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THERMOSTATS AND PERIODICITY

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BY

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SWITCHING SYSTEMS: THERMOSTATS AND PERIODICITY*

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ABSTRACT:

For a variable structure system capable of operating in any of several modes, suppose a 'switching rule' has been established for the transitions. Example: A thermostat is a bimodal system (furnace = ON/OFF) with a switching rule based on the observed sensor value. We discuss the existence and continuous dependence of solutions for a class of such switching systems (including the formulation of thermostats) and examine the existence of periodic solutions.

Key words: differential equations, variable structure, switching, thermostat, global existence, continuous dependence, periodicity.

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1. INTRODUCTION

My introduction to these questions came through a model by Glashoff and Sprekels [3] of a thermostat. Their computational experience with this model suggested that, for any initial state, one rapidly settled down to a periodic regime, cycling between the two (on/off) modes. Direct experience with (physical) thermostats also suggests such periodic behavior. For any physical thermostat a periodic solution is necessarily nontrivial — a constant solution cannot occur for either mode. Nevertheless, having shown the existence of periodic solutions for the model of [3], we observed that this model did permit certain (physically spurious) constant solutions. While it would also be of interest to obtain theoretical confirmation of the entrainment in nontrivial (almost) periodic behavior observed computationally for that model, this led to a search for another model more consistent with actual thermostat behavior.

Another viewpoint came from a paper by Capuzzo-Dolcetta and Evans [2] on optimal control by mode switching. Using a Hamilton-Jacobi-Bellman formulation, this problem leads to a function

$$V(x,j) \quad := \quad \text{optimal cost starting at } x \text{ in mode } j$$

and the obviously necessary condition that along an optimal solution one would not stay in mode j if, for some $k \neq j$, one were to have $V(x,k) < V(x,j) +$ (cost of switching from mode j to mode k) nor would one switch if this inequality were reversed.

These conditions led to the formulation of a <u>switching system</u> as a variable structure (multimodal) system which permits mode transitions subject to a suitable <u>switching rule</u>. We will make this notion more precise in the next section and, in particular, will discuss the structure of the solution set. The principle observation is that to preserve the property that "the limit of solutions is a solution" one must accept nonuniqueness of solutions for the initial value problem. The remainder of the paper discusses some examples and the existence of periodic solutions for such systems.

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2. FORMULATION

Consider an indexed family of <u>modes</u>, which we take to be autonomous flows on an underlying state space X. In general we think of such a mode as given by a differential equation

$$\dot{x} = f_j(x) \qquad j \in J \tag{2.1}$$

whose solution map is the flow

$$\pi_j(t,\xi) = [x(t) : \quad x(\cdot) \text{ satisfies (2.1) with } x(0) = \xi] \tag{2.2}$$

for $t \geq 0$, $\xi \in X$. The theory of dynamical systems generalizes (2.1) by taking

$$\begin{aligned} &\pi_j : \mathbb{R}^+ \times X \to X \quad \text{continuous, such that} \\ &\pi_j(t,\pi_j(s,\xi)) = \pi_j(t+s,\xi). \end{aligned} \tag{2.3}$$

(Strictly speaking, this is a <u>semi</u> dynamical system as we consider only $t \geq 0$. For brevity we write 'orbit' instead of 'semi-orbit', etc.)

By an (autonomous) switching system we mean an indexed family of such flows $\{\pi_j : j \in J\}$, each satisfying (2.3), together with a 'switching rule' governing admissible transitions between modes. A solution then consists of the pair of functions $[x,j]$ on $\mathbb{R}^+$ with the interpretation that $x(t) \in X$ is the position at 'time' t and $j(t) \in J$ is the (current) mode. We require that $x(\cdot)$ be continuous, $j(\cdot)$ be piecewise constant with transitions satisfying the switching rules and that

(2.4) $\quad x(t+s) = \pi_k(s,x(t)) \quad$ provided $j(\cdot) = k$ on $(t,t+s)$.

We will only consider switching rules of the following form:

(2.5) i. The index function $j: \mathbb{R}^+ \to J$ is piecewise constant with isolated transitions. Thus, at each t both $j(t-)$ and $j(t+)$ are defined; we assume $j(t) = j(t-)$.

ii. For each $k \in J$ there may be a forbidden region $\mathcal{R}_k$ such that

$$t > 0, \quad x(t) \in \mathcal{R}_k \Longrightarrow j(t) \neq k.$$

iii. For each $j,k \in J$ $(k \neq j)$ there is an admissible switching set $\mathcal{S}_{j,k}$ such that

$$[t > 0, \ j(t) = j, \ j(t+) = k] \Longrightarrow x(t) \in \mathcal{S}_{j,k}.$$

One may subsume (2.4ii) in (2.4iii) by taking $\mathcal{S}_{k,k}$ to be the complement of $\mathcal{R}_k$.

Some restrictions on the nature of the sets $\{\mathcal{R}_j, \mathcal{S}_{j,k}\}$ are necessary to make the problem 'reasonable' — meaning, for the moment, only that every initial state $(\xi,k) \in X \times J$ is 'possible'. We will assume that

(2.6) Each $\mathcal{S}_{j,k}$ is closed. Each $\mathcal{R}_k$ is open.

(Note that this is consistent with taking $\mathcal{S}_{k,k}$ to be the complement of $\mathcal{R}_k$.) We also assume that $\bigcap_k \overline{R}_k$ is empty so every initial point can proceed in some mode. We say π_j <u>enters</u> $\mathcal{R}_j$ at $\xi \in \partial\mathcal{R}_j$ if there is some $\hat{\xi} \notin \mathcal{R}_j$ and some $\tau, \varepsilon > 0$ such that

$$\pi_j(t,\hat{\xi}) \notin \mathcal{R}_j \qquad \text{for } 0 \leq t \leq \tau, \quad \pi_j(\tau,\hat{\xi}) = \xi,$$

$$\pi_j(t,\hat{\xi}) \in \mathcal{R}_j \qquad \text{for } \tau < t < \tau + \varepsilon$$

and say π_j <u>is tangential to</u> $\mathcal{R}_j$ at $\xi \in \mathcal{R}_j$ if there is some $\hat{\xi} \notin \overline{\mathcal{R}}_j$ and some $\tau, \varepsilon > 0$ such that

$$\pi_j(\tau,\hat{\xi}) = \xi, \quad \pi_j(t,\hat{\xi}) \notin \mathcal{R}_j \qquad \text{for } 0 \leq t \leq \tau + \varepsilon.$$

(If π_j is nowhere tangential to $\mathcal{R}_j$ we call $\partial\mathcal{R}_j$ <u>transverse</u> to π_j.) Clearly, if $\xi \in \partial\mathcal{R}_j$ is reachable along π_j and π_j enters $\mathcal{R}_j$ at ξ, then one can only continue by switching to another mode. Hence we require

(2.7) $$\xi \in \mathcal{S}_{j,k} \Longrightarrow \xi \notin \mathcal{R}_k \text{ nor does } \pi_k \text{ enter } \mathcal{R}_k \text{ at } \xi,$$

$$\pi_j \text{ enters } \mathcal{R}_j \text{ at } \xi \in \partial\mathcal{R}_j \Longrightarrow \xi \in \bigcup_{k \neq j} \mathcal{S}_{j,k}.$$

In this paper we will actually consider only switching rules (2.5) for which J is finite, (2.6) holds and (2.7) is replaced by the stronger condition

(2.7') $$\bigcup_{k \neq j} \mathcal{S}_{j,k} = \partial\mathcal{R}_j.$$

Thus, switching from the j-th mode can occur only at the boundary $\partial\mathcal{R}_j$ of the forbidden region, is always possible there, and is mandatory if one arrives in the j-th mode at any ξ at which π_j enters $\mathcal{R}_j$.

Remark 1: Note that at any tangential point $\xi \in \partial\mathcal{R}_j$ switching is optional: one may continue from ξ either by π_j or, in view of (2.7'), by π_k for some $k \neq j$. We shall see that the nonuniqueness of the solution is unavoidable: see Example 1 in the next section.

Given a sequence of switching systems as above: $\Sigma_n := [\pi_{j;n} : \mathbb{R}^+ \times X \to X;$ $\mathcal{R}_{j,n}; \mathcal{S}_{j,k;n} : j,k \in \mathcal{J}]$ for $n = 1,2,\ldots,$ we say the sequence (Σ_n) converges to the system $\tilde{\Sigma}$, again as above, providing

(2.8) (i) $\pi_{j;n}(t,\xi) \to \tilde{\pi}_j(t,\xi)$ uniformly on bounded sets in $\mathbb{R}^+ \times X$,

(ii) $\xi \in \tilde{\mathcal{R}}_j \Longrightarrow \xi \in \mathcal{R}_{j;n}$ for $n > \bar{n}(\xi)$,

(iii) for subsequences: $[\xi_m \in \mathcal{S}_{j,k;m},\ \xi_m \to \xi] \Longrightarrow \xi \in \mathcal{S}_{j,k}$.

We wish to show that our assumptions give a certain degree of continuous dependence (on the initial condition $\xi := x(0)$ and on the system) despite the nonuniqueness noted in the Remark.

THEOREM 1: For $n = 1,2,\ldots.$ let Σ_n be a switching system as above satisfying (2.6), (2.7') and let $[x_n(\cdot), j_n(\cdot)]$ be a solution so (2.4), (2.5) hold. Assume there is a uniform (at least on bounded t-intervals) separation for transitions (jumps in j_n):

(2.9) There exists $\tau(T) > 0$ for each $T > 0$ such that

$$t \leq T,\quad j_n(t+) = k \neq j_n(t) \Longrightarrow j_n(s) = k \quad \text{for} \quad t < s < t+\tau(T).$$

Suppose $\Sigma_n \to \tilde{\Sigma}$ in the sense of (2.8) and $\xi_{0;n} := x_n(0) \to \tilde{\xi}_0$ with $j_n(0) = k$ independent of n. Then, for a subsequence, (2.10) $\quad x_n(t) \to \tilde{x}(t)$ uniformly on bounded t-intervals,

$\quad j_n(t) \to \tilde{j}(t)$ (sense made precise below)

with $[x(\cdot), j(\cdot)]$ a solution for $\tilde{\Sigma}$, starting at $[\tilde{\xi}_0, k]$.

<u>Proof</u>: We will repeatedly (recursively) extract subsequences without indicating the implied re-indexing explicitly, concluding with a Cantorial diagonal construction. It is convenient to let $t_{0;n} = 0$ and let

$$t_{\nu;n} := \text{time of } \nu\text{-th transition for } j_n(\cdot) \qquad (1 \le \nu < \bar{\nu}_n)$$

where $\bar{\nu} = \bar{\nu}_n = \infty$ or, if there are only finitely many transitions so j_n is constant from some point on, $\bar{\nu} - 1$ is the number of transitions and $t_{\bar{\nu};n} = \infty$. (Note: We are assuming, above and in this theorem, that 'solution' means a global solution: defined for all $t \in \mathbb{R}^+$. It would also be possible to consider situations in which solutions had a maximal interval of existence $[0, t^*)$.) We also let $j_{\nu;n}$ be the ν-th value of $j_n(\cdot)$ so

$$j_n(t) = j_{\nu;n} \quad \text{for } t_{\nu-1;n} < t \le t_{\nu;n} \qquad (1 \le \nu \le \bar{\nu}_n).$$

We will similarly obtain $\{\tilde{t}_\nu, \tilde{j}_\nu\}$ so that

$$\tilde{j}(t) = \tilde{j}_\nu \quad \text{for } \tilde{t}_{\nu-1} < t \le \tilde{t}_\nu \qquad (1 \le \nu \le \tilde{\nu})$$

and define the sense of $j_n \to \tilde{j}$ in (2.10) as meaning

$$t_{\nu;n} \to \tilde{t}_\nu, \quad j_{\nu;n} = \tilde{j}_\nu \quad \text{for } n > n_\nu$$

for $\nu = 1, \ldots, \tilde{\nu}$ so that

$$j_n(t) = \tilde{j}(t) \qquad \text{for } n > n_{\nu(t)},$$

where $\nu(t) = \min\{\nu : t < \tilde{t}_\nu\}$ for t not in the discrete set $\{\tilde{t}_\nu\}$.

To extract the desired subsequence we proceed as follows:

Starting with the original sequence, let $\tilde{j}_1$ be the common value of $j_n(0)$ and let $\tilde{t}_1$ be any limit point of $\{t_{1;n}\}$ — such a limit point, possibly ∞, always exists — and extract a subsequence (if necessary) so $t_{1;n} \to \tilde{t}_1$. If $\tilde{t}_1 = \infty$ we set $\tilde{\nu} = 1$ and are done. Suppose $\tilde{t}_1 < \infty$; we set

$$\xi_{1;n} \;:=\; x_n(t_{1;n}) \;=\; \pi_{\tilde{j}_1;n}(t_{1;n},\xi_{0;n}).$$

Since $t_{1;n} \to \tilde{t}_1$, $\xi_{0;n} \to \tilde{\xi}_0$ we must have, using (2.8i), $\xi_{1;n} \to \tilde{\xi}_1 :=$ $\tilde{\pi}_j(\tilde{t}_1,\tilde{\xi}_0)$, again with $j = \tilde{j}_1$. Since each $t_{1;n}$ is a transition for $j_n(\cdot)$, one must have $\xi_{1;n} \in \mathcal{S}_{j,k;n}$ $(j = \tilde{j},\ k = k_n)$ and, since $\mathcal{J}$ is finite, there must be some $k \in \mathcal{J}$ occurring infinitely often in the sequence (k_n). Extracting a subsequence if necessary, we may thus assume $\xi_{1;n} \in \mathcal{S}_{j,k;n}$ with $j = \tilde{j}_1$ and k independent of n; let $\tilde{j}_2$ be this common value of k.

With the same logic as above we proceed inductively. Suppose we have (after several stages of extracting sequences)

$$t_{\nu;n} \to \tilde{t}_\nu < \infty, \qquad \xi_{\nu;n} \;:=\; x_n(t_{\nu;n}) \to \tilde{\xi}_\nu.$$

Since each $t_{\nu;n}$ is a transition for $j_n(\cdot)$ we must have $\xi_{\nu;n} \in \mathcal{S}_{j,k;n}$ with $j = \tilde{j}_\nu$ and, as above, can assume a common index k which we take to be $\tilde{j}_{\nu+1}$. Letting $\tilde{t}_{\nu+1}$ be any limit point (corresponding, of course, to the subsequence already obtained) of $t_{\nu+1;n}$ we extract a further subsequence and assume $t_{\nu+1;n} \to \tilde{t}_{\nu+1}$. If $\tilde{t}_{\nu+1} = \infty$ we set $\tilde{\nu} = \nu+1$ and are done. Otherwise we consider

$$\xi_{\nu+1;n} \;:=\; x_n(t_{\nu+1;n}) \;=\; \pi_{\tilde{j}_{\nu+1};n}(t_{\nu+1;n} - t_{\nu;n},\xi_{\nu;n})$$

and note that

(2.11) $$\xi_{\nu+1;n} \to \tilde{\xi}_{\nu+1} = \tilde{\pi}_{\tilde{j}_{\nu+1}}(\tilde{t}_{\nu+1} - \tilde{t}_\nu, \tilde{\xi}_\nu)$$

so the induction can proceed.

By (2.9) we have $(t_{\nu+1;n} - t_{\nu;n}) \geq \tau(T)$ whenever $t_{\nu;n} \leq T$ so $t_{\nu;n} \geq T$ if $\nu \geq T/\tau(T)$ whence $\tilde{t}_\nu \geq T$ if $\nu \geq T/\tau(T)$. Thus, either $\tilde{t}_{\tilde{\nu}} = \infty$ for some $\tilde{\nu}$ or $\tilde{t}_\nu \to \infty$. Setting $\nu(t) := \min\{\nu : t \leq \tilde{t}_\nu\}$ (except that $\nu(0) := 1$; note that $\nu(\cdot)$ is well defined on $\mathbb{R}^+$) we define

(2.12) $$\tilde{j}(t) = \tilde{j}_{\nu(t)}$$

$$\tilde{x}(t) = \tilde{\pi}_{\tilde{j}(t)}(t - \tilde{t}_{\nu(t)-1}, \tilde{\xi}_{\nu(t)-1}).$$

Clearly $\tilde{j}(\cdot)$ satisfies (2.5i) and, using (2.11) inductively, $\tilde{x}(\tilde{t}_\nu) = \tilde{\xi}_\nu$ for each ν.

By the construction, if $t \notin \{\tilde{t}_\nu\}$ so $\tilde{t}_{\nu-1} < t < \tilde{t}_\nu$ for some ν we have (as $t_{\nu-1;n} \to \tilde{t}_{\nu-1}$ and $t_{\nu;n} \to \tilde{t}_\nu$) $t_{\nu-1;n} < t < t_{\nu;n}$ for large enough n and so $j_n(t) = j_{\nu;n} = \tilde{j}_\nu =: \tilde{j}(t)$ giving convergence $j_n \to \tilde{j}$ in the sense desired for (2.10). We already know from (2.11) that

$$x_n(t_{\nu;n}) =: \xi_{\nu;n} \to \tilde{x}(\tilde{t}_\nu) = \tilde{\xi}_\nu.$$

We also know, using $\xi_{\nu-1} = \tilde{x}(\tilde{t}_{\nu-1})$ together with (2.3) in (2.12), that $\tilde{x}$ is continuous and satisfies (2.4). To show the convergence $x_n \to \tilde{x}$ with local uniformity in t is slightly messy near the switching times. It is sufficient to do this on intervals $\tilde{t}_\nu \leq t \leq \tilde{t}_{\nu+1}$. Assuming $\nu \geq 1$ (The modification for $0 \leq t \leq \tilde{t}_1$ is simple.), we set $j := \tilde{j}_\nu$, $k = \tilde{j}_{\nu+1}$ and for each t and $n = 1,2,\ldots$ let

$$\xi_n := \xi_{\nu-1;n} = x_n(t_{\nu-1,n}),$$

$$r_n = r_n(t) := \min\{t_{\nu;n} - t_{\nu-1;n},\ t - t_{\nu-1;n}\},$$

$$s_n = s_n(t) := \max\{0,\ t - t_{\nu-1;n}\}$$

and similarly define $\tilde{\xi}$, $\tilde{r} = \tilde{r}(t)$, $\tilde{s} = \tilde{s}(t)$. Clearly,

$$r_n(t) \to \tilde{r}(t), \qquad s_n(t) \to \tilde{s}(t) \qquad \text{uniformly in } t,$$

$$x_n(t) = \pi_{k;n}(s_n, \pi_{j;n}(r_n, \xi_n))$$

$$\tilde{x}(t) = \tilde{\pi}_k(\tilde{s}, \tilde{\pi}_j(\tilde{r}, \tilde{\xi}))$$

for $\tilde{t}_\nu \le t \le \tilde{t}_{\nu+1}$. From (2.8i) and the known convergence $\xi_n \to \tilde{\xi}$ it follows that $x_n(t) \to \tilde{x}(t)$ uniformly in t.

To complete the proof we need only verify that $[\tilde{x}, \tilde{j}]$ satisfies (2.5ii) and (2.5iii). The only switches for $\tilde{j}(\cdot)$ occur at $\tilde{t}_\nu$ $(\nu = 1, \ldots, \nu-1)$ and we set $j := \tilde{j}_\nu$ and $k := \tilde{j}(\tilde{t}_\nu+) = \tilde{j}_{\nu+1}$. By construction $j_{\nu;n} = j$ and $j_{\nu+1;n} = k$ for large n) so $\xi_{\nu;n} = x_n(t_{\nu;n}) \in \mathcal{S}_{j,k;n}$. Since $\xi_{\nu;n} \to \tilde{\xi}_\nu$, this implies $\tilde{x}(\tilde{t}_\nu) = \xi_\nu \in \tilde{\mathcal{S}}_{j,k}$ by (2.8iii). Thus (2.5iii) holds. To verify (2.5ii) we suppose, to the contrary, that $\tilde{x}(t) \in \tilde{\mathcal{R}}_k$ with $\tilde{j}(t) = k$. If $t \notin \{\tilde{t}_\nu\}$ one has $j_n(t) = k$ for large n. By (2.8ii) one then has $\tilde{x}(t) \in \mathcal{R}_{k;n}$ for large n. Since each $[x_n, j_n]$ is a solution for Σ_n, (2.5ii) gives $x_n(t) \notin \mathcal{R}_{k;n}$. As $x_n(t) \to \tilde{x}(t)$ with $x_n(t) \notin \mathcal{R}_{k;n}$ and $\tilde{x}(t) \in \mathcal{R}_{k;n}$, there must be points $\eta_n \in \partial\mathcal{R}_{k;n}$ with $\eta_n \to \tilde{x}(t)$. (If X is a linear space one can find η_n on the straight segment joining $x_n(t)$ to $\tilde{x}(t)$; for X a manifold this can be done in local coordinates for n large so $x_n(t)$ is close to $\tilde{x}(t)$. Rather than introduce an extraneous topological condition, we simply assume X has such a form.) By (2.7') this gives $\eta_n \in \mathcal{S}_{k,m;n}$ $(m = m_n$ but, as earlier, finiteness of $\mathcal{J}$ permits us to extract a subsequence and take a common m, independent of n) so applying (2.8ii) again,

$$\tilde{x}(t) = \lim_n \eta_n \in \tilde{\mathcal{S}}_{k,m}.$$

Since $\tilde{\mathcal{S}}_{k,m} \subset \partial\tilde{\mathcal{R}}_k$ by (2.7') and $\tilde{\mathcal{R}}_k$ is open, this contradicts the assumption that $\tilde{x}(t) \in \tilde{\mathcal{R}}_k$. Hence (2.5ii) holds for $[\tilde{x}, \tilde{j}]$ at $t \notin \{\tilde{t}_\nu\}$. Since $\tilde{\mathcal{R}}_k$ is open, continuity of $\tilde{x}$ at $\tilde{t}_\nu$ ensures also that $\tilde{x}(\tilde{t}_\nu) \notin \tilde{\mathcal{R}}_k$ for $k := \tilde{j}(\tilde{t}_\nu) = \tilde{j}_\nu$.

Hence we have shown that the pair $[\tilde{x},\tilde{j}]$, constructed as the limit of a subsequence, is a solution, starting at $\tilde{\xi}_0$, of the switching system $\tilde{\Sigma}$. ∎

Remark 2: Note that if the original sequence converges as in (2.10) then no extraction of subsequences is needed in the proof above. Thus, the argument shows the continuity result:

The limit of solutions is a solution

whenever such a limit exists — as well as ensuring existence of such a limit for a subsequence whenever $x_n(0) \to \tilde{\xi}_0$ and $\Sigma_n \to \tilde{\Sigma}$, subject to (2.9). Verification of (2.9) is of a nature rather different from that of the other hypotheses but we note, e.g., the argument for a lower bound on interswitching intervals obtained from (3.8) in the proof of Theorem 2.

Remark 3: Observe that the finiteness of the index set $\mathcal{J}$ was not really used in the proof above except to show that with $\xi_{\nu;n} \in \mathcal{S}_{j,k;n}$ with $k = k_n$ and $\xi_{\nu;n} \to \tilde{\xi}_\nu$ one can select a subsequence with a single k. For this it would be sufficient to assume only that for each j one has finiteness of $\mathcal{J}_j := \{k \in \mathcal{J} : \mathcal{S}_{j,k} \neq \emptyset\}$ (with each $\mathcal{J}_j$ independent of n in the theorem) or even a local finiteness condition for neighborhoods of $\partial\mathcal{R}_j$.

The condition (2.7) was strengthened to (2.7') but (with an attendant simplification of the proof!) this change is unnecessary if instead (2.8ii) is strengthened to

$$\text{(2.8ii')} \quad \xi \in \tilde{\mathcal{R}}_j \Longrightarrow [\text{for some neighborhood } N \text{ of } \xi \text{ one has } N \subset \mathcal{R}_{j;n} \text{ for all } n > \bar{n}(N)].$$

Finally, we remark again that instead of insisting on global solutions one could permit solutions to have a (natural) interval of existence $[0,t^*)$ shorter than $\mathbb{R}^+$. The considerations involved in treating intervals with t^* depending not only on the initial point but on the particular branch are of some interest. If the individual modes π_j always give global solutions this could still occur (see Example 2 below) but then only in connection with the possibility of finite limit points of switches. It is the set of technical difficulties attendant on the possibility that the set of switching times not be discrete — involving modification of the basic notion of 'solution' — which we avoid here at the expense of having to verify (2.5i), (2.9).

3. SOME EXAMPLES (INCLUDING THERMOSTATS) AND EXISTENCE

A thermostat is a device, consisting of a sensor (thermometer) and a switch, whose operation, slightly simplified, can be described as follows. One has a pair of (possibly adjustable) set points: $\theta_1 < \theta_2$. Supposing the thermostat to control a furnace, the switch would be OFF for high thermometer readings θ but when θ dropped below θ_1 it would become ON. Rising temperature would not turn the switch OFF until θ reached the higher level θ_2 while later falling temperatures would turn the switch ON only when θ had dropped as far as θ_1 again.

This 'dead zone' between θ_1 and θ_2 is introduced to avoid the undesirable effects on the furnace (or air conditioner or fan or ...) of too frequent switching in the event of temperature oscillations crossing and re-crossing a single switching point. These effects might correspond to the switching costs treated in [2] — although physically they may relate to the interruption of start-up transients, etc., and so would depend on the length of time since the previous transition. Some devices have, instead of a 'dead zone' (θ_1,θ_2), a single set point $\bar{\theta}$ and a timer rendering the device insensitive for an interval τ following any transition; such devices are not modeled by the present treatment although their mathematical formulation might be of future interest.

The operation of such a thermostat is modeled here by a switching system with two modes: π_1 and π_2, corresponding to ON and OFF positions of the switch. Without considering, at the moment, the nature of the state space X or the dynamics described by π_1, π_2 (see Example 5, below), we view the temperature θ at the thermostat sensor as a functional on the state: $\theta(t) := \theta[x(t)]$. Then

$$(3.1) \qquad \mathcal{S}_{1,2} := \{\xi : \theta[\xi] = \theta_2\}, \quad \mathcal{R}_1 := \{\xi : \theta[\xi] > \theta_2\},$$

$$\mathcal{S}_{2,1} := \{\xi : \theta[\xi] = \theta_1\}, \quad \mathcal{R}_2 := \{\xi : \theta[\xi] < \theta_1\},$$

corresponding to (2.5ii, iii), (2.6), (2.7'). Note that verification of (2.5i) (and even the existence of global solutions — compare Example 2, below) depends on the particular dynamics π_1, π_2.

For the remainder of this paper we restrict our attention to switching systems with only 2 modes ($J = \{1,2\}$) as above and with switching rules (2.5) satisfying (2.6), (2.7'). We present, next, two 'cautionary' sets of examples.

EXAMPLE 1: Consider, first, a thermostat as in (3.1) beginning at $t = 0$ with a state $[\xi_0, 1]$ so the furnace is initially ON (say $\theta[\xi_0] < \theta_1 < \theta_2$). Suppose $\hat{\theta}(t) := \theta[\pi_1(t,\xi_0)]$ looks as in Figure 1, tangent to $\theta = \theta_2$ at $t = t_*$ so π_1 is tangential to $\mathcal{R}_1$ (in the sense of the definition given in Section 2) at $\xi_* := \pi_1(t_*,\xi_0)$. The continuation along π_1 is shown as giving $\hat{\theta}$ crossing $\theta = \theta_2$ at t_{**} so π_1 enters $\mathcal{R}_1$ at $\xi_{**} := \pi_1(t_{**},\xi_0)$. The moment of decision, then, is $t = t_*$ at $x(t_*) = \xi_*$: as we have formulated the switching rule (2.5) one has the option of switching to π_2 at t_*

(giving $\hat{\theta} = \theta[x(\cdot)]$ continuing along $\hat{\theta}_*$) or continuing with π_1 and then, necessarily, switching to π_2 at t_{**} (giving $\hat{\theta} = \theta[x(\cdot)]$ continuing along $\hat{\theta}_{**}$).

Could one (reasonably!) modify the switching rule so as to enforce a particular choice at times such as t_* ? Such an additional 'selection rule' would, of course, make the orbit $x(\cdot)$ uniquely determined by its initial value $\xi_0 = x(0)$. We see, however, that the continuity property of Remark 2 would be lost if this were done. Consider a perturbation of θ_2 to $\theta_2' = \theta_2 - \varepsilon$ (or equivalently, a perturbation of ξ_0 to $\xi_0' > \xi_0$). In such a case $\hat{\theta}$ would cross the set point θ_2' at t_1' and one would unquestionably switch to π_2, continuing along $\hat{\theta}'$. As $\theta_2' \to \theta_2$ (equivalently, as $\xi_0' \to \xi_0$) one would have $t_1' \to t_*$ and $\hat{\theta}' \to \hat{\theta}_*$. The continuity property would require that switching at t_* (i.e., at $x(t_*) = \xi_*$) must be permitted. On the other hand, a perturbation of θ_2 to $\theta_2'' = \theta_2 + \varepsilon$ (or equivalently, a perturbation of ξ_0 to $\xi_0'' < \xi_0$) would mean that the function $\hat{\theta}$ would stay bounded away from the set point θ_2'' near t_* and unquestionably one would wait until the crossing at t_1'' to switch to π_2, continuing along $\hat{\theta}''$. Now as $\theta_2'' \to \theta_2$ (or $\xi_0'' \to \xi_0$) one has $t_1'' \to t_{**}$ and $\hat{\theta}'' \to \hat{\theta}_{**}$. The continuity property would require that it must be permitted to remain on π_1 until t_{**} (i.e., <u>not</u> to switch at ξ_*). To preserve the continuity property we are forced to accept the nonuniqueness!

<u>Remark 4</u>: The thermostat model in [3] is quite different in that switching depends not only on θ but on $\dot{\theta}$ as well: the heavy line in Figure 2 (segments A, B, C) is the boundary between the regions in the phase plane for which the switch is ON (lower right) and OFF (upper left). (Actually the model in [ ] introduces a function $\sigma(\theta,\dot{\theta})$ with values 0, 1 in the ON, OFF regions,

respectively and, convexifying, with $\sigma = [0,1]$, set-valued, on this boundary. The advantage of this model mathematically is that the initial-value problem is well-posed: $x(t)$ is unique and continuously dependent on $x(0)$.) The switching behavior is here exactly as desired — except at the segment B ! The loop drawn in Figure 2 represents the same $\hat{\theta}(\cdot)$ as in Figure 1; by a slight abuse of 'notation' it may also be taken to represent $\theta[x(t)]$ which here has essentially the same shape. Note that in this model one necessarily switches OFF at $t_1 = t_*$ — but one then switches ON at $t_2 = \tilde{t}$ and <u>again</u> OFF at $t_3 = t_{**}$! Perturbing θ_2 no longer changes t_1 much (one still crosses B at t_* if θ_2 is increased slightly and crosses C at a nearby point if θ_2 is decreased slightly.) so the subsequent evolution of the system is perturbed only slightly. Indeed, any transitions which can be eliminated or introduced by a perturbation occur in pairs close together and so the perturbation has little effect. The less satisfactory aspect of this model is that it does not describe in any realistic sense the operation of actual thermostats: one should not, in fact, have transitions on crossing the segment B and the ON/OFF state cannot really be determined in a historyless fashion as a function on the phase plane.

EXAMPLE 2: Suppose each of the modes π_1, π_2 is asymptotically stable with a strong global attractor. It is tempting to conjecture that any switching rule will give a stable system. Figure 3 provides a counterexample in the plane: As drawn, two flows each spiral in to the respective stable points but the solution of the switching system, represented by the zigzagging curve going off to the upper right, is here unbounded. While not entirely clear on a finite diagram, this example can obviously be such that <u>every</u> orbit of the switching system remaining between the lines $\mathcal{S}_{1,2}$ and $\mathcal{S}_{2,1}$ (possibly after an initial arc — say, starting in $\mathcal{R}_2$ along π_1 until it hits $\mathcal{S}_{1,2}$) and if the speeds along π_1, π_2 are taken to increase suitably as one goes 'out', zigzags to ∞ at the upper right in finite time. In the sense of our definition, requiring global solutions, this switching system would have no solutions at all! Note that a diagram much like this can still be obtained with <u>linear</u> modes

$$(3.2) \qquad \pi_j: \dot{x} = A_j x + u_j \qquad (j = 1,2)$$

in $\mathbb{R}^2$, although in that case one would not obtain blowup in finite time (and the diagram also changes in that one would have unbounded solutions both going to the upper right and to the lower left as well as a periodic solution).

EXAMPLE 3: To see the variety of possible 'behaviors' on $\partial\mathcal{R}_j$ consider Figure 4. The points ξ_1 in (a), (b) give π_1 entering $\mathcal{R}_1$. The points ξ_2 in (b) and ξ_1 in (e) are tangential points. The point ξ_3 in (b), ξ_1 in (c), and the points of the interval $(\xi_1,\xi_2]$ in (e) do not count as tangential points for π_1 — and have not been classified at all — since one cannot be there in the mode π_1. For (d), π_1 enters $\mathcal{R}_1$ at ξ_2 but the points of $[\xi_1,\xi_2)$ are tangential. For (e) one has π_1 entering $\mathcal{R}_1$ at points of $(\xi_2,\xi_3]$ although these can be reached along π_1 only if one were to <u>start</u> in the region bounded by $[\xi_1,\xi_2,\xi_3,\xi_1]$; similarly, points of $(\xi_1,\xi_2]$ are tangential in (c).

EXAMPLE 4: (The 'linear' case) Consider a switching system with

$$(3.3) \qquad \pi_j: \dot{x} = Ax + u_j \qquad (j = 1,2)$$

on a (finite-or infinite-dimensional) Banach space X. We have $u_1, u_2 \in X$ and assume

(3.4) (i) A generates a C_0 semigroup S on X with

(ii) $\|S(t)\| \leq Me^{\omega t}$ $(t > 0)$.

We take the switching rule (2.5) to be given by (3.1) with $0 \neq \theta \in X^*$ so $\mathcal{R}_1, \mathcal{R}_2$ are disjoint open half spaces bounded by the parallel hyperplanes $\mathcal{S}_{1,2} = \partial\mathcal{R}_1$, $\mathcal{S}_{2,1} = \partial\mathcal{R}_2$.

THEOREM 2: Let Σ be a 'linear' switching system as above (i.e., (2.4), (2.5), (3.1) with $0 \neq \theta \in X^*$, and (3.3) with (3.4)). Then there are global solutions for all initial states $[\xi,k]$, each equicontinuous (over $x(\cdot)$ for all solutions with the same initial ξ) uniformly on bounded t-intervals. If $\omega < 0$ in (3.4ii) then for each $\xi \in X$ the set of orbits starting at ξ is bounded and equicontinuous uniformly on $\mathbb{R}^+$. Further, there is then a bounded invariant set $\mathcal{B}$ such that <u>every</u> solution for Σ eventually enters and then stays in $\mathcal{B}$.

<u>Proof</u>: We begin with the observation that (3.3) gives

$$(3.5) \qquad \dot{x} = Ax + u_{j(t)}, \qquad x(0) = \xi$$

so, from semigroup theory, (cf., e.g., [4]), one has the representation

$$(3.6) \quad x(t) = S(t)\xi + \int_0^t S(t-s)u_{j(s)}\, ds$$

$$= x_0(t) + \int_0^t \sigma_1(s)[S(t-s)u_1]\, ds + \int_0^t \sigma_2(s)[S(t-s)u_2]\, ds$$

where $x_0(t) = \pi_0(t;\xi) := S(t)\xi$ is the solution of

$$\dot{x}_0 = Ax_0, \qquad x_0(0) = \xi$$

and $\sigma_1(s) := 2 - j(s)$, $\sigma_2(s) := j(s) - 1$. Note that π_0 satisfies (2.3) and that each σ_j is $\{0,1\}$-valued and piecewise constant. It follows from (3.6) that for $0 \le t < \bar{t} \le T$ one has

$$(3.7) \qquad \|x(\bar{t}) - x(t)\| \le \|x_0(\bar{t}) - x_0(t)\| + \int_t^{\bar{t}} \|S(\bar{t} - s)\, u_{j(s)}\|\, ds$$

$$\le \|x_0(\bar{t}) - x_0(t)\| + M_T(\bar{t} - t)$$

where $M_T := M \max\{1, e^{\omega T}\} \max\{\|u_1\|, \|u_2\|\}$, using (3.4ii). Since x_0 is continuous in t, hence uniformly continuous on $[0,T]$, this shows equicontinuity, uniform on $[0,T]$, for the set of all possible solutions of (3.5) using 'arbitrary' $j(\cdot)$.

We now wish to show that we can take $j(\cdot)$ to satisfy (2.5) — in particular, (2.5i). Suppose at any time $t \ge 0$ one were to have the state $[x(t) =: \xi_1, 1]$ with $\xi \notin \bar{\mathcal{R}}_1$ so $\theta[\xi_1] =: \hat{\theta}_1 < \theta_2$. By continuity there is an interval $[t, t+h)$ on which $x(\cdot)$ stays out of $\bar{\mathcal{R}}_1$: either one <u>never</u> gets to $\bar{\mathcal{R}}_1$ or there is a 'next time' $\bar{t}$ for which $x(\bar{t}) \in \partial\mathcal{R}_1$ and one (possibly) switches to π_2. Then $x(\bar{t}) = \xi_2 \in \partial\mathcal{R}_1$ gives $\theta[\xi_2] = \theta_2$ so, clearly,

$$(3.8) \qquad 0 < \theta_2 - \hat{\theta}_1 = \theta[\xi_2] - \theta[\xi_1] \le \|\theta\|\, \|\xi_2 - \xi_1\|,$$

$$\|\xi_2 - \xi_1\| = \|x(\bar{t}) - x(t)\| \le \|x_0(\bar{t}) - x_0(t)\| + M_{\bar{t}}(\bar{t} - t).$$

Thus one obtains a <u>lower bound</u> on $(\bar{t} - t)$ in terms of $(\theta_2 - \theta_1)$ and a bound on t, using the uniform continuity of x_0 on bounded t-intervals. A similar argument holds if the state considered is $[x(t) =: \xi_1, 2]$ with $\xi_1 \in \bar{\mathcal{R}}_2$. In particular, one obtains a lower bound on interswitching intervals (lengths of intervals of constancy for $j(\cdot)$) within any $[0,T]$.

An explicit construction of the solution $[x,j]$ proceeds as follows:

(i) If π_k enters $\mathcal{R}_k$ at $\xi_* := x(t_*)$ or if $t_* = 0$ with $x(0) \varepsilon \mathcal{R}_k$, then switch j (from k to $k' := 3-k$ if j is $\{1,2\}$-valued). If π_k is tangential at ξ_*, <u>choose</u>: either continue with $j(\cdot) = k$ or switch j to k'; if $\pi_k(s,\xi_*) \varepsilon \partial\mathcal{R}_k$ for $0 \leq s < \bar{s}$ then a transition (from $j(\cdot) = k$ to k') can be chosen at any time $t_* \leq t \leq t_* + \bar{s}$ — or not at all (then) if π_k leaves $\bar{\mathcal{R}}_k$ at $t_* + \bar{s}$.

(ii) Proceed with $j(\cdot) = k =$ constant ('new' value of k) until one hits $\partial\mathcal{R}_k$ at $t = t_*$ (new value of t_*; if this never occurs one keeps $j(\cdot) = k$ on the 'rest' of $\mathbb{R}^+$). Go back to step (i).

The preceding paragraph ensures that the resulting transition times are isolated as required by (2.5i) while the estimate (3.7) with $t = 0$ shows one cannot have blowup in finite time since x_0 is well-behaved. Thus solutions exist. (It is not difficult to see that all possible solutions are obtainable by the procedure above, making appropriate choices at step (ii) when/if one encounters tangential points.)

Now suppose $\omega < 0$ in (3.4ii) so π_0, π_1, π_2 are asymptotically stable. As $\|x_0(t)\| \leq Me^{\omega t}\|\xi\| \to 0$ one has continuity of x_0 uniformly on $\mathbb{R}^+$ and, noting that M_T is now independent of T, (3.7) gives equicontinuity of x uniformly on $\mathbb{R}^+$. From (3.6)

$$\|x(t)\| \leq Me^{\omega t}\|\xi\| + \bar{M} \tag{3.9}$$

with $\bar{M} := M \max\{\|u_1\|,\|u_2\|\}/|\omega|$ so for all initial $\xi \varepsilon X$ and all $j(\cdot)$ — satisfying (2.5) or not, i.e., for all solutions of (3.5) — one has

$$x(t) \varepsilon \mathcal{B}_0 := [\text{ball of radius}(\bar{M}+1)] \qquad \text{for large enough } t.$$

As $\mathcal{B}_0$ need not itself be invariant, we consider

$$\mathcal{B} := \{x(t) \text{ given by (3.6): } t \geq 0, \ j \text{ as in (2.5i)}, \ \xi \in \mathcal{B}_0\}.$$

Note that $\mathcal{B}$ is bounded by (3.9) with $\|\xi\| \leq \overline{M}+1$ and is clearly a strong attractor as $\mathcal{B}_0 \subset \mathcal{B}$. Any $\hat{\xi} \in \mathcal{B}$ has, by definition, the form

$$\hat{\xi} = S(\hat{t})\xi_0 + \int_0^t S(\hat{t}-s)u_{\hat{j}(s)}\, ds \qquad (\hat{j} \text{ on } [0,\hat{t}], \ \xi_0 \in \mathcal{B}_0).$$

Thus, for any solution $[x,j]$ starting at $\hat{\xi}$ one has

$$\begin{aligned} x(t) &= S(t)\hat{\xi} + \int_0^t S(t-s)u_{j(s)}\, ds \\ &= [S(t+\hat{t})\xi_0 + \int_0^t S(t+\hat{t}-s)u_{\hat{j}(s)}\, ds] + \int_t^{t+\hat{t}} S(t+\hat{t}-s)u_{j(s-\hat{t})}\, ds \end{aligned}$$

which is again in $\mathcal{B}$. Thus $\mathcal{B}$ is invariant under Σ in the sense that all orbits starting at $\xi \in \mathcal{B}$ stay in $\mathcal{B}$. ■

THEOREM 3: Under the hypotheses of Theorem 2 with $\omega < 0$ in (3.4ii), there is a compact set K_ξ — depending continuously on the initial $\xi \in X$ — such that all orbits starting at ξ remain in K_ξ. If, in addition, the operator A is such that

(3.4iii) $\quad S(\varepsilon)$ is compact for some $\varepsilon > 0$, hence for all $t \geq \varepsilon$,

then there is a compact convex invariant set $\mathcal{B}$.

(Compare this result with the situation given in Example 2 using (3.2) in $X = \mathbb{R}^2$. The only difference in the hypotheses is the requirement here that $A_1 = A_2 = A$, leading to the representation (3.6).)

<u>Proof of Theorem 3</u>: We begin by adapting a compactness argument from [5]. As x_0 is continuous with $x_0(t) \to 0$ one has

$$\hat{K}_\xi = 0 \cup [\text{range of } S(\cdot)\xi]$$

compact. Now, for $u \in X$ (e.g., for $u = u_1$ or u_2) let

$$\mathcal{M} = \mathcal{M}(u) := \{ \int_0^t \sigma(s)S(t-s)u \, ds : t > 0, \ \sigma: [0,T] \to \{0,1\} \text{ measurable}\}.$$

It is convenient to re-write

$$(3.10) \qquad \int_0^t \sigma(s)[S(t-s)u] \, ds = \int_0^\infty \hat{\sigma}(s)\hat{S}(s) \, de^{-\bar{\omega}s}$$

where $\bar{\omega} := -\omega/2 > 0$ and

$$(3.11) \qquad \hat{S}(s) := e^{\bar{\omega}s} S(s)u/\bar{\omega},$$

$$\hat{\sigma}(s) := \{\sigma(t-s) \text{ on } [0,T], \ 0 \text{ for } s > t\}.$$

Since $\|\hat{S}(s)\| \le M\|u\| \, e^{-\bar{\omega}s}/\bar{\omega} \to 0$, we have (as for $\hat{K}_\xi$ above) that $\tilde{\mathcal{M}} :=$ [closed convex hull of the range of $\hat{S}$] is compact in X. Since $\hat{\sigma}$ is $\{0,1\}$-valued and $0 \in \tilde{\mathcal{M}}$, the integrand on the right hand side of (3.10) is always in $\tilde{\mathcal{M}}$; since $de^{-\bar{\omega}s}$ gives total measure 1 to $[0,\infty)$, this is an (integral) convex combination in $\tilde{\mathcal{M}}$ and so gives values in $\tilde{\mathcal{M}}$. Thus $\mathcal{M}(u) \subset \tilde{\mathcal{M}} = \tilde{\mathcal{M}}(u) =$ compact. We then have compactness in X of $K_\xi := \hat{K}_\xi + \tilde{\mathcal{M}}(u_1) + \tilde{\mathcal{M}}(u_2)$ which, by (3.6), contains all orbits starting at ξ. It is clear that $\hat{K}_\xi$ depends continuously on ξ — say, in the sense of Hausdorf metric — uniformly on bounded sets whence K_ξ is also continuously dependent on ξ. (Somewhat more generally, it follows from this that for compact $\mathfrak{X} \subset X$ one again has $K_{\mathfrak{X}} := \cup \{K_\xi : \xi \in \mathfrak{X}\}$ compact in X with $\mathfrak{X} \longmapsto K_{\mathfrak{X}}$ continuous in Hausdorf metric.)

Now add the hypothesis (3.4iii). Starting with $\mathcal{B}_0$ as in the proof of Theorem 2, define

$$\mathcal{C}_0 \quad := \quad \{S(t)\xi : \ t \geq 0, \ \ \xi \in \mathcal{B}_0\},$$

$$\mathcal{C} \quad := [\text{closed convex hull of } \ S(\varepsilon)\mathcal{C}_0].$$

Note that $\mathcal{C}_0$ is bounded so $S(\varepsilon)\mathcal{C}_0$ is precompact and $\mathcal{C}$ is compact. Note that $\mathcal{C}_0$ and so $S(\varepsilon)\mathcal{C}_0$ is invariant under $S(t)$ for every $t \geq 0$ so its convex hull is also invariant under $S(t)$ and so is $\mathcal{C}$. Now let

$$\mathcal{B} \quad := \quad \mathcal{C} + \tilde{M}(u_1) + \tilde{M}(u_2).$$

Clearly $\mathcal{B}$ is convex and every orbit enters $\mathcal{B}_0 \subset \mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{B}$; we need only show invariance. Since (3.12) gives $S(t)\hat{S}(s) = e^{-\bar{\omega}t}\hat{S}(t+s)$, one has $S(t)I \in e^{-\bar{\omega}t}\tilde{M}$ for $I \in M$. Thus, with $I \in \tilde{M}$ and $\hat{I}$ given by (3.10), one has

$$[S(t)I+\hat{I}] \ \varepsilon \ [e^{-\bar{\omega}t}\tilde{M} + \left(\int_0^t de^{-\bar{\omega}s}\right)\tilde{M}] \quad = \quad \tilde{M}.$$

Much as in Theorem 2, we now note that for $[x,j]$ starting at $\hat{\xi} \in \mathcal{B}$ one has $\hat{\xi} = \xi + I_1 + I_2$ with $\xi \in \mathcal{C}$ and $I_j \in \tilde{M}_j := \tilde{M}(u_j)$ so, from (3.6),

$$\begin{aligned} x(t) \quad &= \quad S(t)\hat{\xi} + \hat{I}_1 + \hat{I}_2 \\ &\qquad\qquad \text{with } \hat{I}_j \text{ as in (3.10) with } u = u_j \\ &= \quad S(t)\xi + [S(t)I_1 + \hat{I}_1] + [S(t)I_2 + \hat{I}_2] \\ &\varepsilon \quad \mathcal{C} + \tilde{M}_1 + \tilde{M} \quad = \quad \mathcal{B} \end{aligned}$$

which proves the invariance of $\mathcal{B}$. ■

4. MORE EXAMPLES

EXAMPLE 5: We consider a mildly realistic heat control problem involving a thermostat. In Figure 5, the sketch (a) gives a floor plan of a room while (b) shows the 'rear' wall. The heat loss is through the window glass (W in (b)). The radiator (R in (b)) is controlled by the thermostat mounted as indicated by P_* in (a).

We treat this as a boundary control problem. Thus, we have the heat equation

$$(4.1) \qquad \dot{x} = \Delta x \qquad \text{in } \Omega \subset \mathbb{R}^3$$

with boundary conditions

$$(4.2) \qquad \begin{array}{llll} \text{on } W: & -\partial x/\partial n = \alpha x & & \text{(radiative heat loss)}, \\ \text{on } R: & x = \bar{\theta}_j & & \text{for } j = 1,2, \text{ (heating if } j = 1), \\ \text{on } \Gamma: & -\partial x/\partial n = 0 & & \text{(insulation)} \end{array}$$

($\Gamma := \partial\Omega \setminus [W \cup R]$). The choice of $j(t)$ in (3.13R) is to be governed by the thermostat — as in (3.1) with the thermometer reading as θ: for a temperature distribution $\xi(\cdot)$ on Ω we let

$$(4.3) \qquad \theta[\xi] := \xi(P_*).$$

The condition 'furnace ON' = 'radiator hot' is the mode $x\big|_R = \bar{\theta}_1$ while $x\big|_R = \bar{\theta}_2$ corresponds to 'radiator cold'; thus, $\bar{\theta}_1 > \bar{\theta}_2$. On the other hand, the set points $\theta_1 < \theta_2$ are, as in (3.1), the thresholds for 'turn ON', 'turn OFF'. Let ξ_1 be the steady state solution if one were in the 1-st (ON) state and ξ_2 the steady state for OFF: thus

$$(4.4) \qquad -\Delta\xi_j = 0 \text{ on } \Omega, \qquad (3.13) \text{ holds for } \xi_j \text{ with } \bar{\theta}_j \text{ used.}$$

We obviously have $\xi_1 > \xi_2$ on Ω (by the strong maximum principle for (3.15)). For a reasonable situation we must suppose

$$\theta[\xi_1] \ := \ \xi_1(P_*) \ > \ \theta_2 \ > \ \theta_1 \ > \ \xi_2(P_*) \ =: \ \theta[\xi_2].$$

To complete the mathematical description of the model we must specify the space X. For the present problem, the smoothing properties of (4.1) are such that this specification — so long as it does not impose regularity inconsistent with consideration of nontrivial solutions satisfying (4.2) — primarily specifies the sense of our notions of convergence. We will choose to take $X = L^2(\Omega)$ and to interpret (4.1), (4.2) in the sense of semigroup theory. Thus, we consider $A: \mathcal{D} \subset X \to X$ given by

$$\text{(4.5)} \qquad A\xi \ := \ \Delta\xi \in X \qquad \text{for } \xi \in \mathcal{D},$$

$$\mathcal{D} \ := \ \{\xi \in H^2(\Omega) : \ (\xi_\nu + \alpha\xi)\big|_W = 0, \ \ \xi\big|_R = 0, \ \ \xi_\nu\big|_\Gamma = 0\}.$$

This is closed, densely defined, negative definite, self adjoint and so is the infinitesimal generator of an analytic semigroup $S(\cdot)$ of compact operators.

The situation is much as in Example 4 (We have (3.4i,ii,iii) with $\omega < 0$.) with two technical differences: (a) $\theta[\cdot]$ is not continuous on X and (b) the control, i.e., the distinction between the ON/OFF modes π_1, π_2, does not appear in the equation but enters the dynamics through the boundary conditions — a possibility not envisioned in the abstract formulation of Example 4.

The difficulty (a) is only a minor annoyance. Since $S(\cdot)$ is an analytic semigroup one has $S(t)\xi \in \mathcal{D}(A^n)$ for arbitrary n so θ should cause no problem. More cogently, the nature of (4.2Γ) permits one to treat P_* as if it were an interior point (say for an enlarged Ω with everything reflected as even across that 'front' wall containing P_*) and interior regularity for (4.1) makes $x(t)$ spatially analytic near P_* for $t > 0$ and C^∞ in t. The sensor value is undefined at $t = 0$, which causes no problems, or one could modify X slightly to require continuity near P_* initially without otherwise affecting the dynamics for $t > 0$.

The consideration of (b) requires a modification of the representation (3.6) following Balakrishman [1] and Washburn [6]. Let e be the solution of the elliptic problem

(3.18) $$\Delta e = 0; \quad (e_\nu + e)\big|_W = 0, \quad e\big|_R = 1, \quad e_\nu\big|_\Gamma = 0$$

and let

(3 19) $$\tilde{S}(t) = (-A)^\alpha S(t)[(-A)^{1-\alpha} e]$$

with $1 > \alpha > 3/4$, noting that $e \in H^s(\Omega)$ for $s < 1/2$ and, by [ ], this gives $e \in \mathcal{D}((-A)^{s/2})$. Note that

(3.20) $$\|\tilde{S}(t)\| \leq Mt^{-\alpha}e^{\omega t} \qquad \text{for } t > 0$$

with $\omega < 0$. Then the solution of (4.1) and (4.2) is given by

(4.7) $$x(t) = S(t)\,x(0) + \int_0^t \theta_{j(s)}\,\tilde{S}(t-s)\,ds$$
$$= x_0(t) + (\bar{\theta}_1 - \bar{\theta}_2)\int_0^t \sigma(t-s)\tilde{S}(s)\,ds$$

where $x_0(t) := \xi_2 + S(t)x(0)$ is the solution of (4.1), (4.2) with $x\big|_R \equiv \bar{\theta}_2$ using the given initial data and $\sigma(s) := 2 - j(s)$; this replaces (3.6).

We wish to show that: (i) $\theta[x(\cdot)]$ is equicontinuous (over functions $x(\cdot)$ given by (4.7) with $\xi := x(0)$ fixed in X and $\sigma(\cdot)$ ranging over the set C of measurable $\{0,1\}$-valued functions) uniformly on $\mathbb{R}^+$ as in Theorem 2 and that (ii) the set

$$M_0 := \{I := \int_0^t \sigma(t-s)\tilde{S}(s)\,ds : \sigma \in C\}$$

is precompact so the existence of a compact, invariant, global attractor for the switching system can be proved as in Theorem 3. It happens that (i) need not hold quite as in Theorem 2: for arbitrary $\xi \in X = L^2(\Omega)$ one could have $\theta[x(\cdot)]$ oscillating rapidly as $t \to 0+$, since θ is not continuous on X, so that 0 could be necessarily a limit of transition times, violating (2.5i). This could be avoided by modifying X, by assuming the initial ξ is continuous near P_*, or by showing uniformity on $[\varepsilon,\infty)$ for arbitrary $\varepsilon > 0$ so a system which has been in operation for even a brief interval thereafter behaves properly; we adopt this last approach.

Consider separately the terms ξ_2, $S(\cdot)\xi =: x_*$, and the integral giving

$$x_\sigma(t) \quad := \quad \int_0^t \sigma(t-s)\tilde{S}(s)\, ds$$

(Note that (4.6) ensures convergence of this integral.) which sum to $x(\cdot)$. The trick of even reflection across the wall on which the thermostat is mounted shows that P_* may be treated as an interior point. Interior regularity shows $\theta[\xi_2]$ is well-defined. The function x_σ satisfies

$$\dot{x}_\sigma = \Delta x_\sigma, \qquad x_\sigma(0) = 0, \qquad (3.13) \text{ with } x\Big|_R = \sigma(\cdot).$$

One can treat σ, x_σ as existing on all of $\mathbb{R}$, vanishing for $t < 0$, so interior regularity shows $\theta[x_\sigma(\cdot)] := x_\sigma(\cdot.P_*)$ is not just continuous but is in $C^\infty(\mathbb{R})$. In particular, for $\bar{\sigma}(s) := \{1 \text{ on } \mathbb{R}^+;\ 0 \text{ for } s < 0\}$ one has this so $\Phi := \theta[x_{\bar{\sigma}}]^\cdot$ is again in $C^\infty(\mathbb{R})$. One easily sees that Φ is the <u>impulse response function</u> for the input/output map: $\sigma \longmapsto \theta[x_\sigma]$ so that

$$(4.8) \quad \hat{\theta}_\sigma(t) \quad := \quad \theta[x_\sigma(t)] \quad = \quad [\sigma * \Phi](t)$$

$$:= \int_0^t \sigma(t-s)\Phi(s)\, ds \quad = \quad \int_0^t \sigma(s)\Phi(t-s)\, ds,$$

$$\hat{\theta}_\sigma^{(k)}(t) \quad = \quad \int_0^t \sigma(s)\Phi^{(k)}(t-s)\, ds \qquad \text{for } k = 0,1,\ldots\ .$$

The exponential decay of $S(\cdot)$ as $t \to \infty$ gives $\Phi^{(k)}(t) \equiv \mathcal{O}(e^{\omega t})$ for each k and the maximum principle gives $\Phi \geq 0$:

$$(4.9) \qquad \text{(i)} \quad |\Phi^{(k)}(t)| \leq M_k e^{\omega t} \quad \text{for } t \geq 0, \ k = 0,1,\ldots,$$

$$\text{(ii)} \quad \hat{\theta}_\sigma^{(k)}(t) = \mathcal{O}(e^{\omega t}) \quad \text{uniformly in } \sigma \in \mathcal{C},$$

and, of course, (4.9ii) for $k = 1$ gives equicontinuity (over $\sigma \in \mathcal{C}$) for $\hat{\theta}_\sigma$ uniformly on $\mathbb{R}^+$. Finally, the function $S(\cdot)\xi$ is continuous on $[\varepsilon,\infty)$ to $\mathcal{D}((-A)^k)$ for every $\xi \in X$, every $\varepsilon > 0$, and each k so $\theta[S(t)\xi]$ is continuous (indeed C^∞) on $[\varepsilon,\infty)$ and is $\mathcal{O}(e^{\omega t})$ so uniformly continuous on each $[\varepsilon,\infty)$ — with continuity at 0 for, e.g., ξ continuous near P_*. We have just shown that the argument employed in the proof of Theorem 2 can also be used here to show that the transition times determined by (2.5ii,iii) with (3.1) are isolated except possibly at $t = 0+$. For initial data ξ continuous near P_* — so $[S(t)\xi](P_*)$ is continuuous in t at $0+$ — or for any initial data with a slight relaxation of (2.5i) at start-up, we have shown that solutions of the switching system exist as in Theorem 2. A maximum principle argument shows that the bounded set $\{\xi \in X : \xi_2 \leq \xi \leq \xi_1 \text{ pointwise a.e. on } \Omega\}$ is globally attractive and invariant. Alternatively, the existence of a bounded invariant global attractor follows from (3.4ii) much as in Theorem 2.

To show the existence of a compact invariant global attractor one need only show precompactness of

$$\mathcal{M}_0 := \{x_\sigma(t) : \sigma \in \mathcal{C}, \ t \geq 0\}$$

and proceed as in the proof of Theorem 3. Adapting an argument of [ ], we note that

$$(4.10)\qquad x_\sigma(t) = \int_0^\delta \sigma(t-s)\tilde{S}(s)\,ds + \int_\delta^t \sigma(t-s)\tilde{S}(s)\,ds$$

$$= \int_0^\delta \sigma(t-s)\tilde{S}(s)\,ds + s(\delta)\int_0^{t-\delta} \sigma(t-\delta-s)\tilde{S}(s)\,ds$$

$$\varepsilon\ B_\varepsilon + S(\delta)M_0$$

where B_ε is the ball of radius ε (with $\varepsilon \to 0$ as $\delta \to 0$ by (4.6)). Clearly M_0 is bounded by (3.4ii) so $S(\delta)M_0$ is precompact and so totally bounded. Thus, using a finite number of centers one can cover $S(\delta)M_0$ by ε-balls and, by (4.8), each $x_\sigma \varepsilon M_0$ is in one of the corresponding 2ε-balls. As $\varepsilon > 0$ is arbitrary, this shows M_0 is itself totally bounded and so precompact.

Thus, although the present example involves some technical difficulties associated with point observation and boundary control so Theorems 2.3 cannot be applied directly, nevertheless the discussion above shows that the arguments for those theorems, with some minor modifications, can still be used to obtain the same results.

THEOREM 4: Let Σ be the switching system defined by (2.5), (3.1), (4.1), (4.2) with underlying space $\hat{X} := \{\xi \ \varepsilon\ L^2(\Omega) : \xi$ continuous near $P_*\}$. Then all the conclusions of Theorems 2.3 are valid here. ■

One could, of course, consider other variants of this example. E.g., one could use (4.2) for $j = 1$ but include R in Γ (i.e., impose: $x_\nu|_R = 0$) for $j = 2$. Since one now has different semigroups for the two modes, it is not immediately clear that this might not resemble (3.2) more than Example 4. It turns out, however, that (with still more modification which we do not present here) one can show that the same results can still be obtained.

EXAMPLE 6: We consider the state space

$$X := \{\text{piecewise constant, } \{0,1\}\text{-valued functions on } \mathbb{R}^+\}$$

and two flows defined for $j = 1,2$ and $\xi \in X$ by

$$(4.11) \qquad [\pi_j(t,\xi)](s) := \begin{cases} \xi(s-t) & \text{for } s \geq t \\ j-1 & \text{for } 0 \leq s < t. \end{cases}$$

Assume one has given a function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(4.12) (i) $\Phi(0) = 0$, $\int_0^\infty \Phi(s)\,ds = 1$,

(ii) $\sup\{|\Phi(s)| : s \geq t\} \leq \hat{\Phi}(t)$ for $t > 0$ with $\hat{\Phi}$ nonincreasing and integrable: $\int_0^\infty \hat{\Phi} =: M < \infty$,

(iii) $|\Phi(t) - \Phi(s)| \leq K(t-s)\hat{\Phi}(s)$ for $t \geq s \geq 0$; typically one might consider $\hat{\Phi}(t) := Ce^{-\alpha t}$.

Given switching values θ_j $(0 < \theta_1 < \theta_2 < 1)$ we define the switching rules by (3.1) using the observed functional (sensor)

$$(4.13) \qquad \theta[\xi] := \int_0^\infty \Phi(s)\xi(s)\,ds.$$

Note that we can topologize X as a subset of the linear space

$$\hat{X} := \{\xi \text{ measurable on } \mathbb{R}^+ : \int |\xi|\hat{\Phi} =: \|\xi\| < \infty\}$$

and θ, π_j extend continuously to $\hat{X}$, $\mathbb{R}^+ \times \hat{X}$. It is worth noting that (4.12iii) implies differentiability a.e. of Φ (with $|\Phi'| \leq K\hat{\Phi}$) and that the observed sensor value $\hat{\theta}(t) := \theta[\xi(t;\cdot)]$ for any solution $\xi(\cdot)$ must be continuously differentiable with

$$\hat{\theta}^{\cdot}(t) = \int_0^\infty \Phi'(s)\xi(t;s)\,ds. \tag{4.14}$$

Much as with (3.8) one gets a lower bound on the interswitching intervals: if, for example, one switches from π_1 to π_2 at time t_1 and then, proceeding along π_2, next switches back to π_1 at time t_2, then

$$\begin{aligned}
\theta_2 - \theta_1 &= \theta[\xi(t_2) - \xi(t_1)] \\
&= \int_0^\infty \Phi(s)\xi(t_2;s)\,ds - \int_0^\infty \Phi(s)\xi(t_1;s)\,ds \\
&= \int_0^{t_2-t_1} \Phi(s)\,ds + \int_{t_2-t_1}^\infty \Phi(s)\xi(t_1;\, s-[t_2-t_1])\,ds \\
&\qquad - \int_0^\infty \Phi(s)\xi(t_1;s)\,ds \\
&\leq \int_0^{t_2-t_1} |\Phi(s)|\,ds + \int_0^\infty |\Phi(s+t_2-t_1)-\Phi(s)|\,ds \\
&\leq (t_2-t_1)[\max\{|\Phi|\} + MK] = C(t_2-t_1)
\end{aligned} \tag{4.15}$$

so $(t_2-t_1) \geq (\theta_2-\theta_1)/C$; the same estimate holds for interswitching intervals along π_1. For an upper bound on the interswitching interval, we note that if, e.g., one proceeds along π_2 following a switch at time t_1, then

$$\begin{aligned}
\theta[\xi(t_2)] &= \int_0^{t_2-t_1} \Phi(s)\,ds + \int_{t_2-t_1}^\infty \Phi(s)\xi(t_1;\, s-t_2+t_1)\,ds \\
&= 1 - \int_{t_2-t_1}^\infty \Phi(s)[1 - \xi(t_1;\, s-t_2+t_1)]\,ds \\
&\leq 1 - \int_{t_2-t_1}^\infty \hat{\Phi}(s)\,ds \to 1 \quad \text{as } (t_2-t_1)\to\infty
\end{aligned}$$

so for $t_2 - t_1$ is bounded by τ such that

$$\int_\tau^\infty \hat{\Phi}(s)\, ds \leq 1 - \theta_2.$$

A similar estimate holds for intervals along π_1. This proves that (2.5i) holds (for arbitrary initial $[\xi,k]$) with positive lower and upper bounds for the lengths of the interswitching intervals. Hence, global solutions (i.e., solutions on $\mathbb{R}^+$) exist for every initial $[\xi,k]$. Observe that if we introduce the subset

$$X_0 = \{\xi(\cdot) \in X\text{: one has the obtained bounds for the intervals of constancy}\},$$

then X_0 is invariant under the switching system and so could be taken to be the underlying state space.

THEOREM 5: Let Σ_Φ be the switching system with state space $X \times \{1,2\}$ defined by (2.5), (3.1), (4.11), (4.13) in terms of a function Φ satisfying (4.12). Then global (on $\mathbb{R}^+$) solutions exist for each initial $[\xi,k] \in \hat{X} \times \{1,2\}$. There is a compact invariant set $\mathcal{B}$ which is a global weak attractor: for each neighborhood $N \supset \mathcal{B}$, every solution eventually enters and stays in N.

Proof: The first part of the theorem has already been shown for the underlying state space X. For any initial $\xi_0 \in \hat{X}$ we observe that the same method of estimation (4.15) gives a lower bound, except that one now uses

$$\int_0^\infty [\Phi(s+t_2-t_1) - \Phi(s)]\xi(t_1,s)\, ds \leq MK(t_2-t_1)\,\|\xi(t_1)\|_{\hat{X}}$$

which makes the constant C dependent on $\|\xi(t_1)\|$. Since, in any case, $\xi(t;\cdot)$ is $\{0,1\}$-valued on $[0,t)$ one has

$$\|\xi(t)\| = \int_0^t \xi(t;s)\hat{\Phi}(s)\, ds + \int_t^\infty |\xi_0(s)|\hat{\Phi}(s+t)\, ds.$$

The first term is no greater than M and, using the Dominated Convergence Theorem and (4.12ii), the second term goes to 0 as $t \to \infty$. This shows that for any initial $\xi_0 \in \hat{X}$ one has $[\xi(\cdot), j(\cdot)]$ satisfying (2.5i) with asymptotically lower and upper bounds on the lengths of the interswitching intervals approaching those obtained for $\xi_0 \in X$. It is also not too difficult to verify (4.14) for solutions starting at any $\xi \in \hat{X}$.

From the above it follows that for any τ one has $\xi(t;\cdot)$ $\{0,1\}$-valued on $[0,t)$, having norm less than $M+1$ for sufficiently large t, coinciding on $[0,\tau]$ with an element of X_0 (i.e., satisfying the lower and upper bounds for intervals of constancy used to define X_0 — provided X_0 is defined with these specified a bit lower/higher than the best possible bounds obtainable for solutions starting in X).

Since the tail (restriction to (τ,∞)) contributes arbitrarily little to the norm for large enough τ, this shows that: For any solution (starting at an arbitrary $\xi_0 \in \hat{X}$) and any $\varepsilon > 0$, there is a t_* depending only on $\|\xi_0\|$ and ε such that $\|\xi(t) - X_0\| < \varepsilon$ for $t > t_*$. Thus X_0 is a global weak attractor. Let $\xi_n \in X_0$ with $\xi_n \to \xi$ in $\hat{X}$. Let $j_{1,n}$ be the constant value of $\xi_n(\cdot)$ on its first interval of constancy $[0,t_{1,n})$ for $n = 1,\ldots$. Clearly the $\{j_{1,n}\}$ are all the same — say, 1 — from some n on and each $\xi_n(\cdot)$ is then 0 on $[t_{1,n},t_{2,n})$ with $(t_{2,n} - t_{1,n})$ bounded below. Then L^1 convergence ensures $t_{1,n} \to t_1$ for some t_1 with $\xi(\cdot) = 1$ on $[0,t_1)$ a.e. and t_1 within the lower/upper bounds. One then, similarly has $(t_{2,n} - t_{1,n}) \to (t_2 - t_1)$ with $\xi(\cdot) = 0$ on (t_1,t_2) a.e. and $(t_2 - t_1)$ within the lower/upper bounds. Induction then shows $\xi \in X_0$ whence X_0 is closed in X and so complete. Next, given $\varepsilon > 0$ choose $\tau = \tau(\varepsilon)$ large enough that $\int \hat{\Phi} < \varepsilon/2$ on (τ,∞) and let $N = N(\varepsilon)$ be τ divided by the lower bound used for X_0 for constancy intervals so N is an upper bound on the number of switches on $[0,\tau]$. Partition $[0,\tau]$ by $0: \tau_0 < \tau_1 \ldots < \tau_I = \tau$ so that $\int \hat{\Phi} < \varepsilon/2N$ on each subinterval (τ_i, τ_{i+1}). Letting $\mathcal{C}$ be the set of 2^I functions in $\hat{X}$ which are 0 or 1 on each interval (τ_i, τ_{i+1}) and vanish on (τ,∞), we see that every $\xi \in X_0$ is within ε in $\hat{X}$-norm from some one of these centers. Hence X_0 is totally bounded and so compact. Thus, since X_0 is (as noted) invariant. we may take $\mathcal{B} = X_0$. Alternatively, one can also have $\mathcal{B}$ convex by taking the closed convex hull of X_0 in $\hat{X}$. ■

<u>Remark 5</u>: This example is related to the input/output relation (4.8) noted for Example 5 and applying also to Example 4. For each of those 'linear' cases, if one views the system as having had an infinite past history with

$$\xi(t;s) \ := \ \sigma(t-s) \ := \ j(t-s) - 1 \qquad \text{for} \quad s \in \mathbb{R}^+$$

for $t \in \mathbb{R}$, then $(t;\cdot) \in \hat{X}$ and (4.8) gives, for the sensor output (shifted by the lower steady state value):

$$\hat{\theta}(t) \ = \ \int_{-\infty}^{t} \Phi(s)\sigma(t-s)\, ds \ = \ \theta[\xi(t;\cdot)]$$

with Φ the impulse response function determined by the underlying linear dynamics (omitting u_j) and the sensor functional. Typically — in Examples 4.5 in particular — this Φ will satisfy (4.12). (For Example 5 one sees easily, by the Maximum Principle, that $\Phi > 0$ on $(0,\infty)$ and, less easily, that $\Phi' > 0$ on some interval $(0,t_*)$ with $\Phi' < 0$ on (t_*,∞); regularity gives Φ analytic and exponentially decaying on $(0,\infty)$ and C^∞ at 0 when extended as 0 on $\mathbb{R}^-$.)

Consider, finally, a situation in which each $\mathcal{R}_j$ is a global attractor (not necessarily invariant) with compact boundary $\partial\mathcal{R}_j$. While the arguments and result would be essentially the same for any (finite) J, we discuss only the bimodal case $J = \{1,2\}$.

THEOREM 6: Let Σ be a switching system defined by a pair of flows π_1, π_2 on X and 'forbidden regions' $\mathcal{R}_1, \mathcal{R}_2$ with compact boundaries $\partial\mathcal{R}_1 = \mathcal{S}_{1,2}$ and $\partial\mathcal{R}_2 = \mathcal{S}_{2,1}$. (Necessarily we must assume $\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2$ are disjoint.) We suppose all orbits for π_1 eventually enter (and so eventually stay in) the open set $\mathcal{R}_1$ and similarly for $\pi_2, \mathcal{R}_2$. Then solutions exist for all initial states; trajectories satisfying (2.4), (2.5ii,iii) necessarily also satisfy (2.5i). Every solution switches infinitely often between π_1 and π_2 with uniform lower and upper bounds on the interswitching intervals. There is a compact invariant attractive set.

<u>Proof</u>: For any $\xi \notin \overline{\mathcal{R}}_j$ define

$$(4.9) \qquad \tau_j(\xi) := \inf\{t : \pi_j(t,\xi) \in \partial\mathcal{R}_j\} = \sup\{t : \pi_j(t,\xi) \notin \overline{\mathcal{R}}_j\},$$

$$\overline{\tau}_j(\xi) := \inf\{t : \pi_j(t,\xi) \in \mathcal{R}_j\}$$

noting that $\tau_j(\xi) \neq 0$ by the continuity of $\pi_j(\cdot,\xi)$ and that $\pi_j(t,\xi) \in \mathcal{R}_j$ for large enough t (as $\mathcal{R}_j$ was assumed attractive for π_j) so $\overline{\tau}_j(\xi) < \infty$. Clearly, then,

$$0 < \tau_j(\xi) \leq \overline{\tau}_j(\xi)$$

for every $\xi \notin \overline{\mathcal{R}}_j$ — with $\tau_j(\xi) = \overline{\tau}_j(\xi)$ if and only if π_j enters $\mathcal{R}_j$ at $\pi_j(\tau_j(\xi),\xi)$. The continuity of $\pi_j(\cdot,\cdot)$ gives lower semicontinuity of τ_j and upper semicontinuity of $\overline{\tau}_j$ since $\mathcal{R}_j$ is open. Hence by the compactness of

$\partial\mathcal{R}_1$, $\partial\mathcal{R}_2$,

$$\tau_1 := \min\{\tau_1(\xi) : \xi \in \partial\mathcal{R}_2\}, \qquad \overline{\tau}_2^* := \min\{\tau_2(\xi) : \xi \in \partial\mathcal{R}_1\},$$

$$\overline{\tau}_1^* := \max\{\overline{\tau}_1(\xi) : \xi \in \partial\mathcal{R}_2\}, \qquad \overline{\tau}_2^* := \max\{\overline{\tau}_2(\xi) : \xi \in \partial\mathcal{R}_1\},$$

are well-defined so $\min\{\tau_1^*,\tau_2^*\}$ gives a lower bound for interswitching intervals while $\max\{\overline{\tau}_1^*,\overline{\tau}_2^*\}$ gives an upper bound. Existence of solutions for all initial states is then immediate since <u>local</u> existence is ensured <u>a priori</u> and no solution can blow up in finite time (or fail to switch infinitely often) since each π_j gives an orbit eventually entering $\mathcal{R}_j$.

Next, we introduce $K := K_1 \cup K_2$ where

$$K_1 := \{\pi_1(t,\xi) : 0 \le t \le \overline{\tau}_1(\xi), \ \xi \in \partial\mathcal{R}_2\},$$

$$K_2 := \{\pi_2(t,\xi) : 0 \le t \le \overline{\tau}_2(\xi), \ \xi \in \partial\mathcal{R}_1\}.$$

Note that K is invariant and attractive since, for any initial point, we must eventually hit, say, $\partial\mathcal{R}_1$ and switch to π_2 — remaining in $K_2 \subset K$ until switching to π_1 at a point of $\partial\mathcal{R}_2$ and then proceeding along π_1, remaining in $K_1 \subset K$ until again switching, etc. To see that K_1 is compact, suppose $\hat{\xi}_n \in K_1$ so $\hat{\xi}_n = \pi_1(t_{n-1}\xi_n)$ with $\xi_n \in \partial\mathcal{R}_2$ and $0 \le t_n \le \overline{\tau}_1(\xi_n)$. There is then a subsequence for which $\xi_n \to \overline{\xi} \in \partial\mathcal{R}_2$ (by the compactness of $\partial\mathcal{R}_2$) and also $t_n \to \overline{t}$ (by the compactness of $[0,\overline{\tau}_1^*]$). By the continuity of π_1 and upper semicontinuity of $\overline{\tau}_1$ we have (for the subsequence) $\hat{\xi}_n = \pi_1(t_n,\xi_n) \to \pi_1(\overline{t},\overline{\xi}) =: \hat{\xi}$ with $0 \le \overline{t} \le \overline{\tau}_1(\overline{\xi})$ so also $\hat{\xi} \in K_1$. Similarly K_2 is compact and so K is. ∎

5. PERIODICITY

As noted in the Introduction, it was computational experience with the model of [3] which suggested seeking periodic solutions of switching systems. At present, the only positive result for existence of periodic solutions (Theorem 7 below) is for a class of switching systems with no tangential points. If a more general positive result could be obtained, one would expect it for systems as in Theorem 5 with, say, $\bar{\mathcal{R}}_1, \bar{\mathcal{R}}_2$ disjoint balls. We construct a counterexample, however, with no periodic solutions (Example 7).

The simplest possible case of periodicity would be a periodic solution of minimal type: starting, say, from some $\xi_1 \in \partial\mathcal{R}_1$ one proceeds along π_2 until hitting $\partial\mathcal{R}_2$ at ξ_2 whereupon one proceeds along π_1 until hitting $\partial\mathcal{R}_1$ again at ξ_1 and repeats. In certain situations such solutions must exist.

THEOREM 7: Let Σ be a (bimodal) switching system with $\bar{\mathcal{R}}_1, \bar{\mathcal{R}}_2$ compact and convex or else let Σ be as in Theorem 3 with compact $S(t)$. Suppose all orbits of π_1 eventually enter and stay in $\mathcal{R}_1$ and similarly for $\pi_2, \mathcal{R}_2$. Then if $\partial\mathcal{R}_1$ and $\partial\mathcal{R}_2$ are each transverse to the respective flows, one has existence of at least one periodic solution of minimal type.

<u>Proof</u>: In either of the cases considered we start with a compact, convex 'initial set' K_0 and define a map $T: K_0 \to K_0$ with a fixed point $\xi_0 \in \partial\mathcal{R}_2 \cap K_0$ determining a periodic solution of minimal type as above. For the first case we simply take $K_0 := \bar{\mathcal{R}}_2$ while for the 'linear' case we take $K_0 := (\mathcal{B} \cap \partial\mathcal{R}_2)$ where $\mathcal{B}$ is the compact convex invariant attractive set whose existence is ensured by Theorem 3. Now, for $\xi_0 \in K_0$ define $T(\xi_0)$ as follows: starting along π_1 from ξ_0, proceed until hitting $\partial\mathcal{R}_1$ at ξ_1 and then return along π_2 until

hitting $\partial\mathcal{R}_2$ at $\xi_2 =: T(\xi_0)$. Because there are no tangential points it is not hard to show, from the continuity on $\mathbb{R}^+ \times \Omega$ of π_1 and π_2, that the map T is continuous. In the first case one clearly has $T(\xi_0) \in \partial\mathcal{R}_2 \subset K_0$ while in the second one has $T(\xi_0) \in K_0$ by the invariance of $\mathcal{B}$. The Schauder Fixed Point Theorem ensures existence of a fixed point $\xi_0 \in K_0$ of T and it is clear that this ξ_0 defines a periodic solution for Σ. ∎

Corollary: Let Σ be an asymptotically stable 'linear' switching system as in Theorem 3 with $X = \mathbb{R}^2$. Then there is at least one periodic solution.

Proof: Trivially, the only interesting case is that in which the steady state solutions $\xi_1 = -A^{-1}u$, $\xi_2 = -A^{-1}u_2$ are in the forbidden regions $\mathcal{R}_1$, $\mathcal{R}_2$; thus, from (3.1),

$$\langle\theta,\xi_1\rangle > \theta_2, \qquad \langle\theta,\xi_2\rangle < \theta_1.$$

To apply the theorem, one need only show there are no tangential points in this case.

Suppose, to the contrary, one had a tangential point $\xi \in \partial\mathcal{R}_1$ for π_1. Then for a solution of $\dot{x} = Ax + u_1$ passing through ξ we must have, at ξ,

$$\langle\theta,\xi\rangle = \theta_2, \qquad \langle\theta,\dot{x}\rangle = 0, \qquad \langle\theta,\ddot{x}\rangle < 0.$$

Using the equation and setting $z := x - \xi_1$, $\zeta := \xi - \xi_1$, this gives

$$(4.10) \qquad \langle\theta,\zeta\rangle < 0, \qquad \langle\theta,A\zeta\rangle = 0, \qquad \langle\theta,A^2\zeta\rangle < 0.$$

For stability, the spectrum of A must lie in the left half-plane so A satisfies a characteristic equation $A^2 + bA + c = 0$ with $b > 0$, $c \geq 0$.

This, however, would give

$$\langle\theta,A^2\zeta\rangle + b\langle\theta,A\zeta\rangle + c\langle\theta,\zeta\rangle = 0$$

which contradicts (4.10). ■

Note that this argument is strictly two-dimensional and, indeed, it is possible to have tangential points in, say, the three-dimensional case. Nevertheless, we continue to conjecture the existence of periodic solutions for any compact stable 'linear' switching system (see Remark 6).

EXAMPLE 7: For the setting of Theorem 7 we see that the assumption that 'there are no tangential points' is <u>necessary</u>: we construct an example with exactly one tangential point for which there is no periodic solution.

Looking at Figure 6 we see in (a) a sketch of π_1 moving more-or-less radially into $\mathcal{R}_1$ but wiggling trickily through $\mathcal{R}_2$. (For comparison, Figure 7 shows the same example — to within topological equivalence — with the flows radial but the region $\mathcal{R}_2$ no longer a ball.) The orbits of π_1 intersecting $\mathcal{R}_2$ are those between $[a_1]$ and $[e_1]$, entering $\mathcal{R}_1$ between the points A and E. Note that every solution for the switching system must hit $\partial\mathcal{R}_2$ (infinitely often) and so, from some time on, must coincide with a trajectory emanating along π_2 from a point of the arc $\widehat{AE}$. The significant features of this geometry are that $[a_1]$ is tangent to $\partial\mathcal{R}_2$ at $\bar{\alpha}$ as well as at α and $[e_i]$ is tangent to $\partial\mathcal{R}_2$ at ε. The intermediate orbits $[b_1]$, $[c_1]$, $[d_1]$ are indicated only to show a smooth field across $\mathcal{R}_2$ except that $[c_1]$ must 'exit' from $\mathcal{R}_2$ and then re-cross $\partial\mathcal{R}_2$ at a point $\bar{\gamma}$ 'between' $\bar{\alpha}$ and α as well as crossing $\partial\mathcal{R}_2$ at γ between α and ε.

Looking next at the sketch of π_2 in Figure 6(b), we note as the significant features (i) that the orbit $[d_2]$ of π_2 is tangential to $\partial\mathcal{R}_2$ at the same point ε defined above as the point of tangency of $[e_1]$ to $\partial\mathcal{R}_2$ and then enters $\mathcal{R}_2$ at the same point $\bar{\alpha}$ defined above as the 'first' point of tangency of $[a_1]$ to $\partial\mathcal{R}_2$ and (ii) that the two π_2-orbits $[a_2]$ and $[e_2]$ enter $\mathcal{R}_2$ at points γ and $\bar{\gamma}$ defined above as two intersections of the same π_1-orbit $[c_1]$ with $\partial\mathcal{R}_2$. The point ε is the <u>only</u> tangential point in the sense of our original definition.

As noted above, for consideration of (possible) periodicity the only trajectories of interest are those emanating from $\widehat{AE}$; let us follow these trajectories. For $[AD)$ the π_2-orbits hit $\partial\mathcal{R}_2$ to fill in the 'interval' $[\gamma,\varepsilon)$ and then the trajectories 'return' along π_1-orbits hitting $\partial\mathcal{R}_1$ again to fill in $[C,E)$. For $(D,E]$ the π_2-orbits hit $\partial\mathcal{R}_2$ to fill in the 'interval' $(\bar{\alpha},\bar{\gamma}]$ and then the trajectories 'return' along π_1-orbits hitting $\partial\mathcal{R}_1$ again to fill in $(A,C]$. For the trajectory emanating from D one has a choice: $D \mapsto \varepsilon$ along $[d_2]$ and then 'return' along $[e_1]$ to E or $D \mapsto \bar{\alpha}$ along $[d_2]$ and then 'return' along $[a_1]$ to A.

If one now <u>identifies</u> $A \simeq E$, then $\widehat{AE}$ becomes, topologically, a circle. The 'round trip' gives $A \simeq E \mapsto C$ by construction and, with the identification, $D \mapsto A \simeq E$ independent of the choice at ε. Thus, the 'round trip' determines a well-defined map ρ of the 'circle' $\widehat{AE}$ to itself; one easily verifies, using the continuity of π_1, π_2, that this map ρ is continuous as well as injective.

Within the framework above one can adjust the construction to make ρ any desired homeomorphism: $\widehat{AE} \to \widehat{AE}$: parametrize $\widehat{AE}$ as $[0,2\pi]$ (with $0 \approx 2\pi$), modify π_2 so $D := \rho^{-1}\ (A \approx E)$ is the initial point in $\partial\mathcal{R}_2$ of the

π_2-orbit $[d_2]$ as above, and modify π_1 (only 'between' $\widehat{\alpha\varepsilon}$ and into $\mathcal{R}_1$) so each initial point is mapped as desired. In particular, one can construct π_1, π_2 so ρ is a 'rotation' of the 'circle' through an arbitrarily specified angle ω. (It is even possible to accomplish this last with C^∞ flows on $\mathbb{R}^2$.)

Suppose the construction above has been 'tuned' so that ρ corresponds to a rotation with $\omega/2\pi$ irrational. This is then the classic example of a map such that no iterate has a fixed point. Clearly, then, the resulting switching system can have <u>no periodic solutions</u> since any periodic solution must entail existence of a fixed point of some iterate of ρ. We do note that <u>every</u> solution, for this example, is eventually 'approximately periodic': Every solution eventually hits the arc $\widehat{AE}$ of $\partial\mathcal{R}_1$ and (except for a possible one-time choice if it passes through D) then has 'future truncations' continuously dependent, uniformly on compact time intervals, on the successive points $\{\xi_k\}$ at which it subsequently hits $\widehat{AE}$ — and $\{\xi_k\}$ is an almost periodic sequence. (It is conjectured that behavior qualitatively like this is generic!)

On the other hand, suppose one has $\omega/2\pi$ rational so $\rho^{(m)}$ is the identity. One then has a periodic solution of the switching system for every initial point in $\widehat{AE}$. This does not mean, however, that every solution is periodic: for the solutions passing through D one has a choice — next going to ε or to $\bar{\alpha}$ along $[d_2]$ — at each recurrence (every m round trips: $\partial\mathcal{R}_1 \to \partial\mathcal{R}_2 \to \partial\mathcal{R}_1$) with the sequence of choices being completely arbitrary. Note that if one were to choose, say, according to the sequence

$$\varepsilon,\ \bar{\alpha},\ \varepsilon,\ \varepsilon,\ \bar{\alpha},\ \varepsilon,\ \varepsilon,\ \varepsilon,\ \bar{\alpha},\ \ldots,$$

then this solution has no (even approximate) periodicity property.

Another variant of this example takes $\rho(t) := t + \varepsilon \sin(t-\delta) \pmod{2\pi}$ with $\varepsilon, \delta > 0$, small. This makes $t = \delta, \delta+\pi$ fixed points of ρ — corresponding to the periodic solutions of the switching system — with δ unstable and $\delta+\pi$ stable, giving a (globally) attractive periodic solution. (For $\rho(t) := t + \varepsilon \sin k(t-\delta)$ one has 2k fixed points/periodic solutions, alternately unstable and stable. For $\rho(t) := t + \varepsilon \sin(t-\delta)/2$ one has a single asymptotically stable periodic solution.) For $\delta = 0$ the situation is much the same except that one has recurring choices at D, as above.

We note that this construction is not really limited to $\mathbb{R}^2$. Let X be $\mathbb{R}^m$ $(m > 2)$ or even an infinite-dimensional Hilbert space and write it as $X = \mathbb{R}^2 \oplus Y$ so points in X are $x = [\xi,y]$ with $|x|^2 = |\xi|^2 + |y|^2$. Define

$$\hat{\pi}_j(t,[\xi,y]) \quad := \quad [\pi_j(t,\xi),S(t)y]$$

where the π_j are as in the two-dimensional constructions above and $S(\cdot)$ is any (compact) semigroup going asymptotically to 0. The regions $\hat{\mathcal{R}}_j$ are, say, balls with centers $[\xi_j,0]$ where the ξ_j are the centers used in $\mathbb{R}^2$ above and the radii are the same: (In the infinite-dimensional case this could be modified to get compact closure.) Since $S(t) \to 0$ as $t \to \infty$, the only points of interest for possible periodicity lie in the subspace $(y = 0)$ where the system reduces to the previous construction for $\mathbb{R}^2$.

We now return to the question of periodicity for the 'linear' case.

<u>Remark 6</u>: Clearly, any periodic solution — for any system — can be treated as existing for all time (on $\mathbb{R}$, rather than $\mathbb{R}^+$) and so as having an infinite past. Following Remark 5, we may thus treat periodic solutions of 'linear' systems within the framework of Example 6, in terms of the impulse response

function Φ which, we assume, satisfies (4.12). Note that any periodic solution of such a 'linear' switching system gives $j(\cdot)$ periodic and so corresponds to a periodic solution of the system Σ_Φ of Example 6; conversely, any solution of Σ_Φ determines (by stability of the underlying linear dynamics which implies negligibility of 'remotely past initial conditions') a unique solution for the 'linear' system — necessarily periodic if the solution of Σ_Φ is periodic. Next, we observe that if $\xi(\cdot)$ is a periodic solution for Σ_Φ with period τ then $\xi(t,\cdot)$ is periodic on $\mathbb{R}^+$ with period τ (hence extendable as periodic on $\mathbb{R}$) and Σ_Φ acts on it by translation: $\xi(t;s) = \xi_0(s-t)$ for $t,s \in \mathbb{R}$.

If we consider any periodic, piecewise constant, $\{0,1\}$-valued function ξ_0 on $\mathbb{R}$ as 'initial' value and use the formula $\xi(t;s) := \xi_0(s-t)$ to determine the 'input', then the corresponding sensor output $\hat{\theta}(\cdot)$ is given by

$$(4.16) \qquad \hat{\theta}(t) \;:=\; \theta[\xi(t)] \;=\; \int_0^\infty \Phi(s)\xi_0(s-t)\,ds$$

and so is necessarily periodic. Here, inverting the viewpoint above, we have taken Σ_Φ to obtain $\xi(\cdot)$ by the formula as if we knew this produced a solution for Σ_Φ. This procedure is valid if and only if, on setting $j(t) := 1 + \xi(t;0) = 1 + \xi_0(-t)$, the switching rules (2.5), (3.1) are verified. With no loss of generality we can assume translation in time so $\xi_0(0+) = 1$, $\xi_0(0-) = 0$ corresponding to a switch from $j = 2$ to $j = 1$ at time $t = 0$. What is needed, then, for ξ_0 to correspond to a periodic solution of minimal type? Clearly $\xi_0 = 1$ on $(0,a)$ and $\xi_0 = 0$ on $(-b,0)$, thus having a period of length $(a+b)$. Then $j = 1$ on $(0,b)$ and $j = 2$ on $(b, a+b)$, also with period $(a+b)$. For this to correspond to a solution at all (then necessarily a periodic solution of minimal type), one must have

(4.17) (i) $\hat{\theta}(0) = \theta_1$, $\hat{\theta}(a) = \theta_2$

(ii) $\hat{\theta} \leq \theta_2$ on $[0,a]$, $\hat{\theta} \geq \theta_1$ on $[-b,0]$.

Note that for the original 'linear' system this gives

$$x(t) = \int_0^\infty S(s)[u_1 + \xi_0(s-t)(u_2 - u_1)]\, ds.$$

Remark 7: Given some mild compactness condition (as has been part of the conclusions for several of the theorems above), we note that it may be possible to find a periodic solution.

THEOREM 8: Let (Σ_n) be a sequence of switching systems satisfying the hypotheses of Theorem 1 and each having a periodic solution $[x_n(\cdot), j_n(\cdot)]$. Suppose there is a fixed compact set $K \subset X$ such that, for every neighborhood $N \supset X$ one has $x_n(\cdot)$ intersecting N for $n \geq n_*(N)$. Suppose one has a bound on the periods: $\tau_n \leq \tau_*$. Then the limit system Σ also has a periodic solution subject to the same bound.

Proof: Let N_n be a sequence of neighborhoods shrinking to K and (t_n) such that $x_n(t_n) =: \xi_n \in N_n$ with $\hat{\xi}_n \in K$ and $|\xi_n - \hat{\xi}_n| \to 0$ as $n \to \infty$. Extracting a subsequence, we may assume $\hat{\xi}_n \to \bar{\xi} \in K$ so $\xi_n \to \bar{\xi}$. By autonomy we may translate each solution so $\xi_n = x_n(0)$. Extracting a subsequence, we may also assume $\tau_n \to \bar{\tau} \leq \tau_*$. Now, applying Theorem 1 gives, again for a subsequence, $[x_n, j_n] \to [x, j]$ with $[x,j]$ a solution for Σ. Recalling the construction of Theorem 1, since $x_n(\tau_n) = \xi_n \to \bar{\xi}$ and $\tau_n \to \bar{\tau}$, one has $x(\bar{\tau}) = \bar{\xi} = x(0)$. Similarly (possibly requiring a bit of extra care if τ_n is a switching time for each x_n) one has $j(\bar{\tau}) = j(0)$ so $[x,j]$ is periodic with period $\bar{\tau} \leq \tau_*$. (Note that if one has an upper bound on the interswitching times as well as (2.9) then one could bound the number of transitions in a period — e.g., assuming each $[x_n,j_n]$ is of minimal type — rather than bounding the period directly and obtain in the limit a periodic solution subject to the same bound.) ■

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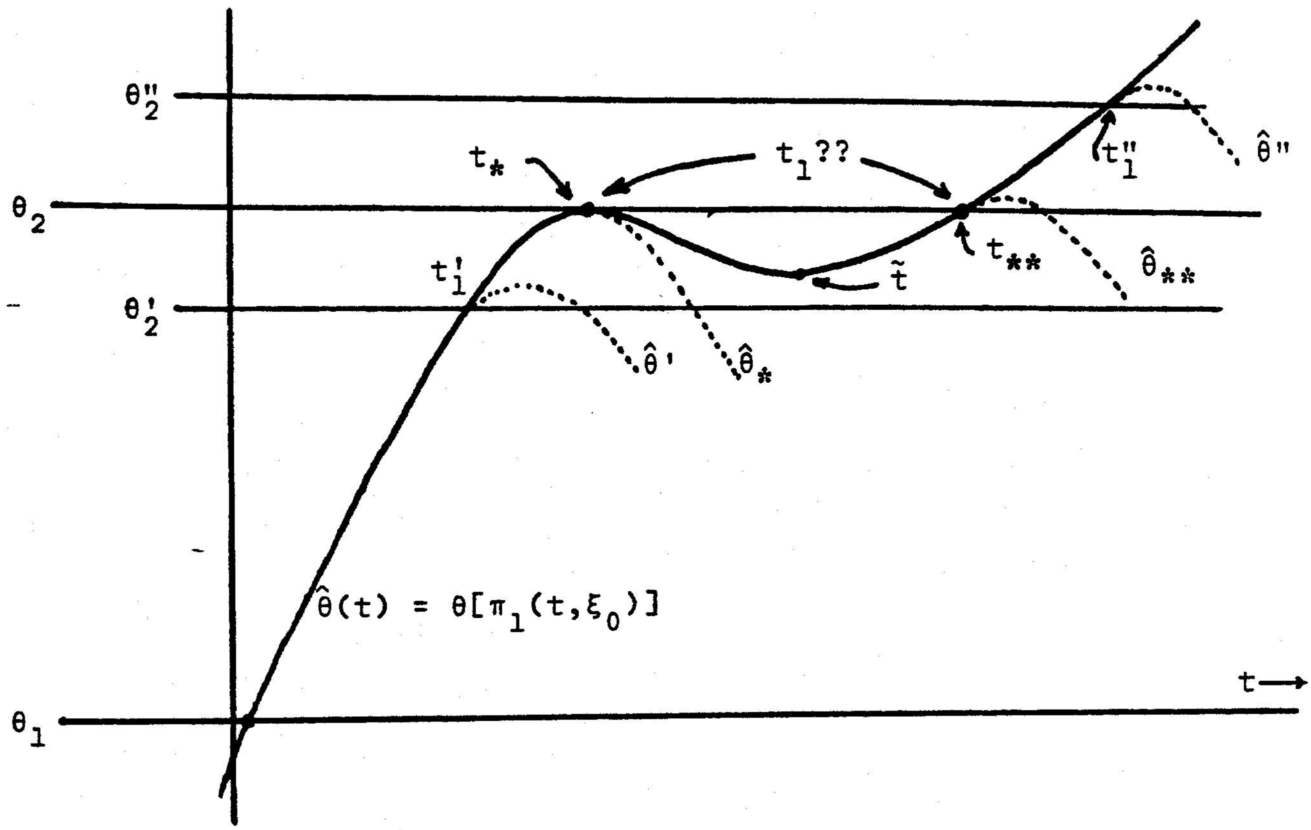
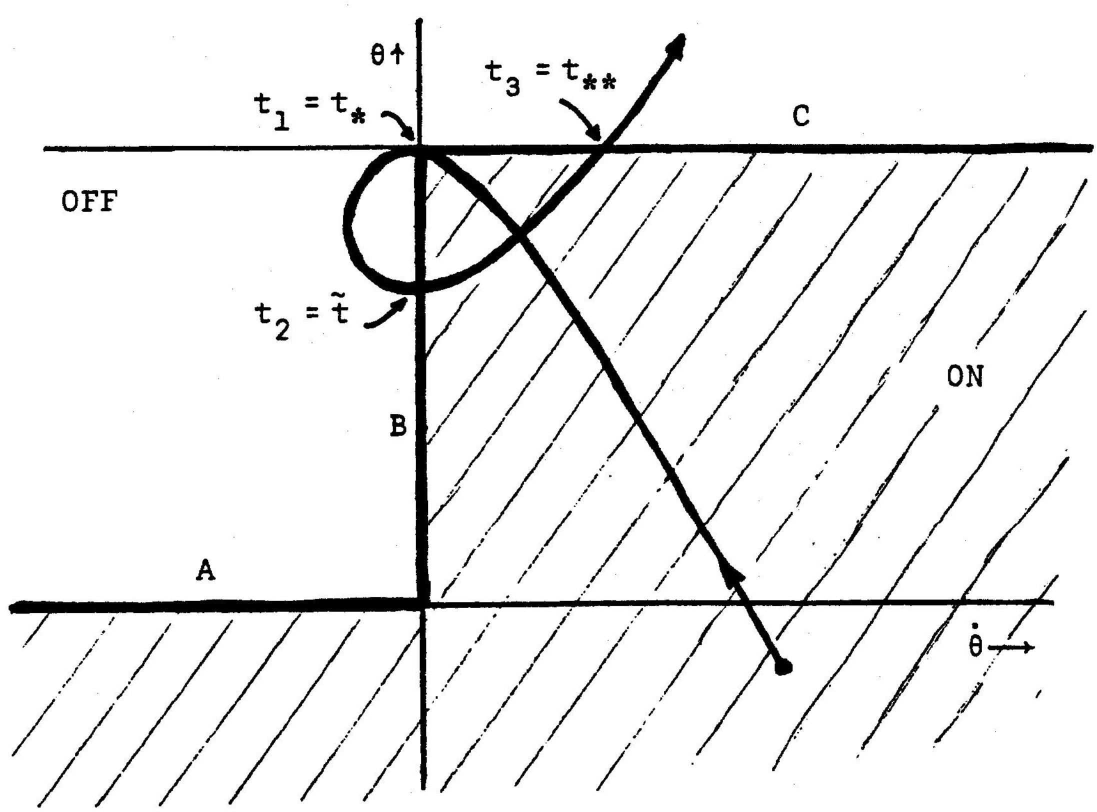


Figure 1.



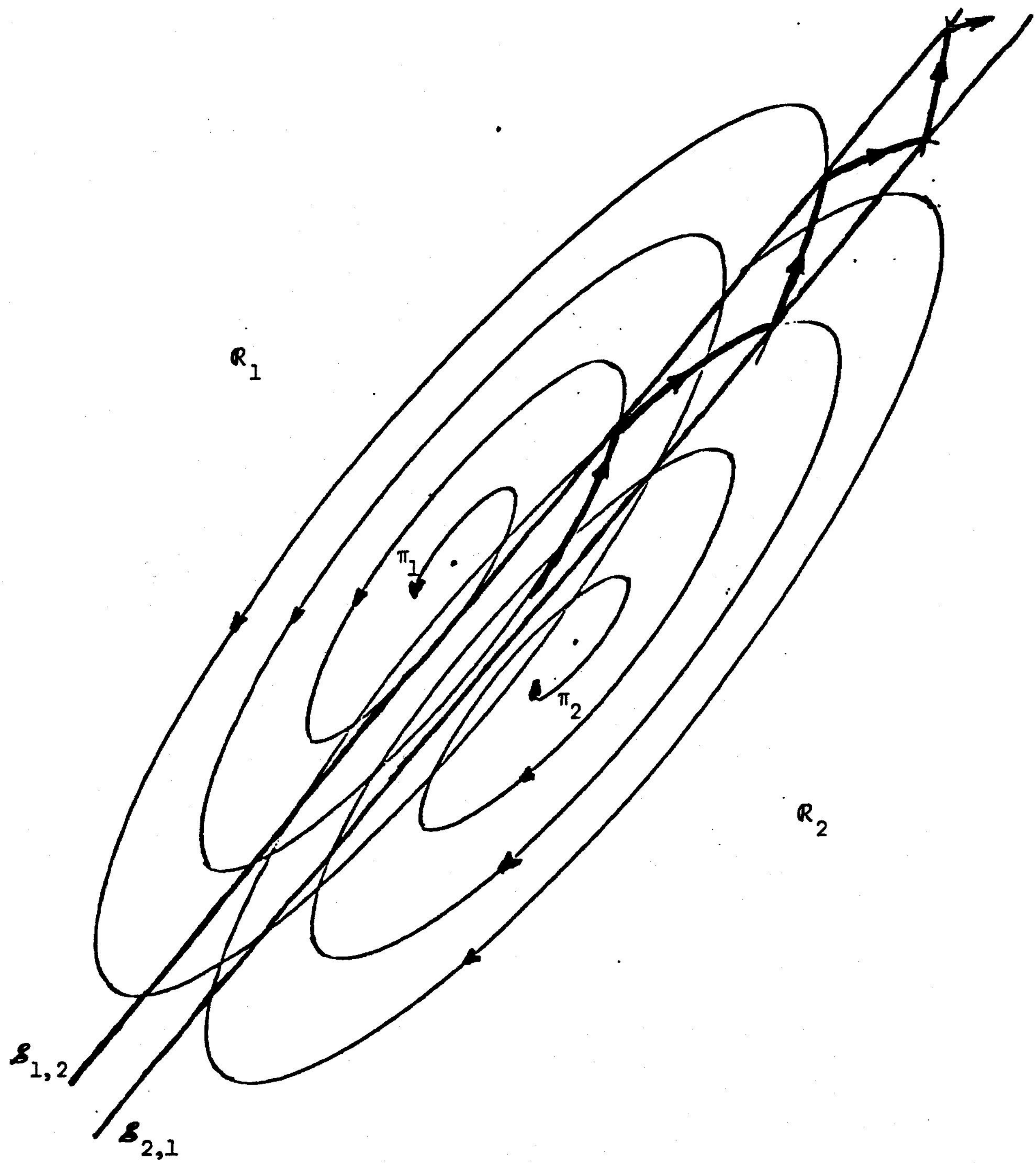
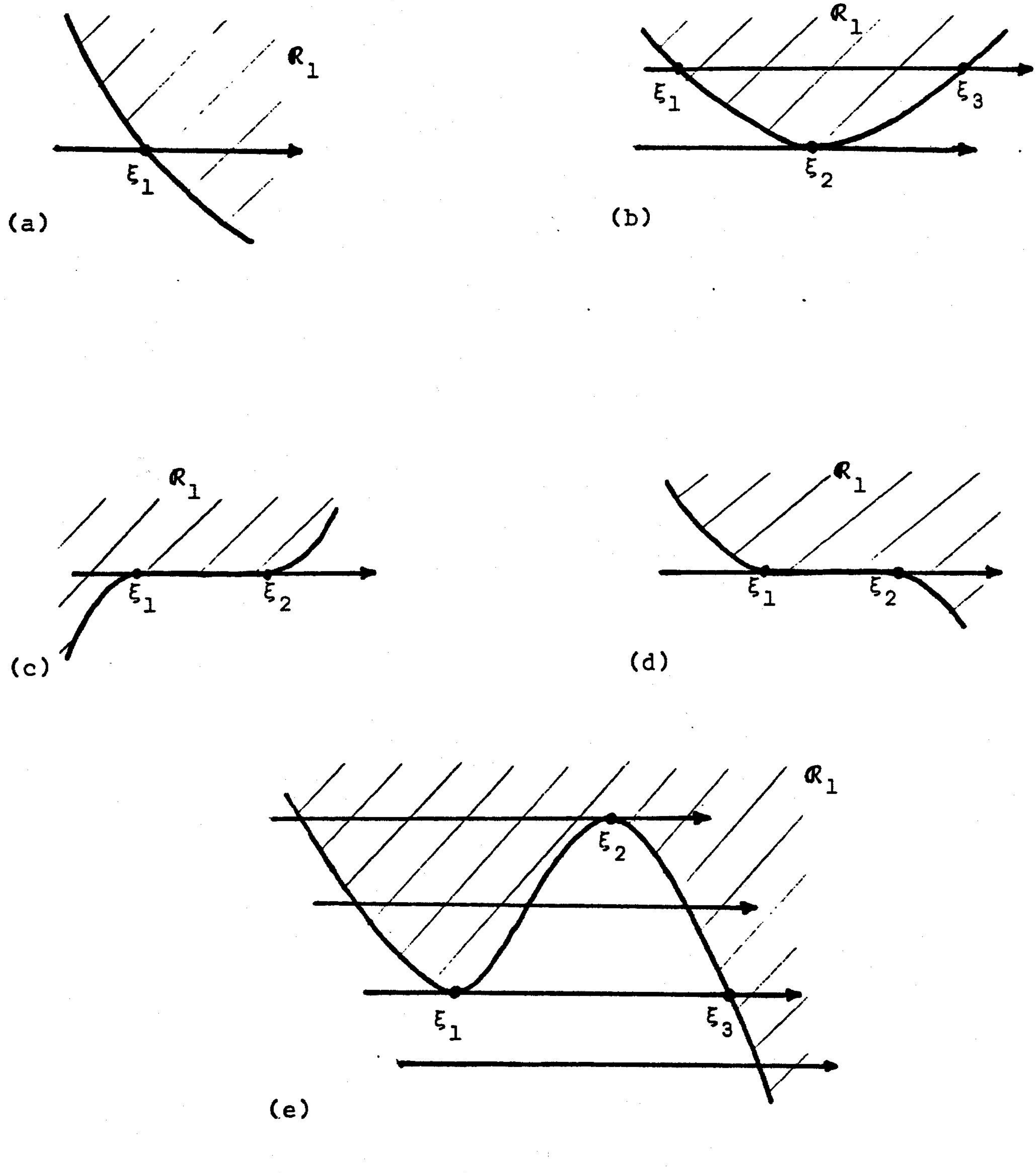
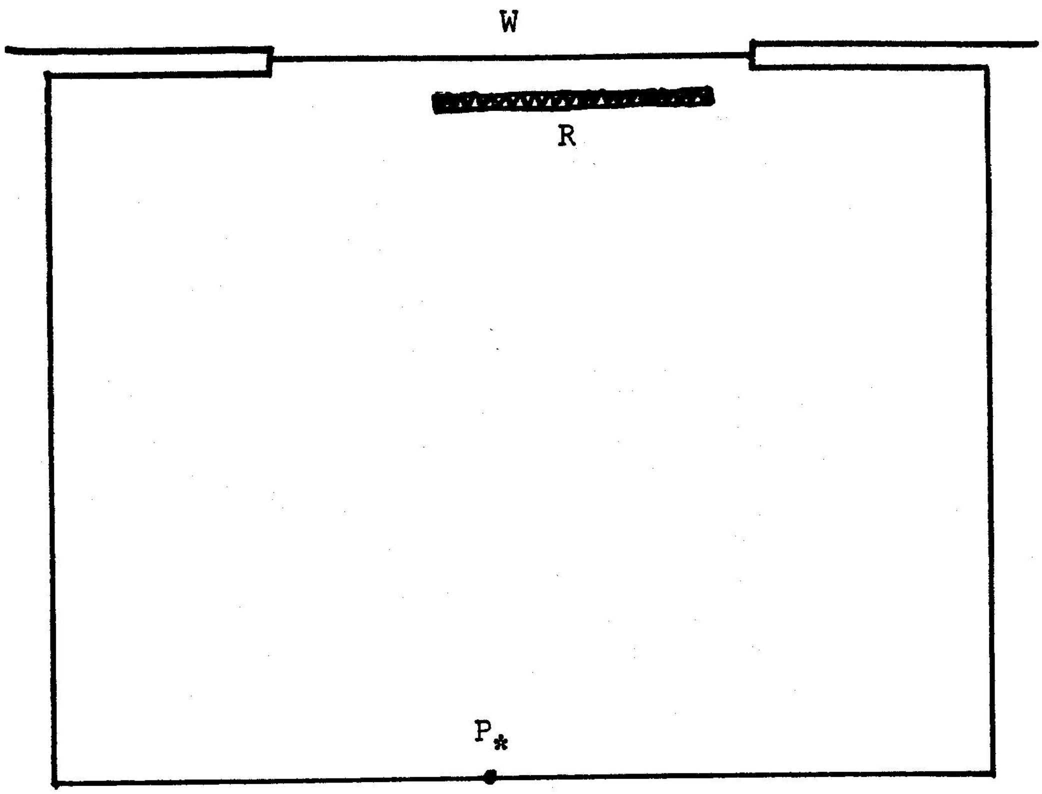


Figure 3.

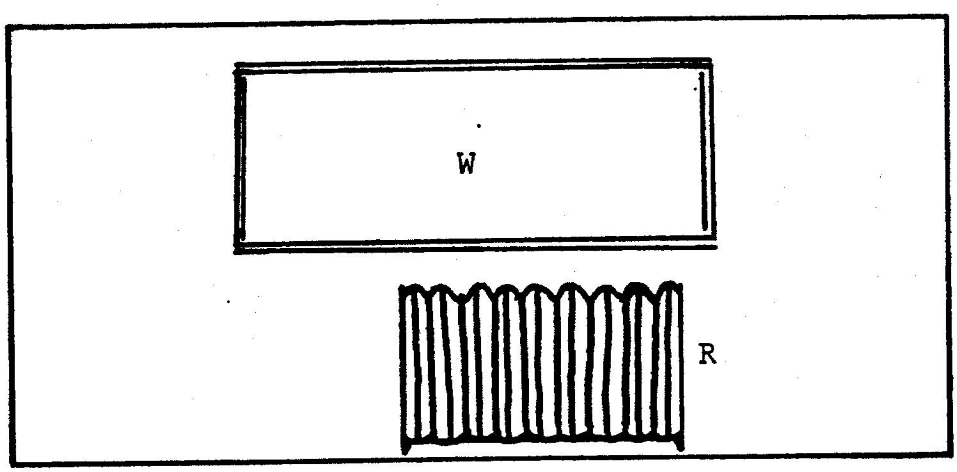


$\mathcal{R}_1$
ξ_1
(a)
$\mathcal{R}_1$
ξ_1
ξ_2
ξ_3
(b)
$\mathcal{R}_1$
ξ_1
ξ_2
(c)
$\mathcal{R}_1$
ξ_1
ξ_2
(d)
$\mathcal{R}_1$
ξ_1
ξ_2
ξ_3
(e)

Figure 4.



(a) Floor Plan



(b) Wall

Figure 5.

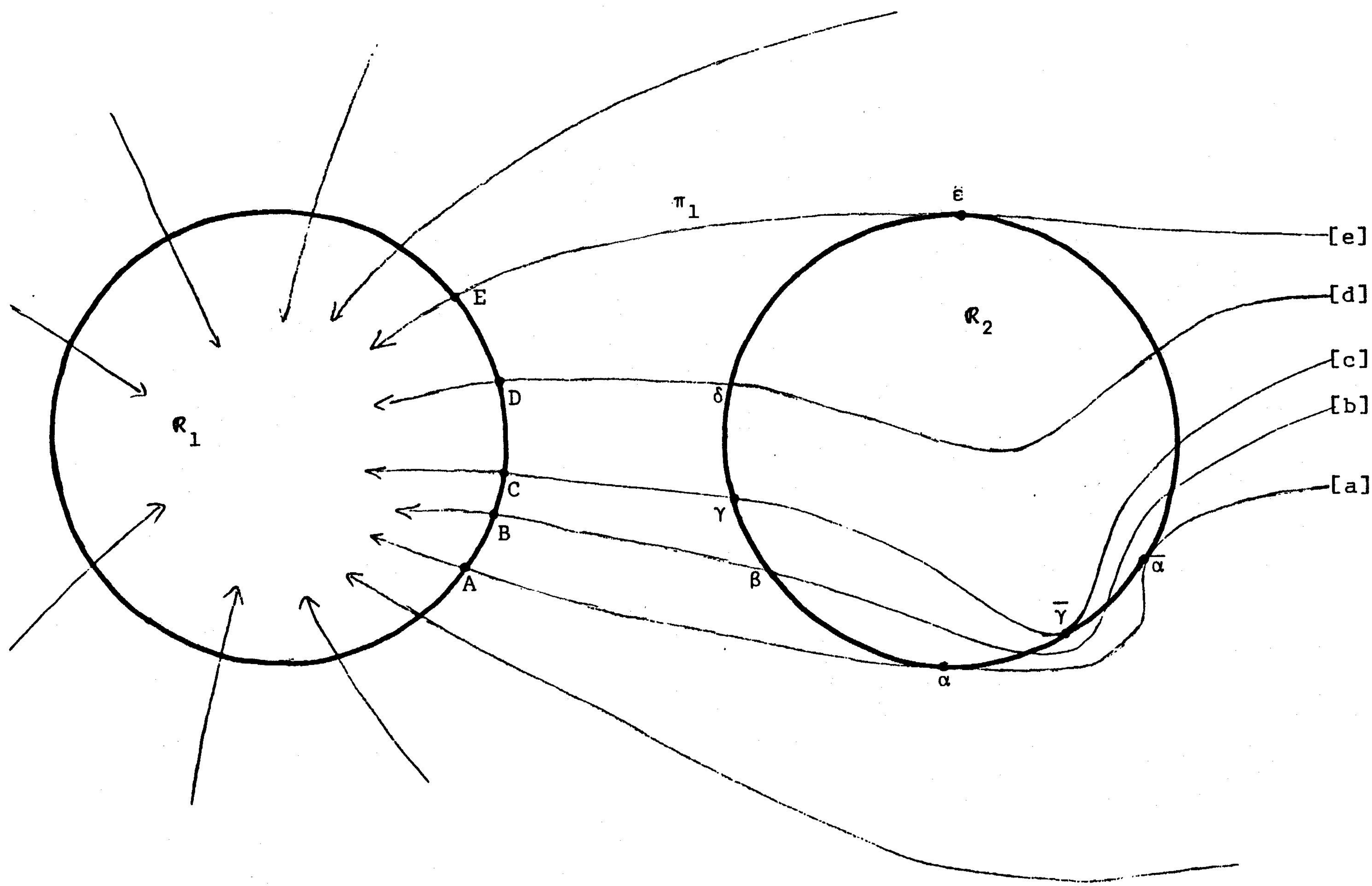


Figure 6(a)

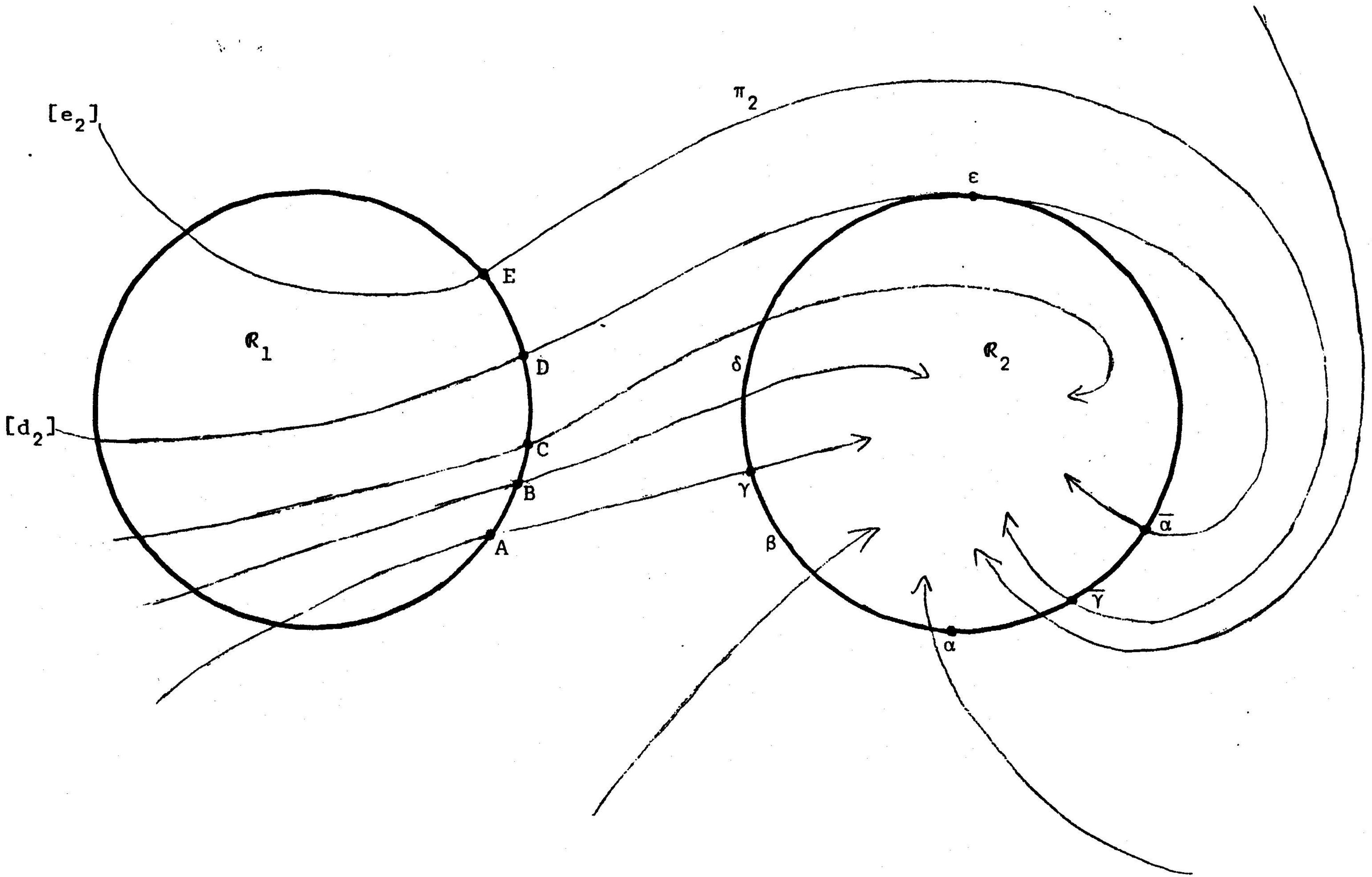


Figure 6(b)