Modeling and Analysis of Modal Switching in Networked Transport Systems

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Abstract. We consider networked transport systems defined on directed graphs: the dynamics on the edges correspond to solutions of transport equations with space dimension one. In addition to the graph setting, a major consideration is the introduction and propagation of discontinuities in the solutions when the system may discontinuously switch modes, naturally or as a hybrid control. This kind of switching has been extensively studied for ordinary differential equations, but not much so far for systems governed by partial differential equations. In particular, we give well-posedness results for switching as a control, both in finite horizon open loop operation and as feedback based on sensor measurements in the system.

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1 Introduction

Despite the aphorism "Nature does not make jumps", it is frequently useful to work, either prescriptively or descriptively, with simplified models which involve switching instantaneously between different modes of evolution. We wish to understand these hybrid systems simultaneously as a design paradigm and as approximating multiscale problems in which the implementation of switching is merely on a faster scale than what is being considered. In particular, we aim at applications from civil engineering, where pumps, valves and other control elements in networked transport systems are to be operated: in view of the macroscopic scale of the continuous dynamics involved, we consider the switching of modes, here representing the operations, as effectively instantaneous. Similar multiscale problems also arise in various communication, information and logistic areas.

In the present context we are primarily concerned with the use of this *modal switching* for control design in the context of material flow governed by the well-known semilinear reaction/transport equation

$$\partial_t u + \partial_s [au] = f(u), \tag{1}$$

embedded into a graph setting where, for example, modal switching might mean opening or closing a valve at one of the nodes. While we view this switching as instantaneous here, we do recognize that in a more detailed modeling it is simply a process taking place on a more rapid time scale than we wish to consider. This is not so obvious in the context of control design, but even modal switching which is designed as instantaneous must somehow be implemented in the real world. A successful switching model is then one where the precise mechanism of these fast dynamics does not cause any unexpected surprises in what we do consider. It is just this concern for what might be happening on the fast time scale — and its implications, especially in the context of feedback control with point sensors — which leads us to the rather technical treatment in Definitions 1, 2. We refer to Remark 2 as well as, e. g., to [19,20] for further elaboration of these modeling considerations.

Thus, we will have a finite set $\mathcal{M} \simeq \{1, \dots, M\}$ of *modes* in which the system may evolve with the expectation of switching between these. For the present we simply consider $\mathcal{M}$ as a set, but note that it plausibly could be given the additional structure of a directed graph, limiting the permissible transitions. So we will have a *discrete state component* $\mu(t) \in \mathcal{M}$ indicating the current mode and describe here, following ideas of [17], the state space we will work with for $\mu(\cdot)$.

Definition 1. *A* switching function $\mu(\cdot)$ *on* $[0, T]$ *is constructed by specifying a finite sequence of* switching times *in*

$$\mathcal{A}_K = \{(\tau_0, \dots, \tau_K) \in \mathbb{R}^{K+1} : 0 = \tau_0 \le \tau_1 \le \cdots \le \tau_K = T\}$$

with an assignment of a mode $\mu_k \in \mathcal{M}$ *to each* interswitching interval $[\tau_{k-1}, \tau_k]$. *The set of all such switching functions will be denoted by* $\mathscr{M}_{\mathrm{pw}}[0, T]$. *A sequence of switching functions* $\{\mu^\nu(\cdot)\}$ *converges to* μ^∞ *if* K^ν *and each* μ_k^ν *are ultimately constant and* $\tau_k^\nu \to \tau_k^\infty$.

Remark 1. In the above definition we abuse notation slightly by assigning modes even to degenerate 0-length interswitching intervals and by permitting several switching times to coalesce while carefully preserving the sequence order of these. Consistently with our concern for the modeling interpretation, we view these values as a residue of genuine intervals on a faster time scale than we are choosing to model. These instantaneous modes cannot affect the dynamics directly, but nevertheless must be retained for our purposes. □

We have defined $\mathscr{M}_{\mathrm{pw}}[0, T]$ and its topology precisely to obtain the following compactness result.

Lemma 1. *The subspace $\mathscr{M}_{\mathrm{pw}}^{K}[0,T] \subset \mathscr{M}_{\mathrm{pw}}[0,T]$ corresponding to a bound on K is sequentially compact.*

Proof. Let μ^{ν} be a sequence in $\mathscr{M}_{\mathrm{pw}}^{K}[a,b]$. Then, with a bound on the number of switching points, we may extract a subsequence (re-indexed by ν), such that $K^{\nu} = K^{\nu'} = \tilde{K} \le K$. With $\tilde{K}$ fixed, a switching function is equivalent to a point in the compact set $\mathcal{A}_{\tilde{K}} \times \mathcal{M}^{\tilde{K}}$. □

As a paradigm for transport problems involving modal switching, we are considering material flow governed by (1) together with boundary and initial conditions $u|_{s=0} = \varphi(t)$, $u|_{t=0} = \bar{u}$; considering (1) on a network, we must also treat the coupling at the nodes. As suggested earlier, the modes considered are often distinguished primarily by alternative nodal couplings (allocations of incoming flux to outgoing edges), although we also include changes in the flow velocities or reaction rates as possible modal transitions. Our goal, then, is to study the effect of such discontinuous modal switching and how this is interacts with the graph geometry of the transport, especially when the switching is used as a feedback control.

While well-posedness and asymptotic behavior of similar systems on graphs without switching modes have been considered in [10,15] using semigroup theory and optimal control of networked transport systems have been considered in [8,9,13,5,12] taking $\omega_{ij}(t)$ or $\varphi_i(t)$ as a control, we will here consider the switching function $\mu(\cdot)$ as a control of the system. We note that working with such discrete-continuous nature of systems governed by *ordinary differential equations* (ODEs) is a rapidly developing area; however, similar systems involving *partial differential equations* (PDEs) have seldom been considered in the literature so far, although noting [3]. For readers not familiar with hybrid ODE systems, we refer to, e. g., [16] or [2] for an introduction.

For the graph setting, we use a notation similar to the one introduced in [10]. We suppose we have a directed graph $G = (V, E)$ with vertices $V = \{v_1, \dots, v_n\}$ and directed edges $E = \{e_1, \dots, e_m\}$, each normalized and identified as $e_j \simeq [0,1]$ $(j = 1, \dots, m)$. The distribution of material along an edge e_j at time $t \ge 0$ is described by the unknown function $u_j(t,s)$ for $s \in [0,1]$. The dynamics on the graph are governed by PDEs like (1) that are coupled at the nodes, together with a switching function $\mu(t)$ as to switch certain properties of the system as desired. The full governing equations are then

$$\begin{cases} \dfrac{\partial}{\partial t} u_j(t,s) + \dfrac{\partial}{\partial s}\Big[a_j^{\mu(t)}(t,s) u_j(t,s)\Big] = f_j^{\mu(t)}(t,s,u_j(t,s)), \;\; s \in (0,1), \;\; t \ge 0 \\ \phi_{ij}^{-} \left(a_j^{\mu(t)} u_j\right)\Big|_{(t,0)} = \omega_{ij}^{\mu(t)}(t) \left(\displaystyle\sum_{k=1}^{m} \phi_{ik}^{+} \left(a_k^{\mu(t)} u_k\right)\Big|_{(t,1)} + \varphi_i^{\mu(t)}(t) \right), \;\; t \ge 0 \\ u_j(0,s) = \bar{u}_j(s), \quad s \in (0,1) \\ \mu(t) \in \mathcal{M} \simeq \{1, \dots, M\}, \quad t \ge 0 \end{cases} \tag{2}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$, where for each fixed $\mu \in \mathcal{M}$

- $a_j^\mu(t,s) > 0$ is the transport velocity of the flow along the edge e_j [Note that, for convenience, we have taken all $a_j^\mu > 0$ so flow is consistently 'from left to right' on each edge.]
- $f_j^\mu(t,s,u_j(t,s))$ is a source/reaction function along the edge e_j
- $(\phi_{ij}^-)_{n\times m}$ and $(\phi_{ij}^+)_{n\times m}$ are incidence matrices for outgoing, respectively incoming, edges e_j at vertex v_i with coefficients

$$\phi_{ij}^- := \begin{cases} 1, & v_i = e_j(0) \\ 0, & \text{otherwise} \end{cases} \qquad \phi_{ij}^+ := \begin{cases} 1, & v_i = e_j(1) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

- $\omega_{ij}^\mu(t)$ are functions expressing the proportion of mass routed at vertex v_i into edge e_j; these satisfy

$$0 \le \omega_{ij}^\mu(t) \le 1 \quad \text{and} \quad \omega_{ij}^\mu(t) = 0 \text{ if } \phi_{ij}^- = 0, \quad t \ge 0 \tag{4}$$

  and we also ask that

$$\sum_{j=1}^m \omega_{ij}^\mu(t) = 1 \quad \text{for all } t \ge 0 \tag{5}$$

- $\varphi_i^\mu(t)$ is a source/drain function at the vertex v_i
- $\bar{u}_j(s)$ is the initial distribution of material along the edge e_j.

We define the outgoing and incoming edge degree of a vertex

$$d^-(v_i) = \sum_{j=1}^n \phi_{ij}^- \qquad d^+(v_i) = \sum_{j=1}^n \phi_{ij}^+ \tag{6}$$

and interpret the system (2) as routing material from the input vertices $\mathcal{I} = \{v_i \,|\, d^+(v_i) = 0 \text{ or } \varphi_i^\mu \ne 0\}$ to the output vertices $\mathcal{O} = \{v_i \,|\, d^-(v_i) = 0\}$ through a network while the system's parameters switch in time. Indeed, at any time $t \ge 0$, condition (5) together with $(2)_2$ implies *Kirchhoff's Law*

$$\sum_{j=1}^m \phi_{ij}^- \left(a_j^{\mu(t)} u_j \right)\Big|_{(t,0)} = \sum_{j=1}^m \phi_{ij}^+ \left(a_j^{\mu(t)} u_j \right)\Big|_{(t,1)} + \varphi_i^{\mu(t)}(t) \tag{7}$$

for each $i = 1, \dots, n$, so we have conservation of flow along the edges and at the vertices in regard to its respective sources f_j^μ or φ_i^μ.

We will be interpreting (2) using the classical *method of characteristics* (cf., e. g. [6, ch.II], [7, ch.VII]), thus considering *generalized solutions* which need not be pointwise differentiable or even continuous except along the characteristics. Both in interpreting the nodal coupling $(2)_2$ and for our results on feedback switching control for which we wish to be able to work with point sensors, we must be strongly concerned with the regularity in time of point evaluations. However (cf., Remark 2), we must accept the introduction and propagation of discontinuities in the solution and so work with a space of piecewise continuous functions. Our interest in feedback control, with state-dependent switching rules, i. e., viewing $\mathcal{M}$ as a state-dependent directed graph, will then require our definition to be similar to Definition 1 in retaining certain aspects of the fine structure so this again will be a nonstandard space.

Definition 2. *A* piecewise continuous function $g(\cdot)$ *on* $[a,b]$ *is constructed by specifying finitely many* partition points $a = p_0 \le p_1 \le \cdots \le p_K = b$ *and, on each partition subinterval, assigning a continuous function* $g_k \in \mathscr{C}[0,1]$*. For degenerate* 0*-length subintervals, we impose the restriction that the assigned* g_k *must be constant. As a function on* $[a,b]$*, we have*

$$g(s) = g_k\left(\frac{s-p_{k-1}}{p_k-p_{k-1}}\right) \quad \text{for } s \in (p_{k-1}, p_k)$$

and $g(s) = [g_{k-1}(1), g_k(0)]$ *if* s *is one of the partition points* p_k*. We may even have* $g(s) = [g_{k-1}(1), g_k, \dots, g_{k+l-1}, g_{k+l}(0)]$ *at coalesced partition points* $p_k = \cdots = p_{k+l}$*; note that the multiple assignment at such a point is not just a set, but preserves the sequence order. The set of all such piecewise continuous functions will be denoted by* $\mathscr{C}_{\rm pw}[a,b]$*. A sequence of such functions* $\{g^\nu(\cdot)\}$ *converges to* $g^\infty \in \mathscr{C}_{\rm pw}[a,b]$ *if* K^ν *is ultimately constant with* $p_k^\nu \to p_k^\infty$ *and each* $g_k^\nu(\cdot) \to g_k^\infty$ *uniformly on* $[0,1]$*. Piecewise continuous functions on the graph will then be in* $\mathscr{C}_{\rm pw}(G) \simeq (\mathscr{C}_{\rm pw}[0,1])^m$*.*

For a given $\mu(\cdot)$, we see that generalized solutions of (2), as given by the method of characteristics, for piecewise continuous data, will lie in the space $\mathscr{C}_{\rm pw}\left([0,T] \to \mathscr{C}_{\rm pw}(G)\right)$ of piecewise continuous functions from $[0,T]$ into $\mathscr{C}_{\rm pw}(G)$.

Remark 2. As in the comment following Definition 1, we have again abused notation slightly by admitting degenerate 0-length partition subintervals and permitting several partition points to coalesce while continuing to assign functions separately to each of the infinitesimal subintervals. We will provide some motivation for this definition and, in particular, comment on the interpretation of the coupling given by $(2)_2$ for solutions of such regularity.

While we defer to the next section a detailed discussion of our use of the method of characteristics, we first note a couple of examples indicating how modal switching, especially in the context of a graph geometry, can introduce discontinuities which then propagate.

> *Example 1.* Consider a 1-edge graph with (1) taken simply as $u_t + u_s = 0$ and with constant initial data $u \equiv 1$. In the first mode we use $u|_{s=0} \equiv 1$, so the solution remains $u \equiv 1$, but at the switching time $t = \tau$ the new mode keeps the equation fixed but now uses input data $u|_{s=0} \equiv 2$. Then a jump discontinuity is introduced at the input node and propagates (along the characteristic $s = t - \tau$) into the edge.
> For comparison, consider a simple 2-edge graph with e_1 feeding into e_2 through the central node ν. We again take constant initial data $u \equiv 1$ for both edges and constant input data $u \equiv 1$ into e_1. For the first mode we

use the equation $u_t + u_s = 0$ in both edges and then switch at the time $t = \tau_1$ to a new mode with the same equation on e_2, but now with the equation $u_t + 2u_s = 0$ on e_1. One easily sees that the solution remains $u \equiv 1$ on e_1, but the flow velocity there has become 2 so the flux into the node has switched from 1 to 2 at $t = \tau$. By Kirchhoff's Law the input flux to e_2 must then be 2 corresponding to $u|_{s=0} \equiv 2$ for e_2 after τ — giving the same jump discontinuity introduced at the node (and propagating into e_2 along the characteristic $s = t - \tau$) as before.
Except for discontinuities present in the initial data, we see from this that new discontinuities will (only) be introduced at nodes at switching times and will then propagate along characteristics: compare [14].

While not true for general elements of $\mathscr{C}_{\rm pw}\left([0,T] \to \mathscr{C}_{\rm pw}(G)\right)$, we easily see that point evaluation of a solution (or the corresponding flux) will produce a function in $\mathscr{C}_{\rm pw}([0,T])$ since the discontinuities propagate only along characteristics. Some comment is needed, however, about how to combine (sum) several of these (specifically coming from evaluations at the endpoints of edges incoming to a single node) as is required for our interpretation of $(2)_2$. When there are no repetitions, the set of partition points for the sum will be the union of the sets for the summands with the sums obtained as usual on the resulting subintervals. One must be careful, however, when there is a repetition.

Example 2. Consider a Y graph with e_1, e_2 directed toward the central node and e_3 directed out; we take $u_t + u_s = 0$ on each edge so, using Kirchhoff's Law, we have $u_3(t,s) = u_1(0, s-t+1) + u_2(0, s-t+1)$ for $0 < s < 1,\ 0 < s-t+1 < 1$. As initial data we take $\bar{u}_1 = H_\epsilon$, $\bar{u}_2 = 1 - H_0$ with $H_\epsilon(s) = \{0 \text{ for } s < 1/2 + \epsilon;\ 1 \text{ for } s > 1/2 + \epsilon\}$ so at $t = 1$ we have $u_3(1,\cdot) = H_\epsilon - H_0 + 1$. For small $\epsilon > 0$ one then has $u_3(1,\cdot) \equiv 1$ except on the infinitesimally small interval $(1/2,\ 1/2 + \epsilon)$ where $u_3(1,\cdot) = 2$. On the other hand, for $\epsilon < 0$ one has $u_3(1,\cdot) \equiv 1$ except on $(1/2 + \epsilon,\ 1/2)$ where, now, $u_3(1,\cdot) = 0$. Taking the limit as $\epsilon \to 0$ for the initial data, without regard for sign, we obtain two limit solutions distinguished by the retention of either 2 or 0 as assigned to the degenerate interval corresponding to treating ϵ alternatively as a positive or negative signed infinitesimal: once the fine scale order is fixed, the sum is clear.

Note that these solutions are distinct, representing information about behavior on an unmodeled fast time scale in looking at the race between e_1 and e_2 as to which discontinuity will first arrive at the node, indeterminate at the level of our modeling. While the distinction is of no direct significance for the transport dynamics, it may be of later consequence for the switching dynamics of feedback, so we must retain both possibilities — accepting as a consequence that the solution would no longer be unique. Such indeterminate races, at worst, lead to uncertainty among a finite number of alternative solutions differing at each t only on a finite set of points of the graph — so we consider the solution to be 'almost unique': the relevant notion of well-posedness is then upper semicontinuity of the solution sets as data is varied. We emphasize that, while our Definition 2 was largely motivated by consideration of modal switching, the nonuniqueness which we have just observed occurs due to the nodal coupling rather than to switching.

We may, of course, encounter still more complicated situations: one could have more than two incoming discontinuities racing to a tie as above along several incoming edges, but there is also the more interesting possibility that one or more of the incoming discontinuities is already multiple with degenerate partition subintervals. If, for example, each of the discontinuity points at 1/2 in the Example 2 were double ($a = b = 1/2$ for e_1 and $A = B = 1/2$ for e_2, with corresponding constant functions assigned to the infinitesimal intervals $[a, b]$ and $[A, B]$), then there would be 6 resulting possibilities (each leading to a quadruple degeneracy with 3 assigned constants) corresponding to the alternative orderings $\{abAB,\ aAbB,\ aABb,\ AabB,\ AaBb,\ ABab\}$ on the fast time scale; we accept each of these as an admissible alternative in accepting multiple solutions. The general situation would be extremely cumbersome to specify in formal detail, but goes the same way. □

With Definition 2 at hand, we later refer, for any time $t \geq 0$, to the *continuous state component* of the system (2) as the vector

$$\mathbf{u}(t) := (u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in \mathscr{C}_{\mathrm{pw}}(G), \tag{8}$$

in opposite to the discrete state component $\mu(t) \in \mathcal{M}$ introduced earlier. Thus the hybrid pair

$$\mathbf{x}(t) := [\mathbf{u}(t), \mu(t)] \tag{9}$$

will be the *full state* of the system.

We will make the following assumptions throughout the paper:

(A1) For every $\mu \in \mathcal{M}$ and $j = 1, \ldots, m$, the given functions $a_j^\mu(\cdot, \cdot)$ are continuous in t and continuously differentiable in s and there are bounds $0 < \underline{a} < \bar{a}$ such that $\underline{a} \leq a_j^\mu(\cdot, \cdot) \leq \bar{a}$. [Thus, by compactness, there exists a Lipschitz-constant $\bar{\bar{a}}$ such that $|a_j^\mu(\cdot, s_1) - a_j^\mu(\cdot, s_2)| \leq \bar{\bar{a}}|s_1 - s_2|$ for all $s_1, s_2 \in [0, 1]$.]

(A2) For every $\mu \in \mathcal{M}$ and $j = 1, \ldots, m$, the given functions $f_j^\mu(\cdot, \cdot, \cdot)$ are continuous; there is a constant $\bar{b} > 0$ such that $|f_j^\mu(\cdot, \cdot, u)| \leq \bar{b}(1+u)$ and there exists a Lipschitz-constant $\bar{\bar{b}}$ such that $|f_j^\mu(\cdot, \cdot, u_1) - f_j^\mu(\cdot, \cdot, u_2)| \leq \bar{\bar{b}}|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$.

(A3) For every $\mu \in \mathcal{M}$ and $(i = 1, \ldots, n,\ j = 1, \ldots, m)$, the given functions $\omega_{ij}^\mu(\cdot)$ are continuous and satisfy (4), (5).

(A4) For every $\mu \in \mathcal{M}$ and $i = 1, \ldots, n$, the given functions $\varphi_i^\mu(\cdot)$ are continuous.

The paper will be organized as follows. In Section 2 we will consider the system (2) with $\mu(\cdot)$ specified as data, i. e., we will show that we have an 'almost unique' solution $\mathbf{u}(\cdot)$ in the sense described in Remark 2, and that the solution set is upper semicontinuous in its dependence on the data. In Section 3, we will consider the switching function $\mu(\cdot)$ not as given a priori, but as the argument of an optimization problem

$$\begin{cases} \min_{\mu(\cdot)\in\mathscr{M}_{\mathrm{pw}}[0,T]} & \mathcal{J}[\mu(\cdot),\mathbf{u}(\cdot)] \\ \quad \text{such that} & \mathbf{u}(\cdot) \text{ solves (2) with } \mu(0)=\bar{\mu} \end{cases} \tag{10}$$

for a fairly general cost function $\mathcal{J}[\cdot]$ including switching costs. In Section 4, we complement the system by an internal feedback law at each t

$$[\mu,\mathbf{y}] \mapsto \left\{ \begin{array}{c} \textit{stay with mode } \mu \textit{ unchanged} \\ \text{or} \\ \textit{switch (immediately) to mode } \mu' \end{array} \right\} \tag{11}$$

specifying the discrete state component while the system evolves in time. The feedback switching structure we consider, with $\mathbf{y}(\cdot)$ given by a finite number of point observations in the graph, parallels the structure occurring for optimal switching with full state observation. We finally discuss some extensions of the switching transport model above for applications and conclude with some remarks in Section 5.

2 The Direct Problem: Taking $\mu(\cdot)$ As Data

We begin by considering well-posedness of the system (2) introduced in Section 1 when the discrete state component $\mu(\cdot)$ is specified a priori as data. It is convenient to consider this first in the setting of a graph G consisting of a single edge ($m=1,\ n=2$) for which the (2) is equivalent to the following *initial boundary value problem* (IBVP)

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) + a^{\mu(t)}(t,s)\dfrac{\partial}{\partial s}[u(t,s)] = \tilde{f}^{\mu}(t)(t,s,u(t,s)), \quad s\in(0,1),\ t\geq 0 \\ u(t,0) = \dfrac{\varphi^{\mu(t)}(t)}{a^{\mu(t)}(t,0)}, \quad t\geq 0 \\ u(0,s)=\bar{u}(s), \quad s\in(0,1) \end{cases} \tag{12}$$

with $\tilde{f}^{\mu}(t,s,u(t,s)) := f^{\mu}(t,s,u(t,s)) - \frac{\partial}{\partial s}a^{\mu}(t,s)$ for all $\mu\in\mathcal{M}$. Moreover, using classical methods of partial differential equations, see e. g. [7], we consider characteristic curves, denoted as $\hat{s}_\sigma(\cdot)$, noting that these may change their slope discontinuously at switching times but remain continuous. These characteristic curves are obtained as solutions of the switched ordinary differential equations

$$\frac{d\hat{s}_\sigma(t)}{dt} = a^{\mu(t)}(t,\hat{s}_\sigma), \quad \hat{s}_\sigma(t_*) = s_* \tag{13}$$

with data $(t_*,s_*)_\sigma$ in the *initial gnomon*

$$\mathcal{G} := \{(t,s)\,|\,t\geq 0,\ s=0\} \cup \{(t,s)\,|\,t=0,\ 0\leq s\leq 1\}. \tag{14}$$

[Geometrically, a *gnomon* is the L-shaped piece of a parallelogram remaining when a similar parallelogram is excised from its corner. We are modifying this usage to consider together the *bottom* and *left side* of the infinite rectangle $[0,\infty)\times[0,1]$, visualizing flow as 'left to right' along the edge so this gnomon precisely contains the input data for transport on the edge.]

We parameterize the gnomon homeomorphically by $\sigma\in[0,\infty)$, using

$$(t_*,s_*)_\sigma=\begin{cases}(0,1-\sigma), & \text{if } \sigma<1\\ (\sigma-1,0), & \text{if } \sigma\geq 1.\end{cases}$$

Then, setting $\hat{u}_\sigma(t)=u(t,\hat{s}_\sigma(t))$, we have $\partial_t\hat{u}_\sigma=\partial_t u+a^\mu\partial_s u$ using (13). Thus, (12) becomes a family of switched ordinary differential equations in t, parameterized by σ,

$$\frac{d\hat{u}_\sigma(t)}{dt}=\hat{f}^{\mu(t)}(t,\hat{u}_\sigma(t)),\quad \hat{u}_\sigma(0)=u|_{(t_*,s_*)_\sigma},\quad (t_*,s_*)_\sigma\in\mathcal{G} \tag{15}$$

where $\hat{f}^{\mu(t)}(t,\hat{u}_\sigma(t))=\tilde{f}^{\mu(t)}(t,\hat{s}_\sigma(t),\hat{u}_\sigma(t))$ with $\hat{s}_\sigma(t)$ given by (13) and

$$u|_{(t_*,s_*)_\sigma}=\begin{cases}\bar{u}(1-\sigma), & \sigma<1\\ \varphi^{\mu(\sigma-1)}(\sigma-1), & \sigma\geq 1.\end{cases} \tag{16}$$

We will be assuming a finite set of discontinuities of the data $u|_{(t_*,s_*)_\sigma}$ on the gnomon $\mathcal{G}$ and denote the set of characteristic curves emanating from these by Γ. It is easily seen that solution discontinuities can occur only along the curves of Γ.

With the above method, a generalized solution (i. e., constructed, as noted earlier, by the method of characteristics so not necessarily differentiable and only piecewise continuous) can be obtained for correspondingly suitable regularity of the initial and boundary data, provided the system (13), (15) has a solution for each σ. With the choice of regularity for the data we have the following.

Lemma 2. *Consider the system* (12) *under assumptions (A1), (A2) and (A4) for $m=1$ and $n=2$. Moreover, assume that $\bar{u}(\cdot)\in\mathscr{C}_{\text{pw}}[0,1]$ and $\mu(\cdot)\in\mathscr{M}_{\text{pw}}[0,T]$. Then there exists a unique solution $u(\cdot,\cdot)$ of the IBVP* (12)*. Further, for fixed $s^*\in[0,1]$, $u(\cdot,s^*)\in\mathscr{C}_{\text{pw}}[0,T]$. Finally, for each $t\in[0,T]$, we have continuous dependence of $u(t,\cdot)$ on $\bar{u}(\cdot)$, $\varphi^\mu(\cdot)$ and $\mu(\cdot)$.*

Proof. Assuming (A1), (A2) and $\mu(\cdot)\in\mathscr{M}_{\text{pw}}[0,T]$, the switched ordinary differential equations (13) and (15) have a unique global solution on $[0,T]$. Moreover, the solutions are continuous also at the switching times τ_k of $\mu(\cdot)$ and reversible (cf. e. g. [18]). So using that these solutions depend continuously on the initial data, the resulting coordinate transformation $(t,s)\leftrightarrow(t,\hat{s}_\sigma(t))$ is a homeomorphism between the domain $[0,T]\times[0,1]$ and the corresponding portion of $[0,T]\times\mathcal{G}$ (where $\mathcal{G}$ is the gnomon). Consequently, the number of discontinuities in $u(t,\cdot)$ is bounded by the number of discontinuities in $\bar{u}(\cdot)$ and $\varphi^{\mu(\cdot)}(\cdot)$ on $[0,1]$ or $[0,T]$, respectively. Then, using that $\varphi^{\mu(\cdot)}(\cdot)\in\mathscr{C}_{\text{pw}}[0,T]$, we have

$u(t^*,\cdot) \in \mathscr{C}_{\mathrm{pw}}[0,1]$ for all $t^* \in [0,T]$. By exchanging the variables t and s, it also follows that $u(\cdot,s^*) \in \mathscr{C}_{\mathrm{pw}}[0,T]$ for all $s^* \in [0,1]$. Finally, a laborious but straightforward use of standard wellposedness theory for the ODEs (13), (15) shows that $u(t,\cdot)$ depends continuously there on the data. □

We now get back to the networked system (2) and collect here the hypotheses we will impose for the uncontrolled problem.

Hypotheses ($\mathbf{H}^1$):

1. The assumptions (A1)–(A4) hold.
2. The discrete state component $\mu(\cdot)$ is given with $\mu(\cdot) \in \mathscr{M}_{\mathrm{pw}}[0,T]$.
3. The initial data satisfies $\bar{\mathbf{u}}(\cdot) \in \mathscr{C}_{\mathrm{pw}}(G)$.

Theorem 1. *Consider the coupled system* (2) *under the hypotheses (H^1: 1–3). Then there exists a solution $\mathbf{u}(\cdot) \in \mathscr{C}_{\mathrm{pw}}([0,T] \to \mathscr{C}_{\mathrm{pw}}(G))$. Further, the solution set is upper semicontinuous in its dependence on the data $\bar{\mathbf{u}}$ and $\mu(\cdot)$.*

Proof. We wish to apply Lemma 2 to each of the edges e_j, $j = 1,\dots,m$. Therefore, it suffices to show that the right hand side of the nodal conditions $(2)_2$ is piecewise continuous for all i,j with $\phi_{ij}^+ \neq 0$. This can be easily seen, noting that under (A4), we have $\varphi_i^{\mu(\cdot)}(\cdot) \in \mathscr{C}_{\mathrm{pw}}[0,T]$ and $u_j(\cdot,1) \in \mathscr{C}_{\mathrm{pw}}[0,T]$ according to Lemma 2, so $\left(a_j^{\mu(\cdot)} u_j\right)\Big|_{(\cdot,1)} \in \mathscr{C}_{\mathrm{pw}}[0,T]$, and that any finite sum of functions in $\mathscr{C}_{\mathrm{pw}}[0,T]$ taken as in Remark 2 is piecewise continuous. Well-posedness of the solution $\mathbf{u}(t,\cdot)$, as interpreted here, follows from the corresponding well-posedness in Lemma 2. Indeed, the problem is almost well-posed in the usual sense since nonuniqueness is impossible except in the coincidental situations discussed in Remark 2. □

We next note certain bounds which we will use subsequently.

Corollary 1. *Under the hypotheses of Theorem 1, there exist uniform bounds for the* transit time *$\Delta t(s_*,s^*)$ of material traveling from points s_* to s^* ($s_* < s^* \in [0,1]$)*

$$(s^* - s_*)/\bar{a} \le \Delta t(s_*,s^*) \le (s^* - s_*)/\underline{a} \tag{17}$$

and there exists a constant $\bar{c}(T)$, such that

$$|\mathbf{u}(t)| \le \bar{c}(T) \quad \textit{for all } t \in [0,T]. \tag{18}$$

These bounds are independent of $\mu(\cdot) \in \mathscr{M}_{\mathrm{pw}}([0,T])$.

Proof. The bound (17) is an easy consequence of (13) under (A1). Similarly, the bound (18) is given by (15) under (A2), i. e.

$$\bar{c}(T) = T\,\bar{a}\,d^+_{\max}\left(\max_j \|\bar{u}_j(\cdot)\|_\infty + \max_{i,\mu} \|\varphi_i^\mu(\cdot)\|_\infty + T\,\bar{\bar{b}}\right), \tag{19}$$

where $d^+_{\max} = \max_i d^+(v_i)$, $\|\bar{u}_j(\cdot)\|_\infty$ and $\|\varphi_i^\mu(\cdot)\|_\infty$ are finite due to $\bar{u}_j \in \mathscr{C}_{\mathrm{pw}}[0,1]$ and $\varphi_i^\mu(\cdot) \in \mathscr{C}_{\mathrm{pw}}[0,T]$. □

3 Optimal Switching

In this section, we wish to consider the possibility of *optimal modal control* for the system (2) — taking the switching function $\mu(\cdot)$ not as given a priori, but as an open loop control, subject to our specification in order to minimize a cost criterion of the form

$$\begin{aligned}\mathcal{J}[\cdot] = \int_0^T \left[c^{\mu(t)}(\mathbf{u}(t)) + \sum_{i\in\mathcal{O}} \delta_i(\mathbf{u}(t)) \right] e^{-\lambda t} dt \\ + \sum_{k=1}^{K} \gamma(\tau_k - \tau_{k-1}; \mathbf{u}(\tau_k); \mu_{k-1} \curvearrowright \mu_k)\, e^{-\lambda\tau_k} \\ + e^{-\lambda T}\psi(\mathbf{u}(T)).\end{aligned} \tag{20}$$

Here, for each fixed $\mu \in \mathcal{M}$,

- $c^\mu(\cdot)$ is a *running cost* involving costs $c_j^\mu(\cdot)$ and $d_i^\mu(\cdot)$ for the distribution of material along the edges or the external input at the nodes, respectively, e. g.,
$$c^\mu(\mathbf{u}(t)) = \sum_{j=1}^{m} \left(\int_0^1 c_j^\mu(s, u_j) ds + \sum_{i=1}^{n} d_i^\mu(\omega_{ij}^\mu(t), \varphi_i^\mu(t)) \right) \tag{21}$$
- $\gamma(\tau; \mathbf{u}; \mu \curvearrowright \mu')$ is a *switching cost* associated with a modal transition $\mu \curvearrowright \mu'$
- $\delta_i(\mathbf{u}(t)) = \delta_i(\beta_i(t) - \sum_{j=1,\ldots,n} \phi_{ij}^- u_i(t, 1))$ models a *demand penalty* at the output vertices $i \in \mathcal{O}$
- $\psi(\cdot)$ is a *terminal cost*
- λ is a *discount rate.*

We will indicate as needed the dependence of $\mathcal{J}[\cdot]$ on the data $\mu(\cdot)$, $\mathbf{u}(\cdot)$, $\mathbf{x}(\cdot)$, $\bar{\mathbf{x}}$, etc. and we write $\mathcal{J}^T$ to indicate dependence on T. Our principal objective here is to show the existence of an optimal control, i. e., solving (10) for $\mathcal{J}[\cdot]$ given as above. We collect here the hypotheses we will impose for this optimal control problem.

Hypotheses ($\mathbf{H}^2$):

1. The assumptions (A1)–(A4) hold.
2. The functions $\beta_i(\cdot)$ are given.
3. For the initial state $\bar{\mathbf{x}} = [\bar{\mathbf{u}}, \bar{\mu}]$, we have $\bar{\mathbf{u}} \in \mathscr{C}_{\mathrm{pw}}(G)$ and $\bar{\mu} \in \mathcal{M}$.
4. The functions $c^\mu(\cdot)$ in (21) are lsc (lower semicontinuous) on the closed set $\{\mathbf{u} : c^\mu(\mathbf{u}) < \infty\}$.
5. The functions $\tau, \mathbf{u} \mapsto \gamma(\tau, \mathbf{u})$ are lsc, there exists a bound $\underline{\gamma} > 0$ such that $\gamma(\cdot) \geq \underline{\gamma}$.
6. The functions $\delta_i(\cdot)$ and $\psi(\cdot)$ are lsc.

7. For some $\mu(\cdot) \in \mathscr{M}_{\rm pw}[0,T]$, defining $\mathbf{u}(\cdot)$ by (2), the cost $J(\mu(\cdot),\mathbf{u}(\cdot))$ is finite.

Theorem 2. *Assume the set of hypotheses (H*^{2}*: 1–7). Then there exists an optimal* $\mu_*(\cdot) \in \mathscr{M}_{\rm pw}[0,T]$ *solving* (10). *Further, the minimized cost* $\mathcal{J}_*[\bar{\mathbf{x}}] = \mathcal{J}[\mu_*(\cdot;\bar{\mathbf{x}})]$ *is lsc in its dependence on the initial data* $\bar{\mathbf{x}}$.

Proof. The proof is fairly standard. Let $\mu^\nu(\cdot)$ be a minimizing sequence in $\mathscr{M}_{\rm pw}[0,T]$. In view of (H^2: 5) a bound on $\mathcal{J}$ implies a bound $\bar{K}$ on the number K of switching points in each $\mu^\nu(\cdot)$. With the bound $\bar{K}$ the space of admissible switching functions $\bar{\mathscr{M}}_{\rm pw}[0,T] = \{\mathscr{M}^{\bar{K}}_{\rm pw}[0,T] \,|\, \mu(0) = \bar{\mu}\}$ becomes compact by Lemma 1 so, extracting a subsequence, we have $\mu^\nu(\cdot) \to \mu_*(\cdot)$ for some $\mu_*(\cdot) \in \bar{\mathscr{M}}_{\rm pw}[0,T]$. The upper-semicontinuity of the solution set in its dependence on $\mu(\cdot)$ in Theorem 1 ensures that the limit of the corresponding solutions $\mathbf{u}^\nu(\cdot)$ is a solution $\mathbf{u}_*(\cdot) \in \mathscr{C}_{\rm pw}(G)$. The lower semicontinuity conditions in (H^2: 4–6) then ensure similar lower semicontinuity for $\mathcal{J}[\cdot]$:

$$\mathcal{J}[\mu^*(\cdot)] \le \liminf_{\nu\to\infty} \mathcal{J}[\mu^\nu(\cdot)] = \inf_{\mu(\cdot)\in\bar{\mathscr{M}}_{\rm pw}[0,T]} \mathcal{J}[\mu(\cdot;\bar{\mathbf{x}})] = \mathcal{J}_*[\bar{\mathbf{x}}]$$

so the minimum $\mathcal{J}_*$ is attained at $\mu_*(\cdot)$. Similarly, considering a sequence of initial data $\bar{\mathbf{x}}_\nu \to \bar{\mathbf{x}}_\infty$ and a corresponding sequence of optimizers μ_*^ν, we see that $\mathcal{J}_*[\cdot]$ is lsc under the same hypotheses. $\square$

We could also consider the corresponding *infinite horizon* problem $T \to \infty$, for which we omit the terminal cost $\psi(\cdot)$ of (20) in defining $\mathcal{J}^\infty$. [Finiteness of $\mathcal{J}^\infty$ may depend on having a large enough discount rate λ in (20).] We also set

$$\mathscr{M}_{\rm pw}[0,\infty) = \{\mu(\cdot)\colon [0,\infty) \longrightarrow \mathcal{M} \,|\, \mu(\cdot)|_{[0,T]} \in \mathscr{M}_{\rm pw}[0,T] \text{ for all } T \ge 0\}. \quad (22)$$

Corollary 2. *Assume the hypotheses (H*^{2}*: 1–7) for each bounded subinterval* $[0,T]$ *and the existence of some admissible global switching function* $\mu(\cdot) \in \bar{\mathscr{M}}_{\rm pw}[0,\infty) = \{\mu(\cdot) \in \mathscr{M}_{\rm pw}[0,\infty) \,|\, \mu(0) = \bar{\mu}\}$ *for which* $\mathcal{J}^\infty[\mu(\cdot)]$ *is finite. Then there exists an optimal* $\mu_*(\cdot) = \mu_*(\cdot,\bar{\mathbf{x}}) \in \bar{\mathscr{M}}_{\rm pw}[0,\infty)$ *for which* $\mathcal{J} = \mathcal{J}^\infty[\mu(\cdot),\bar{\mathbf{x}}]$ *attains its minimum* $\mathcal{J}_*^\infty$. *Further, the minimized cost* $\mathcal{J}_*^\infty[\bar{\mathbf{x}}] = \mathcal{J}^\infty[\mu_*(\cdot;\bar{\mathbf{x}})]$ *is lsc in its dependence on the initial data* $\bar{\mathbf{x}}$.

Proof. Since $\bar{\mathscr{M}}_{\rm pw}[0,\infty)$ is nonempty by assumption, there exists a minimizing sequence for $\mathcal{J}^\infty$. From this sequence we can, as in the proof of Theorem 2, extract a subsequence convergent on $[0,1]$, then extract from that a subsequence convergent on $[0,2]$, etc. (recursively) and by a diagonal argument obtain a minimizing sequence $\{\mu^\nu\}$ convergent (on every $[0,T]$) to a switching function $\mu_*(\cdot)$. Since $\{\mu^\nu\}$ is a minimizing sequence, we have

$$\mathcal{J}^\infty[\mu^\nu] \le \mathcal{J}_*^\infty(\bar{\mathbf{x}}) + \varepsilon_\nu \quad \text{with } \varepsilon_\nu \to 0;$$

for each T we have (again as in the proof of Theorem 2, noting that $\{\mu^\nu\}$ converges on $[0,T]$)

$$\mathcal{J}^T[\mu_*] \le \mathcal{J}^T[\mu^\nu] + \varepsilon_\nu(T) \quad \text{with } \varepsilon_\nu(T) \to 0.$$

Then, for each T,

$$\mathcal{J}^T[\mu_*] \le \mathcal{J}^T[\mu^\nu] + \varepsilon(T) \le \mathcal{J}^\infty[\mu^\nu] + \varepsilon_\nu(T) \le \mathcal{J}_*^\infty + \varepsilon_\nu(T) + \varepsilon_\nu.$$

Letting $T \to \infty$ gives $\mathcal{J}^\infty[\mu_*; \bar{\mathbf{x}}] \le \mathcal{J}_*^\infty[\bar{\mathbf{x}}]$ so this limit switching function μ_* minimizes $\mathcal{J}^\infty[\cdot, \bar{\mathbf{x}}]$ as desired. Similarly, considering a sequence of initial data $\bar{\mathbf{x}}_\nu \to \bar{\mathbf{x}}_\infty$ and a corresponding sequence of optimizers μ_*^ν, we see that $\mathcal{J}_*^\infty[\cdot]$ is lsc under the same hypotheses. □

Remark 3. For an autonomous problem it is easy to see the invariant embedding principle: that the minimizer on $[0, T]$ of the finite horizon truncation $\mathcal{J}^T$ with $\psi(\cdot) = \mathcal{J}_*^\infty[\cdot]$ will be the initial segment of an optimal control for the autonomous infinite horizon problem with continuation optimal on $[T, \infty)$ for initial data $\mathbf{x}(T)$. [A similar, but more complicated statement holds for time-dependent and for finite horizon settings.] A consequence of this—compare the discussion in [4]—is that one can never have $\mathbf{x} = [\mu, \mathbf{u}]$, $\mathbf{x}' = [\mu', \mathbf{u}]$ such that

$$\mathcal{J}_*^\infty[\mathbf{x}'] + \gamma(\mu \curvearrowright \mu') < \mathcal{J}_*^\infty[\mathbf{x}] \tag{23}$$

since in the state $\mathbf{x}$ one could immediately switch modes from μ to μ', incurring the switching cost, and (23) would contradict the defining optimality of $\mathcal{J}_*^\infty[x]$. Thus one certainly remains in the mode μ when the continuous component of the state is in the open set

$$\mathcal{S}^\mu = \left\{ \mathbf{u} \,|\, \mathcal{J}_*^\infty[[\mu, \mathbf{u}]] < \min_{\mu' \ne \mu} \{ \mathcal{J}_*^\infty[[\mu', \mathbf{u}]] + \gamma(\mu \curvearrowright \mu') \} \right\}, \tag{24}$$

but, when (following the μ-dynamics of the transport problem) one arrives at the complement

$$\breve{\mathcal{S}}^\mu = \left\{ \mathbf{u} \,|\, \mathcal{J}_*^\infty[[\mu, \mathbf{u}]] = \min_{\mu' \ne \mu} \{ \mathcal{J}_*^\infty[[\mu', \mathbf{u}]] + \gamma(\mu \curvearrowright \mu') \} \right\} \tag{25}$$

one can/should switch away from μ to some μ' attaining the minimum in (23), although this does not distinguish where switching away from μ is not merely possible but is forced by optimality. In such situations we have obtained nonunique controls, but each provides the same minimum cost. We see that each ∂S^μ is a *switching surface* for leaving that mode: provided the value function $\mathcal{J}_*^\infty[\mathbf{x}]$ is known and one has full state observation (or, almost equivalently, the possibility of accurately reconstructing that), the policy of switching at $\partial\mathcal{S}^\mu$ converts the optimal control problem to feedback form. □

4 Switching by Feedback

In this section we consider the system (2) where the switching function $\mu(\cdot)$ is neither prescribed nor optimized, but event-driven, to be determined by certain switching rules during the system evolution. In particular, we have in mind some approximate implementation of the optimal control problem in feedback form as described in Remark 3 in the last Section, but with the system state now only partially known: our available information is given by S *sensor values*

$$y_l(t) = \mathbf{u}(t, s_l), \quad l = 1, \ldots, S \tag{26}$$

where each s_l is a point in the graph, i. e., $s_l \in e_j$ for some $j = j(l)$, so $\mathbf{u}(t, s_l)$ here means $u_j(t, s_l)$.

For future reference we set $\mathbf{y}(t) = (y_1(t), \ldots, y_S(t))$ and will then set $\hat{s} = \min\{s_l \mid l = 1, \ldots, S\}$, requiring that $0 < \hat{s} \leq 1$ with the interpretation that a sensor placed at a node senses the density at the end of a particular corresponding incoming edge ($s_l = 1$ for that edge).

For a problem in which switching is the only control, a feedback law necessarily has the form of an assignment of the points observed and control actions as in (11), which we must make more precise by providing the rules we will use to specify switching. We begin by assuming:

(A5) For each $\mu \in \mathcal{M}$ there is a disjoint pair of open sets $\mathcal{S}^\mu \subset \mathbb{R}^S$ and $\hat{S}^\mu \subset \mathbb{R}^S$ such that the complement $\check{\mathcal{S}}^\mu$ of S^μ is the union of nonempty closed sets $\mathcal{C}^{\mu \curvearrowright \mu'}$ with each $\mathcal{C}^{\mu \curvearrowright \mu'} \subset \mathcal{S}^{\mu'}$ ($\mu' \in \mathcal{M}$).

Our *switching rules* are then

$$\begin{cases} \text{do not switch} & \text{if } \mathbf{y}(t) \in \mathcal{S}^\mu \\ \text{switch } \mu \curvearrowright \mu' \text{ immediately} & \text{if } \mathbf{y}(t) \in \check{\mathcal{S}}^\mu \text{ (with } \mu' \text{ s. t. } \mathbf{y}(t) \in \mathcal{C}^{\mu \curvearrowright \mu'}) \\ \text{switch } \mu \curvearrowright \mu' \text{ optionally} & \text{if } \mathbf{y}(t) \in \mathcal{C}^{\mu \curvearrowright \mu'} \text{ but } \mathbf{y}(t) \notin S^\mu \cup \hat{S}^\mu. \end{cases} \tag{27}$$

We note that this set of switching rules parallels the structure of Remark 3 and, in its simplest realization, corresponds precisely to the elementary hysteron = 'non-ideal relay' of [11, sect. 28.2]. Due to threshold phenomena in (27), one cannot expect the solution to depend continuously on the system data. However, one does want every limit of solutions again to be a solution, compare Remark 2. This *upper semicontinuity* of the solution set was, after all, at the heart of Theorems 1, 2. Therefore we accept the ambiguity in (27) of optional switching and accept as solutions the continuations for all the optional choices.

We also note the possibility of a multivalued measurement $y(t) = [y^1, \ldots, y^k]$ obtained from observing values assigned to degenerate partition intervals along the lines of Example 2 of Remark 2. We interpret this by applying the switching rules (27) successively to each of the components in that order, as if separated in time — and accept that, even with the simplifying assumption $\mathcal{C}^{\mu \curvearrowright \mu'} \subset \mathcal{S}^{\mu'}$ in (A5), this may result in instantaneous multiple switches $\mu \curvearrowright \mu' \curvearrowright \cdots \curvearrowright \mu''^{\cdots\prime}$ at the time t.

For our main result below, we impose the following hypotheses.

Hypotheses ($\mathbf{H}^3$):

1. The assumptions (A1)–(A5) hold.

2. $\mathbf{y}(\cdot) = (y_1(\cdot), \dots, y_S(\cdot))$ is given as in (26).
3. The initial state $\bar{\mathbf{x}} = [\bar{\mathbf{u}}, \bar{\mu}]$ is in $\mathscr{C}_{\mathrm{pw}}(G) \times \mathcal{M}$.

Theorem 3. *Consider the system* (2) *together with the feedback* (27) *under the hypothesis (H*^{3}*: 1–3). Then this* hybrid system *has a solution* $\mathbf{x}(\cdot) = [\mathbf{u}(\cdot), \mu(\cdot)]$ *with* $\mathbf{u}(t) \in \mathscr{C}_{\mathrm{pw}}(G)$ *on* $[0, T]$ *and* $\mu(\cdot) \in \mathscr{M}_{\mathrm{pw}}[0, T]$ *for all* $T \geq 0$. *Further, this nonempty solution set is upper semicontinuous in its dependence on the initial data* $\bar{\mathbf{x}}$.

Proof. Our major objective is the construction of the switching signal $\mu(\cdot)$, since the construction of the solution is then given by Theorem 1. Our concern here will be to show that the resulting switching signal is piecewise continuous, i. e., that $\mu(\cdot) \in \mathscr{M}_{\mathrm{pw}}[0, T]$ for any $T \geq 0$. In particular, we will have to show, despite the difficulty that the observed state $\mathbf{y}(\cdot)$ itself will have jumps, that no Zeno phenomena can arise, i. e., that the switching times τ_k (when a switch $\mu \curvearrowright \mu'$ is given by the switching rules) do not accumulate. To this end we first observe that the control actions given by (27) are always admissible under assumption (A5). Defining $\varDelta = \hat{s}/\bar{a}$, we recursively construct $\mu(\cdot)$ on the time intervals $[T_m, T_{m+1}]$ (with $T_m = m\varDelta$). For this recursion we may assume that $\mu(\cdot) \in \mathscr{M}_{\mathrm{pw}}[0, T_m]$ so all the argument of Theorem 1 applies on that interval, in particular we know that $\mathbf{u}(T_k) \in \mathscr{C}_{\mathrm{pw}}(G)$, i. e., $\mathbf{u}(T_k)$ has only finitely many jumps.

With this $\varDelta$ it follows from the bounds on the transit times obtained in Corollary 1 that the data at $t \in [T_m, T_m + \varDelta]$ used for each of the sensor points s_l can only come from the 'initial data' $\mathbf{u}(T_m)$. Thus, recalling the arguments of Lemma 2, we have $y_l(\cdot)|_{[T_m, T_{m+1}]} \in \mathscr{C}_{\mathrm{pw}}[T_m, T_{m+1}]$ for all $l = 1, \dots, S$ and so $\mathbf{y}(\cdot)|_{[T_m, T_{m+1}]} \in (\mathscr{C}_{\mathrm{pw}}[T_m, T_{m+1}])^S$. Further, we make use of the a priori bound $\bar{c}(T_{m+1})$ on $\mathbf{u}(\cdot)$ also given by Corollary 1, saying that $\mathbf{u}(t) \leq \bar{c}(T_{m+1})$ independently of $\mu(\cdot)$, and so $\mathbf{y}(t) \leq \bar{c}(T_{m+1})$ for all $t \in [0, T_{m+1}]$. In the presence of this bound the switching sets $\mathcal{C}^{\mu \curvearrowright \mu'}$ are not only closed, but compact, so there is a minimum distance $\delta_* > 0$ in $\mathbb{R}^S$ between any set $\mathcal{C}^{\mu'' \curvearrowright \mu}$ and any $\mathcal{C}^{\mu \curvearrowright \mu'}$. Further, on the union of closed time intervals where $\mathbf{y}(\cdot)$ has none of the finite set of discontinuities, $\mathbf{y}(\cdot)$ is uniformly continuous. Consider, then, an interswitching interval $[t, t+\tau]$ on which $\mathbf{y}(\cdot)$ is continuous. This begins with some modal switch $\mu'' \curvearrowright \mu$ and terminates with $\mu \curvearrowright \mu'$ for some $\mu'', \mu, \mu' \in \mathcal{M}$, so $\mathbf{y}(t) \in \mathcal{C}^{\mu'' \curvearrowright \mu}$ and $\mathbf{y}(t+\tau) \in \mathcal{C}^{\mu \curvearrowright \mu'}$. Hence $|\mathbf{y}(t+\tau) - \mathbf{y}(t)| \geq \delta_*$ and, by the uniform continuity, this gives a lower bound τ_* for the length of the interswitching interval. These intervals are disjoint by definition, so there can be at most $\varDelta/\tau_*$ of them in the finite time $\varDelta$.

For each sequence $\bar{u}^\nu$ converging to $\bar{u}^\infty$ in $\mathscr{C}_{\mathrm{pw}}(G)$, the bounds above are uniform in ν. Thus, by Lemma 1, for each corresponding choice μ^ν consistent with the switching rules (27), there exists a subsequence (again denoted by μ^ν) converging in $\mathscr{M}_{\mathrm{pw}}[0, T]$ to μ^∞. By Theorem 1, the upper semicontinuity of the set of solution components $\mathbf{u}^\nu(\cdot)$ ensures that there exists $\mathbf{u}^\infty$ such that $\mathbf{u}^\nu \to \mathbf{u}^\infty$ and $x^\infty(t) = [\mu^\infty(t), \mathbf{u}^\infty(t)]$ being consistent with (27) for all t due to the assigned limit values to degenerate intervals in $\mu^\infty(t)$ and $\mathbf{u}^\infty(t)$. Thus we have similar upper semicontinuity of the solution set in its dependence on the initial data $\bar{x}$. $\square$

5 Extensions and Final Remarks

In this last section, we wish to present some final remarks and discuss some extensions of the proposed modeling in view of possible applications.

It should have become clear to the reader that, although the modal index μ is global at any time for our system (2), the distinction between one mode and another may be quite limited for many applications, e. g., corresponding to a change $\omega_{ij}^{\mu}(t) \curvearrowright \omega_{ij}^{\mu'}(t)$ at a single vertex or cutting off a single external source $\varphi_i^{\mu}(t) \curvearrowright \varphi_i^{\mu'}(t) = 0$ or changing the flow velocity $a_j^{\mu}(\cdot,\cdot) \curvearrowright a_j^{\mu'}(\cdot,\cdot)$ at a single edge: this is primarily a matter of notational convenience.

A word is in order about our inclusion of degenerate 0-length intervals in the Definitions 1 and 2. For Theorem 1, we did not really need the retained specification of mode assignment for degenerate interswitching intervals since they made no difference to the system dynamics and the retention of value assignments for degenerate intervals of continuity in $\mathscr{C}_{\mathrm{pw}}(G)$ led to the concerns of Remark 2, but also did not affect the observable dynamics there. However, this retention has become significant in the context of Theorem 2. Suppose we have a minimizing sequence μ^{ν} with $\mu \curvearrowright \mu'$ at τ_k^{ν} and $\mu' \curvearrowright \mu''$ at τ_{k+1}^{ν}. If now τ_k^{ν} and τ_{k+1}^{ν} coalesce in the limit, we have to admit the compound switch $\mu \curvearrowright \mu' \curvearrowright \mu''$ as this is then likely to be less costly than the direct switch $\mu \curvearrowright \mu''$. Thus, the retention of the intermediate mode μ' for the now-degenerate interswitching interval $[\tau_k, \tau_{k+1}]$ becomes necessary for our argument. Similarly, the $\mathbf{u}$-dependence of the costs γ requires, for our argument, the retention of assigned limit values for degenerate intervals of continuity of $\mathbf{u}(\cdot)$ since this can be relevant to relating the cost of the compound switch $\mu \curvearrowright \mu' \curvearrowright \mu''$ to an appropriate limit of the sum of the costs for individual switches $\mu \curvearrowright \mu'$, $\mu' \curvearrowright \mu''$ which might depend on just the values taken in the now-degenerate continuity interval (e. g., dependence on the max or min of $\mathbf{u}(\cdot)$ over some edge, compare the example in Remark 2). To take advantage of the assumed lower semicontinuity of these costs it is important to retain the assignment in the limit. We could ignore these considerations under sufficiently restrictive conditions on the switching costs: if these were independent of $\mathbf{u}$ and satisfy $\gamma(0; \mu \curvearrowright \mu'') \leq \gamma(0; \mu \curvearrowright \mu') + \gamma(0; \mu' \curvearrowright \mu'')$, then the retention would no longer be needed for Theorem 2. However, for applications, we not only expect to have some cost associated with each modal transition, but also that this switching cost may depend on the continuous state component of the system. If we, e. g. consider controlling a transport network by switching pumps on/off, then the startup/shutdown sequence of a pump may not only take some (unmodeled) time, but may also involve consumption of fuel or require manual intervention subject to the state of the system. Moreover, the retention of limiting assigned values both for degenerate interswitching intervals and for degenerate continuity intervals finally became explicitly significant for the well-posedness result in Theorem 3.

It is easy to modify our modeling to address another typical problem arising in many applications: in addition to the transport along the edges within the network, material may also be stored (queued) at the vertices v_i. We model the *storage buffer* at the node by an additional state component $U_i(t) \in \mathbb{R}$ ($i \in \{1,\ldots,n\}$) and enforce conservation at the nodes by replacing (5) by the *nodal dynamics*

$$\frac{dU_i(t)}{dt} = \sum_{k=1}^{m} \left[\phi_{ik}^{+} \left(a_k^{\mu(t)} u_k \right)\Big|_{(t,1)} - \phi_{ik}^{-} \left(a_k^{\mu(t)} u_k \right)\Big|_{(t,0)} \right] + \varphi_i^{\mu(t)}(t) \quad t > 0 \tag{28}$$

for $i = 1,\ldots,n$ with an initial condition $U_i(0) = 0$. A major difficulty is then the treatment of constraints on $U_i(\cdot)$, i. e. box constraints of the form

$$0 \le U_i \le \bar{U}_i, \tag{29}$$

representing limited storage capacity in the buffers that must be maintained by (28). A plausible model might lead to a discontinuous switch of the dynamics, e. g. abruptly switching $\omega_{ij}^{\mu}(t) \curvearrowright \omega_{ij}^{\mu'}(t)$ in $(2)_2$ when a finite buffer is filled. For the optimal control problem (10) we may penalize the violation of the constraints (29), but one may also — similarly to observing the values of $\mathbf{u}(\cdot)$ at finitely many points on edges as in Section 4 — take $U_i(t)$ as part of the observed quantities in $\mathbb{R}^S$ and maintain the state constrains (29) by the switching rules (27). Well-posedness of such a closed-loop system under appropriate hypotheses can then be obtained by an argument analogous to that given for Theorem 3.

It should also become clear from our treatment that there would be little difference in considering not a scalar but a switched multicomponent flow with common flow velocities $a_j^{\mu}(t,s)$. Also, the restriction to linear nodal conditions $(2)_2$ is only for expository simplicity, noting that all of our results hold if, in each mode,one were to make non-linear but continuous assignments $(2)_2$ at the nodes. A treatment of the more general problem where the flow velocities are given by matrices $A_j^{\mu}(t,s)$ subject to conditions of strict hyperbolicity and consistency is in preparation, noting [1].

Finally, we mention here that a key assumption for all the treatment in this paper is that the flow velocities $a^{\mu}(t,s)$ in (2) are independent of u, so we were not dealing here with the shock formations which typically arise in non-linear systems. Future work will be devoted to the analysis of switching non-linear systems.

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