha(t_3) - \alpha(t_1) \geq \frac{\delta\nu^2}{c_1 + \delta^2}\left(\frac{1}{\|\mathbf{u}(t_2)\|^2_{\alpha(t_2)}} - \frac{1}{\|\mathbf{u}(t_1)\|^2_{\alpha(t_1)}}\right). \tag{95}$$

Proof Note that the continuity of the map $t \to \|\mathbf{u}(t)\|_{\alpha(t)}$ in the interval (t_1, T) and the fact that $\lim_{t\nearrow T} \|\mathbf{u}(t)\|_{\alpha(t)} = \infty$ imply that there exists a point $t_3 \in (t_2, T)$ satisfying $\|\mathbf{u}(t_3)\|_{\alpha(t_3)} = \|\mathbf{u}(t_1)\|_{\alpha(t_1)}$. Since the norm $|\mathbf{u}(t)|_{\alpha(t)}$ is strictly monotonically increasing, (94) follows from (87), where

$$\mathbf{v} = \mathbf{u}(t_1), \mathbf{w} = \mathbf{u}(t_3), \gamma_1 = \alpha(t_1) \text{ and } \gamma_2 = \alpha(t_3).$$

It now remains to show (95). From the choice of t_3 and (22), and using the fact that

$$\alpha(s; t_0, \alpha_0) = \alpha(s; t, \alpha(t; t_0, \alpha_0)), t_0 < t < s < T_M(t_0, \alpha_0),$$

we have the estimate

$$\|\mathbf{u}(t_1)\|^2_{\alpha(t_1)} = \|\mathbf{u}(t_3)\|^2_{\alpha(t_3)} \leq \frac{\|u(t_2)\|^2_{\alpha(t_2)}}{\sqrt{1 - \frac{(c_1+\delta^2)}{\delta\nu^2}\|u(t_2)\|^2_{\alpha(t_2)}(\alpha(t_3) - \alpha(t_2))}}.$$

Rearranging this inequality and noting that $\alpha(t_3) - \alpha(t_1) \geq \alpha(t_3) - \alpha(t_2)$, we obtain (95). □

Theorem 10.5 *For $T < t_\infty \leq \infty$, the following hold:*

(i) *If I_T is a Leray spike, then we have a strong inverse cascade of energy. More precisely, letting $\beta = \beta_c(T)$ if $\beta_c(T) < \infty$ and $\beta > \alpha_c(T)$ arbitrary if $\beta_c(T) = \infty$, we have a sequence of points $\{t_j\}$ with $t_j \nearrow T$ and $\{m_j\} \subset \mathbb{N}$ such that for any $l \in \mathbb{N}$ with $m_j \leq l \leq \infty$, we have*

$$\sum_{n=m_j}^{l} |R_n\mathbf{u}(t_j)|^2 > e^{2(\beta-\alpha_c(T))\Lambda^{1/2}_{m_j}} \sum_{n=m_j}^{l} |R_n\mathbf{u}(T)|^2. \tag{96}$$

In this case, either $m_j \to \infty$ as $t_j \nearrow T$, or $\mathbf{u}(T)$ is a trigonometric polynomial. The second case can be regarded as an extremely strong inverse energy cascade in the sense that there exists $m \in \mathbb{N}$ with

$$\frac{\sum_{n=m}^{l} |R_n\mathbf{u}(t_j)|^2}{\sum_{n=m}^{l} |R_n\mathbf{u}(T)|^2} = \infty.$$

(ii) *Let $(T, \alpha_c(T))$ be a Leray boundary point which is reached by a solution curve $\alpha(\cdot) := \alpha(\cdot; t_0, \alpha_0)$ of (1) with $T = T_M(t_0, \alpha_0)$. In this case, there exists a sequence $\{s_j\}_{j=1}^{\infty}$ satisfying $t_0 < s_j < s_{j+1} < T$ for all j and $s_j \nearrow T$ such that (94) and (95) hold with $s_{3j-2} = t_1$ and $s_{3j-2} = t_2$ and $s_{3j} = t_3$, for each $j = 1, 2, \ldots$*

Proof Part (i) of the theorem follows immediately from Proposition 10.3.

We will now prove part (ii). Due to part (iv) of Proposition 7.6, we may assume that δ satisfies (37) and $|\mathbf{u}(t)|_{\alpha(t)}$ is strictly increasing. Note also that since $T = T_M(t_0, \alpha_0)$, we have $\lim_{t\nearrow T} \|\mathbf{u}(t)\|_{\alpha(t)} = \infty$. Thus, in view of Theorem 10.4, to conclude the proof of part (ii), we need only to show that $\|\mathbf{u}(t)\|_{\alpha(t)}$ is not strictly monotonically increasing in any interval (T_1, T), where $T_1 < T$. Assume to the contrary that $\|\mathbf{u}(t)\|_{\alpha(t)}$ is strictly monotonically increasing in some interval (T_1, T) and recall that since $(T, \alpha_c(T))$ is a Leray boundary point, we have $\|\mathbf{u}(T)\|_{\alpha_c(t)} = \|\mathbf{u}(T)\|_{\alpha(T)} < \infty$. Following exactly the same proof as in Proposition 6.2, we see that $e^{\alpha(t)A^{1/2}}\mathbf{u}(t) \to e^{\alpha(T)A^{1/2}}\mathbf{u}(T)$ in V as $t \nearrow T$. This means that $\sup_{[T_1,T)} \|\mathbf{u}(t)\|_{\alpha(t)} < \infty$ and the solution $\alpha(t)$ can be extended beyond T. This contradicts the condition that $T = T_M(t_0, 0)$. □

Remark 12 The relations (94) [respectively (96)] show that in the time interval (t_1, t_3) (respectively, (t_j, T)), the energy of the higher modes has had a significant drop.

Acknowledgments This research was partially supported by NSF grants DMS11-09532 (A. Biswas) and DMS11-09784 (C. Foias). Animikh Biswas would also like to thank the Mathematics Department at Texas A&M University for support while he visited College Station on several occasions during the preparation of this manuscript.

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