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The residue of model reduction[1][2]

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1. Our intention is to explore some possibly overlooked consequences of the classical observation *"Natura in operationibus non facit saltus,"* — which we take to mean that

- *Any apparent discontinuity occurring in the real world is actually a continuous process having 'fine structure' on a more rapid time scale*

(perhaps omitting quantum mechanics). The significance of this observation for hybrid systems is that the nominal description involving discontinuity is merely a convenient approximation at the relevant time scale which involves the (unmodelled?) neglect of dynamics on any faster time scales — whose details are then necessarily lost in the process of model reduction. We will argue that some residue of these details must be retained to understand, in certain contexts, what will actually occur when these hybrid strategies are to be implemented in the real world.

For example, at a familiar level we think of the thermostat in an electric heater as simply switching the element on or off discontinuously, but it is certainly possible to 'open the box' and consider in more detail, if desirable, the

[1] in *Hybrid Systems III. Verification and Control*, (LNCS #1066; R. Alur, T.A. Henzinger, E.D. Sontag, eds.) pp. 201–207, Springer-Verlag, Berlin (1996). [Proc. Rutgers Conf. on Hybrid Systems — Oct., 1995.]

[2] This is a slightly expanded version of a talk given at the Workshop on Hybrid Systems held at Rutgers University in October, 1995, and is to appear in the proceedings: *Hybrid Systems III*, (R. Alur, T. Henzinger, and E. Sontag, eds.), Springer–Verlag Lecture Notes in Computer Science.

moderately complicated internal operation of the thermostatic switch itself — or, for that matter, similarly to ask about the 'switching transient' involved in current flow to the heater element, rise time for its heating, etc.

From this point of view, the nominal description of the situation is a 'reduced order model', very much in the sense of the 'outer solution' of singular perturbation theory. Since the transitions between discrete values of logical variables are essential to the nature of hybrid systems, we might expect some likelihood that these considerations could become relevant for hybrid systems in appropriate contexts. In particular, we concentrate our attention in this note to the setting of 'chattering modes', for which switching is intentionally frequent so it is plausible to anticipate significant cumulative effects of the individually negligible switching transients, etc. As we shall see, there is then some possibility of (perhaps unpleasant) surprises if these are ignored.

The key to our analysis is the consideration of *time scales.* Suppose we are faced with a situation in which, on the 'natural' time scale, we have frequent switching between several available elementary modes

$$\dot{\xi} = F_1 \qquad \text{or} \qquad \dot{\xi} = F_2 \qquad \text{or} \qquad \dots \tag{1}$$

(These are, of course, vector ODEs in some relevant state space $\mathcal{X}$.) We will refer to the composite zig-zag dynamics as a *chattering mode* and we seek a simplified (averaged) description, called the *sliding mode*, which provides an acceptable approximation for the chattering mode. It is even more likely that the simplified sliding mode represents the 'intention' under consideration at the level of control design and it is the chattering mode which is to be considered as an (implementable) acceptable approximation (cf., e.g., [9] or [5]) at the natural (design) time scale.

2. An easy analysis shows that in the case of rapid switching, solutions of (1) can be well approximated by considering

$$\dot{\xi} = \hat{F} := \Sigma_j \alpha_j F_j \tag{2}$$

where the coefficients α_j of this convex combination are the (local) *fractions of time spent* in each of the modes of (1) — provided there is some intermediate time scale for which these fractions are suitably definable yet short enough to take each F_j as approximately constant. Our point is that this relation of (2) to (1) necessarily comes from some specific *implementation.* If — e.g., as in [5] — the chattering mode would be explicitly constructed to provide explicitly specified time fractions α_j, then this relation is clear. In realistic cases such an (open loop) explicit construction may well be a burden and the control design may provide for an *implicit* determination (closed loop) of the switching times and so of the coefficients. All the usual arguments for the preference of closed loop over open loop control design applies to this point.

It is specifically in these situations that there are possible traps for the unwary in the determination of the correct sliding mode (2) to provide an acceptable approximation of reality — or, conversely, how one might design an implicit control structure providing an appropriate chattering mode.

The simplest analysis corresponds to a bimodal control specification:

$$(\mathbf{C0}) \qquad \begin{cases} \text{IF } x > 0, & \text{THEN: } \dot{\xi} = [u_1, v_1] \\ \text{ELSE (IF } x > 0), & \text{THEN: } \dot{\xi} = [u_2, v_2] \end{cases}$$

where we assume $u_1 < 0 < u_2$ and have written $\xi = [x, z] \in \mathcal{X}$ so x is one 'coordinate' (with z complementary in $\mathcal{X}$) or, more generally, is a sensor value. The surface $x = 0$ is the switching surface and we alternate between these modes — giving (1) with $F_j = [u_j, v_j]$. The 'inwardness condition' $u_1 < 0 < u_2$ ensures that we must zigzag across $x = 0$ so, as averaged in (2), we must have $x \equiv 0$ so $\dot{x} = 0$, requiring $\alpha_1 u_1 + \alpha_2 u_2 = 0$: in this situation the sliding mode is uniquely determined from **(C0)** with no further analysis needed.

At this point we note, as a warning, anecdotal evidence [6] that possible 'traps' may arise: a real apparatus was constructed corresponding to a 'control law' of the form **(C0)** with scalar z (2-dimensional dynamics) in which v_1, v_2 had the same sign yet the physically observed motion along $x = 0$ paradoxically went in the opposite direction.

To see how such an apparent paradox might occur and might be explained by details of the implementation which were neglected in the apparently complete description above, we consider a more complicated situation involving four (constant) fields in $\mathcal{X} = \mathbb{R}^3$:

$$F_1 \equiv \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \qquad F_2 \equiv \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \qquad F_3 \equiv \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \qquad F_4 \equiv \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}.$$

Our control intention is to move along the z-axis ($x = y = 0$) in the positive direction. To this end we employ the control specification:

$$(\mathbf{C1}) \quad \begin{cases} \text{IF } x > 0, \text{ THEN:} & (\mathbf{C2}) \begin{cases} \text{IF } y > 0, \text{ THEN: } \dot{\xi} = F_1 \\ \text{IF } y < 0, \text{ THEN: } \dot{\xi} = F_4 \end{cases} \\ \text{ELSE } (x < 0): & (\mathbf{C3}) \begin{cases} \text{IF } y > 0, \text{ THEN: } \dot{\xi} = F_2 \\ \text{IF } y < 0, \text{ THEN: } \dot{\xi} = F_3 \end{cases} \end{cases}$$

We can apply the same analysis as for **(C0)** to simplify **(C2)** and **(C3)** to obtain sliding modes in the plane $y = 0$, noting that we arrive at this plane for each of the alternatives ($x > 0$, $x < 0$). For $x > 0$, the condition that the y-component of the sliding mode must vanish implies coefficients $\alpha_1 = 2/3$, $\alpha_4 = 1/3$ for the convex combination of F_1, F_4 and we obtain similarly the coefficients

$\alpha_2 = 1/3$, $\alpha_3 = 2/3$ for the convex combination of F_2, F_3 when $x < 0$. Thus we have the sliding modes

$$\hat{F}_2 = \begin{bmatrix} -5/3 \\ 0 \\ 2 \end{bmatrix}, \qquad \hat{F}_3 = \begin{bmatrix} 5/3 \\ 0 \\ 2 \end{bmatrix},$$

respectively, for these alternatives. Inserting these sliding modes in **(C1)** , we obtain the simplified control specification in the plane $y = 0$:

$$(\mathbf{C1'}) \quad \begin{cases} \text{IF } x > 0, \text{ THEN:} & (\mathbf{C2'}) \;\; \dot{\xi} \approx \hat{F}_2 \\ \text{ELSE } (x < 0): & (\mathbf{C3'}) \;\; \dot{\xi} \approx \hat{F}_3 \end{cases}$$

In this case a repetition of the same analysis now requires that the x-component of the convex combination of $\hat{F}_2$, $\hat{F}_3$ should vanish, giving coefficients $\hat{a}_2 = \hat{a}_3 = 1/2$. The resulting sliding mode gives the desired motion along the z-axis with velocity $+2$. So far, so good.

At this point we note that the control specification:

$$(\mathbf{C4}) \quad \begin{cases} \text{IF } y > 0, \text{ THEN:} & (\mathbf{C5}) \;\; \dot{\xi} = \{F_1 \text{ IF } x > 0; \text{ELSE } F_2\} \\ \text{ELSE } (y < 0): & (\mathbf{C6}) \;\; \dot{\xi} = \{F_4 \text{ IF } x > 0; \text{ELSE } F_3\} \end{cases}$$

is logically equivalent to **(C1)** — each says, in a slightly different way, that one is to use F_j when in the j^{th} quadrant of the x, y-plane. Now. however, applying the same method of analysis[3] to **(C4)** as was previously applied to **(C1)** now gives the sliding modes

$$\hat{F}_6 = \begin{bmatrix} 0 \\ -5/3 \\ -1 \end{bmatrix}, \qquad \hat{F}_6 = \begin{bmatrix} 0 \\ 5/3 \\ -1 \end{bmatrix},$$

in the plane $x = 0$ for $y > 0$ and $y < 0$, respectively, to give the simplified control specification in the plane $x = 0$:

$$(\mathbf{C4'}) \quad \begin{cases} \text{IF } y > 0, \text{ THEN:} & (\mathbf{C5'}) \;\; \dot{\xi} \approx \hat{F}_5 \\ \text{ELSE } (y < 0): & (\mathbf{C6'}) \;\; \dot{\xi} \approx \hat{F}_6 \end{cases}$$

and so to imply a resulting motion along the z-axis with velocity -1, i.e., in the direction opposite to what had been obtained earlier. If the sliding mode computed as for **(C1)** were in fact correct (so the motion actually occurring had $v = 2$ along the z-axis), then **(C4')** would present a paradoxical behavior within the plane $y = 0$, much as for the 'experimental' situation of [6].

[3] It is also interesting to note that the plane $x = 2y$ is invariant if one uses only F_1 and F_3 and, again switching at $x = y = 0$, the analysis then gives motion along the z-axis with velocity $+5$.

3. Our principal concern in this section will be the resolution of the different results associated with **(C1)** and **(C4)** despite the apparent logical equivalence. The real understanding of **(C0)** — and so of **(C1)** and **(C4)** — comes from a somewhat more detailed consideration of the actual implementation which, for such a control specification as: "IF $x > 0$, THEN $\dot{\xi} = F_1$," would involve both a *sensor* X, tracking x and an *actuator* A producing F_1.

At this point we re-emphasize the intended significance of the sliding mode as providing a more easily computable formulation which is expected to provide a realistic approximation of satisfactory accuracy to 'what would, in fact, occur' when the control program is implemented in the real world or, conversely if one begins with an sliding mode at the design stage, the responsibility to ensure that the implementation will (approximately) produce what is intended when the ambiguities inherent in the discussion above show that this may not be 'automatic'.

Even apart from consideration of possible time sampling and/or quantization effects for X, the simplest version of the switching process, as implemented, will still have some delay δ_1' in the actual sensing of the state condition: if $x = 0$ at time t_0, then the controller will actually have the established state $(x > 0)$ at a time $t_1 = t_0 + \delta_1'$ (with $x = \varepsilon_1' > 0$) at which the actuator A is nominally set to F_1. One will then have some switching transient for the actuator corresponding to an evolution $(t = t_1 + \tau)$

$$\dot{\xi} = F_{21}^*(\tau) \qquad (0 < \tau < \delta_1'') \tag{3}$$

with $F_{21}^* = F_2$ at $\tau = 0$ and $F_{21}^* \approx F_1$ after the further delay δ_1''; in a multimodal setting the switching transient and δ_1'' will obviously depend on the mode from which one is switching. Thus, $\delta_{21} = \delta_1' + \delta_1''$ represents the time scale for switching from F_2 to F_1, etc.

Provided[4] δ_1', δ_2' are taken so $\delta_j'' \ll \delta_j'$ (making (1) a plausible approximation), we see that the total period Δ_{21} for which the control state is $(x > 0)$ — i.e., the interval from t_1 when this becomes the state to t_0^* when again $x = 0$ to $t_1^* = t_0^* + \delta_2'$ when the state has been switched to $(x < 0)$ — and the corresponding period Δ_{12} will each be of the same order of magnitude as δ_{21}; in this case we get (2) with coefficients $[\alpha_1, \alpha_2] = [\Delta_{21}/\Delta, \Delta_{12}/\Delta]$ (here $\Delta = \Delta_{21} + \Delta_{12}$ is the total 'cycle time') as well as the approximate relation $u_1\Delta_{21} + u_2\Delta_{12} \approx 0$ — leading to the same results as for the original (less detailed) analysis but with a more refined understanding of the justifying assumptions.

For the analysis of **(C1)** we observe from our most recent discussion that the approximate reduction to **(C1')** is justifiable only if the time scales for

[4]With no such assumption it would be perfectly possible that by $t_2 = t_1 + \delta_1''$ one would already have $x < 0$ so one would abide in eternal transiency without the nominal description (1) ever becoming even approximately true. For simplicity we assume that — as is often done in practice for related reasons — δ_1' is artificially increased, as necessary. An analysis without this assumption would certainly be possible but could be expected to be significantly more complicated.

the separate consideration of **(C2)** and **(C3)** are each quite rapid compared to the alternation between them, i.e., if $\delta_{14}, \delta_{23} \ll \hat{\delta}_{23}$ where it is not difficult to see that $\hat{\delta}_{23}$ is comparable to δ_{12}, δ_{34}. Effectively, this analysis is justifiable provided the sensor Y were extremely fast compared to the sensor X and, conversely, the analysis given for **(C4)** would be justifiable provided the sensor X were extremely fast compared to the sensor Y. These are *implementation assumptions* and it is the distinction between them which resolves the paradox; compare the discussion in [7].

In this context we may understand the title of this paper as suggesting that the actual result of implementing such a control fragment as **(C1)** is impossible without taking into account some features of the implementation, some residue of reality which must be retained in the model reduction process as providing a 'selection principle'. The nature of this 'residue' is clear when, as just indicated, there might be a 'time scale separation' for the effects of the two switching surfaces. When this simplifying assumption is inapplicable — say, if X, Y are of comparable speed then the determination of a suitable sliding mode becomes much more problematic. Some partial analysis is presented in [7], but this very much remains work-in-progress.

We also note that another type of implementation may be plausible for this nominal context of four fields used in the quadrants defined by two intersecting switching surfaces. In some applications one might plausibly have a 'blending' of the fields F_j — corresponding to a 'fuzzy logic' interpretation of the state conjunctions. This is analyzed in [1], where it is shown that there is a computable sliding mode without any restriction, as above, that there be a time scale separation for the sensors. We note also the analyses of **(C1)** noted in [2] for a stochastic and for a delay interpretation.

4. We conclude with an observation that, apart from the correctness of the sliding mode approximation to the hybrid dynamics, there is a cumulative effect of the switching transients (3) — more precisely, of the distinction between these and the nominal (1) — to be taken into account for, e.g., consideration of total costs for optimization. E.g., for a thermostatically controlled gas furnace these 'switching costs' include both the control effect of 'rise time' delay and the 'waste' of gas at ignition of the gas flame. Note that these are ignored in the nominal (reduced) description which treats the situation as jumping instantly to the new mode as if in steady operation. The switching costs are individually small, but one must question the justification for their total neglect in a context of frequent switching.

Let us consider, for example, a control optimization problem with a cost functional of the form

$$\mathcal{J}_0 = \int_0^\infty e^{-\lambda t}\varphi(x,u)\,dt \tag{4}$$

so φ represents running costs as a function of (continuous) state x and control u. We are thinking of a setting in which this is minimized by some x_* corresponding to a 'relaxed control' u_*, expressible in the form $\Sigma_j \alpha_j u_j$ in terms of 'accessible' modes u_j so

$$\mathcal{J}_0^* = \int_0^\infty e^{-\lambda t}\Sigma_j \alpha_j \varphi(x_*,u_j)\,dt \tag{5}$$

and, following [5], one would approximate by a chattering mode in which (with suitable proportions) one cycles through the modes $\{u_j\}$ with cycle time Δ. Note that this gives the controlled trajectory $\hat{x}$ with, approximately, $\hat{x} \approx x_* + \Delta \cdot \xi$ for an appropriate (highly oscillatory) 'perturbation function' ξ. It is not too difficult to see that the resulting perturbation of $\mathcal{J}_0$ will be quadratic: one expects this to be 0 to first order here — even if, due to the control constraints, one would not have vanishing of the first order variation of $\mathcal{J}_0$. Thus, a control specification of this nature would give

$$\mathcal{J}_0 = \mathcal{J}_0^* + a\Delta^2 \tag{6}$$

with a computable coefficient a so long as we continue to ignore the switching costs $\mathcal{J}_1$. Clearly we can minimize (6) by taking Δ to be as small as possible, consistent with feasibility.

We now consider the total cost $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$ — i.e., including consideration of the switching cost — with, say,

$$\mathcal{J}_1 = \sum\nolimits_\nu e^{-\lambda t_\nu}\varepsilon\psi\left(x(t_\nu); j_\nu \hookleftarrow j_{\nu-1}\right) \tag{7}$$

where the sum is taken over all the switching times t_ν and $\varepsilon\psi$ is the associated cost of a single mode transition. Keeping ψ as 'order of 1', we have introduced $\varepsilon \ll 1$ here to indicate that this switching cost must be small or we could not reasonably be using the kind of control strategy we are describing here. Now suppose, for convenience, we chatter with round robin rotation of J modes ($1,\ldots,J$, in cyclic order) and cycle time Δ. Taking the leading term with respect to Δ, we would then have

$$\mathcal{J}_1 \approx (\varepsilon/\Delta)\int_0^\infty e^{-\lambda t}\left[\Sigma_r \psi_r(x)\right]\,dt \tag{8}$$

where we have set $\psi_r(x) := \psi(x; r \hookleftarrow r-1)$. Combining this with what we obtained just above in (6), one sees that the optimal choice of the cycle time can be simply computed. The total cost now takes the form

$$\mathcal{J} = \mathcal{J}_0^* + a\Delta^2 + b\varepsilon/\Delta \tag{9}$$

and this can be minimized with respect to Δ by elementary Calculus to obtain

$$\Delta_{opt} = [C\varepsilon]^{1/3} \quad \text{with } C = \frac{1}{2a}\int_0^\infty e^{-\lambda t}\,[\Sigma_r \psi_r(x)]\; dt. \tag{10}$$

[Thus, the optimal cycling frequency is $\mathcal{O}([\text{unit switching cost}]^{-1/3})$.] Note that in [5] the parameter Δ may itself be viewed as a control variable and we have here shown how its optimization is related to the switching cost when that is to be taken into account.

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