A convection/reaction/switching system

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Abstract

We consider a spatially distributed hybrid system consisting of a convection/reaction system in which the reaction switches discontinuously in time between modes, independently at each spatial point on reaching "switching threshholds." The model involves a novel formulation for evolution of the free boundary between the modal regions.

Key words: partial differential equations, system, nonlinear, convection/reaction, hybrid system, bioremediation

1 Introduction

We are here concerned with convection/reaction systems which switch discontinuously between alternate modes of the system (i.e., different reaction functions) independently at each spatial point on reaching certain "switching threshholds". We begin with the problem:

$$\begin{array}{lrcl} i. & \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial s} & = & -f_j(t,s,v) \\ ii. & j(\cdot,s) & = & W[v(\cdot,s)] \end{array} \tag{1.1}$$

on $\mathcal{Q}_0 = [0,T] \times [\underline{s},\bar{s}]$. Here (1.1-$i$) is a convection/reaction equation with reaction given by $-f_j$ specifying the choice of reactive mode by the index j. On the other hand, $j = j(t,s)$ is to be determined from $v(\cdot,s)$, independently at each spatial point $s \in [\underline{s},\bar{s}]$, by (1.1-$ii$) through the "elementary hysteron" $W[\cdot]$ of [1]; we will discuss this switching rule in more detail in Section 3.

Of course we must adjoin input boundary conditions for v to (1.1-i) and adjoin initial conditions for each component:

$$\begin{array}{c} v(t,\underline{s}) = v_*(t) \\ v(0,s) = \overset{\circ}{v}(s), \qquad\qquad j(0,s) = j_0(s). \end{array} \tag{1.2}$$

For our treatment here it will be important that we assume $\overset{\circ}{v}(\cdot)$ decreasing in s and that j_0 has the form

$$j_0(s) = \begin{cases} 1 & \text{if } \underline{s} \le s < s_* \\ 0 & \text{if } s_* < s \le \bar{s} \end{cases} \tag{1.3}$$

for some $s_* \in [\underline{s}, \bar{s}]$ with $\alpha_- \le \overset{\circ}{v}(s_*) \le \alpha_+$.

The system (1.1) is only a special case of the more general form which we will wish to consider later, but it already exhibits the essential novel features of the situation:

- The index j is needed to indicate modal selection, so this is a *hybrid system*: the state at each point is $[v(t,s), j(t,s)]$, which has both the continuous component v, taking values in $\mathbb{R}$, and also the discrete component j, taking values in $\{0,1\}$.
- The constitutive 'switching function' $W[\cdot]$ appearing in (1.1-ii) is not really a function, but a discontinuous, hysteretically history-dependent, input/output relation.

This form of hysteretic interaction seems new in connection with partial differential equations, although we note [7], [5], [8].

Our first step in treating (1.1) will be the substitution[1]

$$\tau = t - s: \qquad \tilde{v}(\tau,s) = v(\tau+s,s), \quad \tilde{\jmath}(\tau,s) = j(\tau+s,s) \tag{1.4}$$

whence, with

$$\tilde{v}^*(\tau) = v_*(\tau+\bar{s}), \qquad \tilde{f}_j(\tau,s,r) = f_j(\tau+s,s,r), \tag{1.5}$$

we have

$$\begin{array}{llcl} i. & \dfrac{\partial \tilde{v}}{\partial s} & = & -\tilde{f}_{\tilde{\jmath}}(\tau,s,\tilde{v}), \qquad \tilde{v}(\tau,\underline{s}) = \tilde{v}^*(\tau) \\ ii. & \tilde{\jmath}(\cdot,s) & = & W[\tilde{v}(\cdot,s)]. \end{array} \tag{1.6}$$

[1] Note that we are *not* changing to Lagrangian coordinates here: instead, we keep spatial points unchanged while shifting time for each s. This is important to obtain (1.6-ii).

Note that this has transformed the convection equation into a family of ordinary differential equations in s, parametrized by τ, while (1.6-ii) is a family of switchings with respect to τ, parametrized by s. Of course, (1.4) has transformed $\mathcal{Q}_0$ into

$$\tilde{\mathcal{Q}}_0 = \{(\tau, s) : \underline{s} \le s \le \bar{s},\ -s \le \tau \le T - s\}$$

and we must consider (1.6) on $\tilde{\mathcal{Q}} = [\underline{\tau}, \bar{\tau}] \times [\underline{s}, \bar{s}]$ where we take $\underline{\tau} = -\bar{s}$ and $\bar{\tau} = T - \underline{s}$ to have $\tilde{\mathcal{Q}}_0 \subset \tilde{\mathcal{Q}}$; we discuss the requisite data for (1.6) in Section 4 (Remark 4.2).

Our second step, the key to our treatment, is consideration of a free boundary problem, determining the regions where the index takes each value. Under the assumptions that the initial data $\mathring{v}, j_0$ are as above and that $\tilde{f} > 0$, we will see that there will be a separator $\sigma(\tau)$, later to be characterized by a double obstacle problem, such that

$$\tilde{\jmath}(\tau, s) = \begin{cases} 1 & \text{if } \underline{s} \le s < \sigma(\tau) \\ 0 & \text{if } \sigma(\tau) < s \le \bar{s} \end{cases} \tag{1.7}$$

so the problem (1.1) for the hybrid pair of unknowns (v, j) can be converted to a problem seeking the pair of continuously-valued unknowns (v, σ).

After briefly presenting an example in Section 2, we will discuss $W[\cdot]$ and the characterization (1.7) in Section 3 and then in Section 4 will complete the demonstration of well-posedness for the problem (1.6) and so also for the convection/reaction/switching problem (1.1)-(1.2).

Remark 1.1. It is not very difficult to generalize this to n-dimensional convection. Suppose we wish to consider the equation

$$\frac{\partial v}{\partial t} + \nabla \cdot (v\mathbf{v}) = \ -f(t, x) \tag{1.8}$$

in a spatial region $\Omega_0 \subset \mathbb{R}^m$ with (specified) velocity field $\mathbf{v} = \mathbf{v}(x)$.

The relevant notion of "geometric admissibility" for $[\Omega_0, \mathbf{v}]$ is intuitively clear, but awkward to describe satisfactorily. Given a point $x \in \Omega_0$ we can solve the ordinary differential equation: $dx/ds = \mathbf{v}(x)$ to obtain both a flowline and a parametrization by $s \in (\underline{s}, \bar{s})$ along that flowline. We are then assuming that this flow moves smoothly through the region Ω_0 from each input point $x_0 = x(\underline{s}) \in \partial\Omega_0$ to an outflow point $x(\bar{s}) \in \partial\Omega_0$, taking finite time $[\bar{s} - \underline{s}]$.

The second, more awkward, aspect of this admissibility is that there should be a local transverse variable $y \in Y \subset \mathbb{R}^{m-1}$ parametrizing the family of flowlines so Ω_0 appears as a manifold Ω with local coordinates $[s, y]$, diffeomorphically related to the original x. While looking to this generality in principle, we will actually treat explicitly only the more restricted situation in which, for each such $y \in Y$, there is just one associated flowline segment $[\underline{s}(y), \bar{s}(y)]$ so Ω_0 maps smoothly and invertibly to

$$\Omega = \{x = [s, y] \in \mathbb{R}^m : y \in Y,\ \underline{s}(y) < s < \bar{s}(y)\}. \tag{1.9}$$

Without further loss of generality, then, we assume we already begin with a 'nice' bounded region $\Omega_0 = \Omega$ in the form (1.9).

We are then specifying the boundary data for the convection on the set of input points $\Gamma_0 = \{[\underline{s}(y), y] : y \in Y\} \subset \partial\Omega$. Note that with this coordinatization $x = [s, y]$ the convective flow is at unit speed (with respect to s) along the flowlines $y =$*constant* so (1.8) becomes the family of spatially one-dimensional problems

$$\begin{aligned} &\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} = -\hat{f}(t, s, y, v), \\ &v(t, \underline{s}(y), y) = v^*(t, y), \quad v(0, s, y) = \overset{\circ}{v}(s, y) \end{aligned} \tag{1.10}$$

with $\hat{f} = f + (\nabla \cdot \mathbf{v})v$, parametrized by $y \in Y$.

Of course we can also take f in (1.8) to depend also on j, v and then, at each spatial point, couple this with the modal switching rule $W[\cdot]$ to consider a family, parametrized by $y \in Y$, of convection/reaction/switching systems (1.1). Assuming Ω is given as in (1.9) and after again making the substitution $\tau = t - s$, we have the system

$$\begin{array}{lll} i. & \dfrac{\partial \tilde{v}}{\partial s} = -\hat{f}_{\tilde{\jmath}}(\tau + s, s, y, \tilde{v}), & \tilde{v}(\tau, \underline{s}, y) = v^*(\tau + \bar{s}, y) \\ ii. & \tilde{\jmath}(\cdot, s, y) = W[\tilde{v}(\cdot, s, y)] & \end{array} \tag{1.11}$$

(plus initial conditions) for the hybrid pair $[\tilde{v}(\tau, s, y), \tilde{\jmath}(\tau, s, y)]$. Each of these problems$_y$ would then be treated, as described earlier, by converting to a problem for v and $\sigma(\cdot, y)$. ■

We will finally wish to couple (1.11) with another quasilinear problem for an $\mathbb{R}^K$-valued unknown w. Our fully coupled system will then have the general form

$$
\begin{array}{lrcl}
i. & \dfrac{\partial v}{\partial s} & = & -f_j(t,x,v,w), \\
ii. & j(\cdot,x) & = & W[v(\cdot,x)], \\
iii. & \dfrac{\partial w}{\partial t} - \mathbf{L}w & = & g_j(t,x,v,w)
\end{array}
\tag{1.12}
$$

for $t > 0$, $x = (s, y) \in \Omega$. In incorporating (1.11) — i.e., (1.8) after the time shift — with the switching $\tilde{\jmath} = W[\tilde{v}]$, we have omitted the ˜s over v, j and are now writing t rather than τ. Note that the index j appears in (1.12-*iii*) well as (1.12-*i*) and that (1.11) is coupled with (1.12-*iii*) through this and the dependence of f_j, g_j on both v and w — but the switching (1.12-*ii*) depends only on the component v, not on w.

Under appropriate hypotheses we will obtain well-posedness for this fully coupled hybrid system in Section 5.

2 A motivating example: application to a bioremediation model

We begin by describing an ODE version of this model — considering only the reactive aspect with no spatial dependence. This is given by the hybrid system

$$
\begin{array}{c}
\dot{\alpha} = u - f_j(\alpha,\beta) \qquad \dot{\beta} = jg(\alpha,\beta) \qquad \dot{\pi} = -cj\beta \\
\text{switching rule: } j = W[\alpha].
\end{array}
\tag{2.1}
$$

Here $\beta = \beta(t)$ is the biomass, which can either be in an ACTIVE or in a DORMANT state, indicated by $j = 1, 0$, respectively and π is the level of a pollutant, cometabolized by the bacteria when active. The modal state transitions are determined by the concentration $\alpha = \alpha(t)$ of some critical nutrient through the input/output map $W[\cdot]$:

- The bacteria become dormant when this concentration $\alpha(t)$ drops below a critical threshhold value α_-,
- The bacteria are then re-activated when $\alpha(t)$ subsequently rises above a higher threshhold α_+.

The nutrient is fed into the system (input $u = u(t)$) and is also degraded in time and metabolized by active bacteria.

One might now consider a distributed version[2] of (2.1): at each spatial point x of a region Ω we have a biomass concentration $\beta(\cdot, x)$ and pollutant level $\pi(\cdot, x)$ with the activity state $j(\cdot, x)$ for the bacteria; both bacteria and pollutant are assumed to be attached to that point. As in (2.1), these satisfy

$$\dot{\beta} = jg(\alpha, \beta), \quad \dot{\pi} = -cj\beta; \qquad j = W[\alpha] \tag{2.2}$$

(as an independent evolution in time for each $x \in \Omega$, i.e., ordinary differential equations parametrized by x). We then postulate Ω as a subsurface region with a (known) groundwater flow having stationary stream velocity $\mathbf{v} = \mathbf{v}(x)$, carrying the nutrient convectively in solution — with specified flux at the boundary points where the flow enters Ω. Assuming the same form for the reactions as in the model (2.1), we then have the ordinary differential equations in (2.2) coupled through the convection/reaction equation

$$\alpha_t - \nabla \cdot [\alpha \mathbf{v}] = -f_j(\alpha, \beta). \tag{2.3}$$

[The nutrient supply now appears in the input boundary condition, rather than in the equation (2.3) itself.]

Remark 2.1. The systems (2.1) and (2.2), (2.3) are, of course, oversimplified caricatures of biologically correct models. For example, the equation $\dot{\pi} = -c\beta$ (for ACTIVE bacteria) is crude — and is actually impossible in that $\dot{\pi}$ necessarily vanishes after π gets to 0, perhaps requiring a further discontinuity in the dynamics at this point... More significantly, the discontinuous switching is intended as an approximate reduction of complexity where a biologically more realistic description would involve a complicated process developing on a faster time scale than we wish to consider. At present we are less concerned with the realism of these models than with the mathematical problems which they pose. ■

For this bioremediation problem, then, we are thus considering the coupled system (2.2)–(2.3) — to hold in $\mathcal{Q}_0 = (0, T) \times \Omega$) with initial data and

[2] A version of the model (2.1) was considered in [3] in the context of optimal control, selecting the nutrient supply rate $u(t)$ for effective bioremediation to optimize a cost/benefit criterion, balancing the cost of the (expensive) nutrient with reduction of the pollutant level. An earlier version of the spatially distributed problem (2.2), (2.3) was presented in [5] — again in an optimal control context — with a brief indication of the well-posedness argument.

suitable (boundary) source data for α. If we consider this for one-dimensional Ω as in [5]), then we recognize the system as (1.6), coupled with the ordinary differential equations $\dot{\beta} = jg(\alpha, \beta)$, $\dot{\pi} = -cj\beta$, parametrized by $s \in \Omega$ — giving (1.12-*iii*) on taking $w = [\beta, \pi]^{\mathsf{T}}$ with $L = 0$ and $g_j = j[g(\alpha, \beta), c\beta]^{\mathsf{T}}$.

3 The switching function $W[\cdot]$

Our modal switching will be determined independently at each spatial point s by switching rules (cf., e.g., [4]) equivalent to the *elementary hysteron* of [1]. We consider a causal map W from continuous scalar inputs $t \mapsto \omega(t)$ to the corresponding $\{0, 1\}$-valued outputs $t \mapsto \chi(t)$ for $\underline{\tau} \le t \le \bar{\tau}$: after specifying a pair of threshhold values (with $\alpha_- < \alpha_+$) and a consistent initial value $\chi(\underline{\tau})$, the output will be characterized by the conditions:

$$
\begin{array}{lll}
i. & \chi & = \begin{cases} 0 & \text{when } \omega < \alpha_- \\ 1 & \text{when } \omega > \alpha_+ \end{cases} \\
ii. & \chi & \text{is constant except for switching:} \\
 & & \begin{cases} 0 \curvearrowright 1 & \text{when } \omega \text{ increases across } \alpha_+ \\ 1 \curvearrowright 0 & \text{when } \omega \text{ decreases across } \alpha_- \end{cases}
\end{array}
\tag{3.1}
$$

[This is not a pointwise function: $W : \omega(t) \mapsto \chi(t)$ since the determination in (3.1) of the output $\chi(t)$ is history-dependent at times when the input $\omega(t)$ lies between the threshholds; following [1], we may emphasize the rate-independence of this history dependence.]

It is not difficult to see that (3.1) does, indeed, define $\chi = W[\omega]$ — e.g., in $\mathcal{L} = L^1(\underline{\tau}, \bar{\tau})$ — for each $\omega \in \mathcal{C} = C[\underline{\tau}, \bar{\tau}]$ and that the resulting map is causal. One must be a bit careful here about transversality — the meaning of “across” in (3.1-*ii*) — e.g., if $\chi(\tau-) = 0$ and ω rises to α_+ at time $\tau \in (\underline{\tau}, \bar{\tau})$ without actually crossing (i.e., if $\omega(t) \le \alpha_+$ on $[\tau, \tau + \varepsilon]$), then $\chi \equiv 0$ on $[\tau, \tau + \varepsilon)$. The possibility of such “anomalous points” as τ means that W cannot be continuous as a map: $: \mathcal{C} \to \mathcal{L}$.

Remark 3.1. This possibility of ambiguous anomalous points is a source of significant technical difficulty for the general theory. The treatment in [4] effectively modifies (3.1) to replace W by its closure, the minimal upper semicontinuous set-valued extension. This does provide some useful continuity, but has the consequence that the output of $W[\cdot]$ may turn out to be nonunique for certain input functions. We need not address this point here, since such possible nonuniqueness will not affect us in the context of (1.1): we will see that this could occur for j only on a nullset and cannot affect v at all. We also observe that (with ω continuous, so uniformly continuous on the compact set $[\underline{\tau},\bar{\tau}]\times[\underline{s},\bar{s}]$) "Zeno points" cannot arise here — i.e., one cannot have a limit of switching points within any bounded t-interval and, indeed, one has a uniform positive lower bound for the length of interswitching intervals.

Of particular importance for us, however, is the fact, already noted in [1], that W is isotone: assuming consistent initial data we have

$$\begin{array}{c} \textit{for continuous functions } \{\omega_k\} \textit{ with } \chi_k = W[\omega_k]: \\ \textit{If } \omega_1 \le \omega_2 \textit{ pointwise on } [\underline{\tau},\bar{\tau}], \textit{ then also } \chi_1 \le \chi_2. \end{array} \tag{3.2}$$

[To see this, note that the conclusion could be falsified only by having both $\chi_k(\tau-)=0$ with χ_1 switching $0 \curvearrowright 1$ at τ while χ_2 does not switch or the reverse of that — neither of which is consistent with (3.1) if $\omega_1 \le \omega_2$.] ■

The novel feature of our present concerns is the consideration of a *family* of such hysterons, parametrized by $s \in [\underline{s},\bar{s}]$, with corresponding initial data $\chi_0(s)$ and continuous input families $\omega(\cdot,s)$ subject to the assumption that $\chi_0(\cdot)$ and, for each fixed t, the function $s \mapsto \omega(t,s)$ is monotone decreasing:

$$s_1 < s_2 \quad \Rightarrow \quad \left\{ \begin{array}{ll} \chi_0(s_1) \ge \chi_0(s_2) & \text{and} \\ \omega(t,s_1) > \omega(t,s_2) & \text{for each } t \in [\underline{\tau},\bar{\tau}]. \end{array} \right. \tag{3.3}$$

Applying W as above, independently with respect to s, we obtain output $\chi(\cdot,s) = W[u(\cdot,s)]$ parametrized by s. Note from (3.2) that χ will also be monotone in s — $\chi(t,\cdot)$ is nonincreasing for each fixed t — so for each $t \in [\underline{\tau},\bar{\tau}]$ there exists some separating point $\hat{\sigma}(t) \in [\underline{s},\bar{s}]$ such that[3]

$$\chi(t,s) = \left\{ \begin{array}{ll} 1 & \text{for } \underline{s} \le s < \hat{\sigma}(t) \\ 0 & \text{for } \hat{\sigma}(t) < s \le \bar{s} \end{array} \right. \tag{3.4}$$

It is not immediately clear at this point — but will follow from Theorem 3.4 below — that the separation function $\hat{\sigma}(\cdot)$ in (3.4) is continuous.

[3] Compare (1.7). This says nothing about $\chi(t,s)$ when $s = \hat{\sigma}(t)$, but that information will not really be needed for our purposes. It is, of course, possible that $\hat{\sigma}(t) = \underline{s}$ or $\hat{\sigma}(t) = \bar{s}$ so (3.4) would not be a true separation.

Our major task in this section will be to find an alternative construction of the free boundary. I.e., assuming the form of (3.4), we seek an alternative construction of the separation function $\hat{\sigma}(\cdot)$, from the continuous input function $\omega : [\underline{\tau}, \bar{\tau}] \times [\underline{s}, \bar{s}] \to \mathbb{R}$, satisfying (3.3).

To this end we begin by introducing functionals $\Phi_\pm$, acting on strictly decreasing functions $s \mapsto \hat{\omega}(s)$, by

$$\Phi_-[\hat{\omega}] = \begin{cases} s \in [\underline{s}, \bar{s}] & \text{if } \hat{\omega}(s) = \alpha_+ \\ \underline{s} & \text{if } \hat{\omega}(\underline{s}) < \alpha_+ \\ \bar{s} & \text{if } \hat{\omega}(\bar{s}) > \alpha_+ \end{cases} \qquad \Phi_+[\hat{\omega}] = \begin{cases} s \in [\underline{s}, \bar{s}] & \text{if } \hat{\omega}(s) = \alpha_- \\ \underline{s} & \text{if } \hat{\omega}(\underline{s}) > \alpha_- \\ \bar{s} & \text{if } \hat{\omega}(\bar{s}) < \alpha_- \end{cases}$$

For future reference we note the easy estimate

$$d\hat{\omega}/ds \le -\beta, \ |\omega - \omega'| \le \delta \quad \Rightarrow \quad \begin{array}{l} |\Phi_-(\omega) - \Phi_-(\omega')| \le \delta/\beta \\ |\Phi_+(\omega) - \Phi_+(\omega')| \le \delta/\beta. \end{array} \tag{3.5}$$

Given the continuous input function ω, we next set $\varphi_\pm(t) = \Phi_\pm[\omega(t, \cdot)]$ so

$$[\varphi_-(t), \varphi_+(t)] = \mathcal{K}(t) := \{s \in [\underline{s}, \bar{s}] : \alpha_- \le \omega(t, s) \le \alpha_+\} \tag{3.6}$$

for $t \in [\underline{\tau}, \bar{\tau}]$. Note that the assumed continuity of ω ensures that $\varphi_\pm(\cdot)$ are each continuous.

From *any* given pair $[\varphi_-, \varphi_+]$ of continuous functions with $\varphi_- \le \varphi_+$ pointwise on $[\underline{\tau}, \bar{\tau}]$ and a consistent initial value $\sigma(\underline{\tau})$, we can solve a double obstacle problem to obtain a function σ on $[\underline{\tau}, \bar{\tau}]$, characterized by

i. $\varphi_-(t) \le \sigma(t) \le \varphi_+(t)$ for $\underline{\tau} \le t \le \bar{\tau}$
ii. $\sigma(\cdot)$ is constant when that is possible subject to *i.*, so it is increasing or decreasing only when this is forced — i.e., it increases when $\sigma(t) = \varphi_-(t)$ with f_- increasing and it decreases when $\sigma(t) = \varphi_+(t)$ with f_+ decreasing. (3.7)

We may write $\sigma(t) = \sigma(t; \varphi_-, \varphi_+)$ to indicate explicitly the dependence of σ on the pair of functions $\varphi_\pm$.

Remark 3.2. We recognize (3.7) as a one-dimensional version of Moreau's *sweeping process* (cf., e.g., [2]) determined by the moving convex set $t \mapsto \mathcal{K}(t) = [\varphi_-(t), \varphi_+(t)]$. Assuming some regularity, (3.7-*ii*) effectively requires that $\dot{\sigma} \ge 0$ when $\sigma < \varphi_+$ and $\dot{\sigma} \le 0$ when $\sigma > \varphi_-$ so σ satisfies the variational inequality: $(\zeta - \sigma)\dot{\sigma} \ge 0$ for $\zeta \in [\varphi_-, \varphi_+]$. From, e.g., [6, Theorem 4.5] — note also the treatment of the generalized play operator in [7, Proposition III.2.5] — we have existence of unique solutions for (3.7) with the well-posedness estimate:

$$|\sigma(t) - \tilde{\sigma}(t)| \leq \max\left\{|\sigma(0) - \tilde{\sigma}(0)|,\ \max_{0 \leq t' \leq t}\{\Delta(t')\}\right\} \tag{3.8}$$

where $\quad \sigma(\cdot) = \sigma(\cdot; \varphi_-, \varphi_+), \qquad \tilde{\sigma} = \sigma(\cdot; \tilde{\varphi}_-, \tilde{\varphi}_+),$
$\Delta(t) = \max\{|\varphi_-(t) - \tilde{\varphi}_-(t)|,\ |\varphi_+(t) - \tilde{\varphi}_+(t)|\}.$ ■

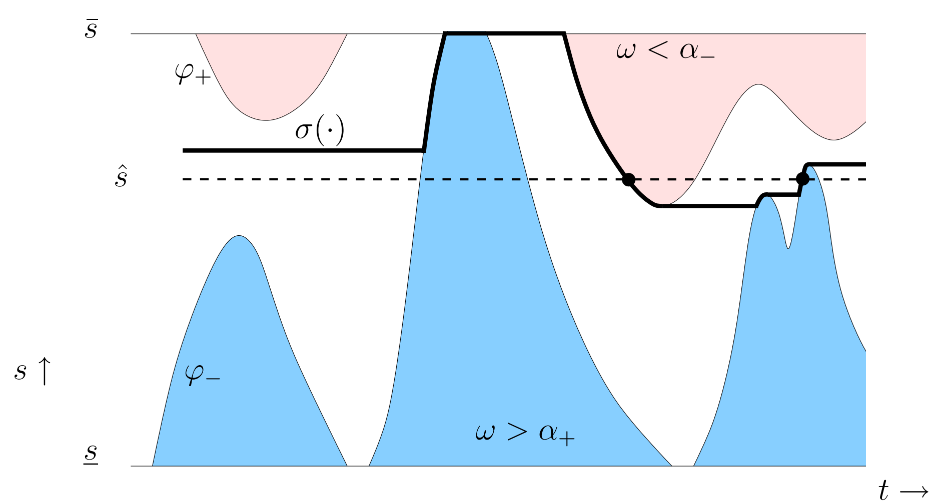

Figure 1: $\sigma(\cdot)$ and switching along $s = \hat{s}$

Later we will also need the following observation:

Lemma 3.3. *Let $\sigma = \sigma(\cdot; \varphi_-, \varphi_+)$ and $\hat{\sigma} = \sigma(\cdot; \hat{\varphi}_-, \hat{\varphi}_+)$ be obtained as in (3.7) with the same initial data at $\underline{\tau}$. Suppose, for each t, one has:*

$$\varphi_-(t) \leq \hat{\sigma}(t) \leq \varphi_+(t), \qquad \begin{cases} \hat{\varphi}_-(t) = \varphi_-(t) & \text{when } \hat{\sigma}(t) = \hat{\varphi}_-(t), \\ \hat{\varphi}_+(t) = \varphi_+(t) & \text{when } \hat{\sigma}(t) = \hat{\varphi}_+(t). \end{cases} \tag{3.9}$$

Then $\sigma, \hat{\sigma}$ are identical on $[\underline{\tau}, \bar{\tau}]$.

Proof: Given $\hat{\sigma}(\cdot)$, consider the use of (3.7) to construct σ. As t increases, our hypotheses (3.9) just ensure that $\sigma(\cdot)$ is forced to increase or decrease precisely when $\hat{\sigma}(\cdot)$ had been forced to increase or decrease by (3.7-*ii*), so they remain the same on all of $[\underline{\tau}, \bar{\tau}]$. ■

We may further remark that this gives, in particular,

$$\begin{aligned} \sigma(\cdot;\varphi_-,\varphi_+) \wedge \bar{s} &= \sigma(\cdot;\varphi_- \wedge \bar{s}, \varphi_+ \wedge \bar{s}), \\ \sigma(\cdot;\varphi_-,\varphi_+) \vee \underline{s} &= \sigma(\cdot;\varphi_- \vee \underline{s}, \varphi_+ \vee \underline{s}). \end{aligned} \tag{3.10}$$

We now turn to the principal substantive result of this section. The significance for us of Theorem 3.4 is that, subject to (3.3), the input ω equivalently determines the indicial output j of $W[\cdot]$ by

$$(3.6) \longrightarrow (3.7) \longrightarrow (3.4) : \omega \mapsto \varphi_\pm \mapsto \sigma \equiv \hat{\sigma} \mapsto \chi = j$$

without directly using the switching rules (3.1) and, further, that we will have the estimate (3.8) available.

Theorem 3.4. *Let ω be a continuous scalar function on $[\underline{\tau}, \bar{\tau}] \times [\underline{s}, \bar{s}]$ satisfying* (3.3)*. Define $\chi(\cdot, s) = W[\omega(\cdot, s)]$ on $[\underline{\tau}, \bar{\tau}]$ for each $s \in [\underline{s}, \bar{s}]$ with monotone initial data $\chi(\underline{\tau}, \cdot)$ and use* (3.4) *to obtain $\hat{\sigma}(\cdot)$. Also, define $\varphi_\pm$ from ω as in* (3.6) *and, with the consistent initial datum $\sigma(\underline{\tau})$, use* (3.7) *to obtain $\sigma(\cdot)$. These functions σ and $\tilde{\sigma}$ are then identical.*

Proof: Consider an arbitrary $s_* \in [\underline{s}, \bar{s}]$. For exposition, let us assume, e.g., that $\underline{s} < s_* < \sigma(\underline{\tau})$ so, by assumption, $\chi(\underline{\tau}, s_*) = 1$. If, the switching rules (3.1) cause $\chi(\cdot, s_*)$ to switch $1 \curvearrowright 0$ at some $t_1 > \underline{\tau}$, this can only be because the corresponding input $\omega(\cdot, s_*)$ decreases across α_-. This, however, means that we must have $\omega(t, s_*) \geq \alpha_-$ for $t \approx t_1-$ and $\omega(t, s_*) < \alpha_-$ for $t \approx t_1+$ whence $\varphi_-(t_1+) < s_*$. Of course, this ensures that $\sigma(\cdot)$ has been forced to decrease across s_* by t_1 so the switch $1 \curvearrowright 0$ of $\chi(\cdot, s_*)$ does not falsify (3.4) there: $\chi(t_1+, s_*) = 0$ with $s_* > \sigma(T_1+)$. Conversely, if $\sigma(\cdot)$ decreases across s_* at t_1, it can only be because $\varphi_-(t_1+) < s_*$ so $\omega(t, s_*) < \alpha_-$ for $t \approx t_1+$ and, by (3.1), we must have $\chi(\cdot, s_*)$ switching $1 \curvearrowright 0$ at t_1 so again (3.4) would not be falsified.

Proceeding this way along $s = s_*$ as t increases, we may argue similarly that (3.4) continues to hold for $\underline{\tau} \leq t \leq \bar{\tau}$, noting that the corresponding argument holds for switchings $0 \curvearrowright 1$ and $\sigma(\cdot)$ decreasing across s_*. As s_* was arbitrary, the asserted result follows. ■

Remark 3.5. While the assumed monotonicity (3.3) with respect to the parameter s was significant for our arguments, replacing the decrease by strict monotone increase

$$s_1 < s_2 \quad \Rightarrow \quad \begin{cases} \chi_0(s_1) \le \chi_0(s_2) & \text{and} \\ \omega(t, s_1) < \omega(t, s_2) & \text{for each } t \in [\underline{\tau}, \bar{\tau}]. \end{cases}$$

would permit corresponding considerations. [One either parallels the arguments above or applies these results with sign reversals for ω and for $\alpha_\pm$.] ■

4 Solving the simple system

Our primary task in this section is to show well-posedness for the system

$$\begin{array}{llcll} i. & \dfrac{\partial \tilde{v}}{\partial s} & = & -\tilde{f}_{\tilde{\jmath}}(\tau, s, \tilde{v}), & \tilde{v}(\cdot, \underline{s}) = \tilde{v}^*(\cdot) \\ ii. & \tilde{\jmath}(\cdot, s) & = & W[\tilde{v}(\cdot, s)] & \tilde{\jmath}(\underline{\tau}, \cdot) = \chi_0(\cdot) \end{array} \tag{4.1}$$

with hybrid state $[\tilde{v}, \tilde{\jmath}] \in \mathbb{R} \times \{0, 1\}$ at each $(\tau, s) \in \tilde{\mathcal{Q}} = [\underline{\tau}, \bar{\tau}] \times [\underline{s}, \bar{s}]$. We defer to Remark 4.2 a discussion of the relation of this to the original convection/reaction/switching system (1.1)-(1.2) before the substitution (1.4).

Theorem 4.1. *Assume*

- $\tilde{f}_j$ *($j = 0, 1$) are each strictly positive, continuous and Lipschitzian in $\tilde{v}$;*
- *the data $\tilde{v}^*(\cdot)$ is continuous;*
- *for some*[4] $s_* \in [\underline{s}, \bar{s}]$ *one has*

$$\chi_0(s) = \begin{cases} 1 & \text{for } \underline{s} \le s < s_* \\ 0 & \text{for } s_* < s \le \bar{s}. \end{cases} \tag{4.2}$$

[4]We impose a consistency condition: If $\tilde{v}^*(\underline{\tau}) \le \alpha_-$ we require $s_* = \underline{s}$; else we solve $d \mathring{v} / ds = -\tilde{f}_1(\underline{\tau}, s, \mathring{v})$ with $\mathring{v}(\underline{s}) = \tilde{v}^*(\underline{\tau})$ and require that $s_* = \bar{s}$ if $\mathring{v}(\bar{s}) \ge \alpha_+$.

Then the system (4.1) has a unique solution on $\tilde{\mathcal{Q}}$. Further, this solution depends continuously on the data $\tilde{v}^*$ and the constitutive functions $\tilde{f}_j$.

Proof: It is interesting that, under the given hypotheses, a solution can be constructed directly, rather than via a fixpoint problem. The construction on $\tilde{\mathcal{Q}}$ of the pair $\tilde{v}, \tilde{\jmath}$ proceeds as follows:

1. For each τ, solve on $[\underline{s}, \bar{s}]$ the ordinary differential equation (4.1-i) with fixed index $j \equiv 1$:

$$\frac{d\hat{v}}{ds} = -\tilde{f}_1(t, s, \hat{v}), \qquad \hat{v}(\underline{s}) = v(\tau, \underline{s}) = \tilde{v}^*(\tau), \tag{4.3}$$

   to obtain $\hat{v}$ on $\tilde{\mathcal{Q}}$. Note that the assumption that $\tilde{f}_1$ is Lipschitzian in its third argument ensures solvability of (4.3).

2. The positivity of $\tilde{f}_1$ ensures that each $\hat{v}(\tau, \cdot)$ is strictly decreasing in s; thus we can define $\hat{\varphi}_\pm(\tau) = \Phi_\pm(\hat{v}(\tau, \cdot))$ (as in (3.6), but using $\hat{v}$ as the input function) so

$$[\hat{\varphi}_-(\tau), \hat{\varphi}_+(\tau)] = \{s \in [\underline{s}, \bar{s}] : \alpha_- \le \hat{v}(\tau, s) \le \alpha_+\}; \tag{4.4}$$

3. Solve the double obstacle problem (3.7) (using the initial condition $\hat{\sigma}(\underline{\tau}) = s_*$) to obtain $\hat{\sigma} = \sigma(\cdot; \hat{\varphi}_-, \hat{\varphi}_+)$;

4. Use (3.4) in reverse to construct the index component $\tilde{\jmath}$:

$$\tilde{\jmath}(\tau, s) = \begin{cases} 1 & \text{when } \underline{s} \le s < \hat{\sigma}(\tau) \\ 0 & \text{when } \hat{\sigma}(\tau) < s \le \bar{s}; \end{cases} \tag{4.5}$$

5. Having obtained $\tilde{\jmath}$, we can finally solve on $[\underline{s}, \bar{s}]$ the ordinary differential equation (4.1-i) for each τ, obtaining $\tilde{v}$ on $\tilde{\mathcal{Q}}$.

We remark that the assumed continuity of the data and constitutive functions, together with the continuity of σ given by (3.7), ensure the continuity on $\tilde{\mathcal{Q}}$ of the constructed $\tilde{v}$.

We must now show that the pair $\tilde{v}, \tilde{\jmath}$ just constructed constitutes a solution of (4.1) — in particular, we must show that this $\tilde{\jmath}$ gives $\tilde{\jmath}(\cdot, s) = W[\tilde{v}(\cdot, s)]$, satisfying the switching rules (3.1), using the input $\tilde{v}(\cdot, s)$ for each

$s \in [\underline{s}, \bar{s}]$. In view of Theorem 3.4, this is equivalent to showing the correctness of $\hat{\sigma}$. The difficulty, of course, is that $\hat{\sigma}$ was obtained using $\hat{v}$ as input function — rather than $\tilde{v}$, which was as yet unknown. Thus, to justify our construction we must show that this distinction is nugatory; we note that this can be done by appealing to Lemma 3.3 to show that $\hat{\sigma} = \sigma$, where, given $\tilde{v}$ on $\tilde{\mathcal{Q}}$,

$$\sigma = \sigma(\cdot; \varphi_-, \varphi_+) \qquad \varphi_\pm = \Phi_\pm(\tilde{v}(\cdot, \cdot))$$

if we can only show that $\varphi_\pm, \hat{f}_\pm$ satisfy the hypotheses (3.9) of that lemma.

To this end we compare the defining differential equations — (4.3) for $\hat{v}$ and (4.1-*i*) for $\tilde{v}$ — noting first that $\tilde{v} \equiv \hat{v}$ on $[\underline{s}, \hat{\sigma}(\tau)]$ for each τ. To verify (3.9) we fix τ and, somewhat tediously, check the various cases. We begin with the prototypical case: $\underline{s} < \hat{\varphi}_- < \hat{\varphi}_+ < \bar{s}$.

> We have, then, $\hat{v}(\hat{\varphi}_-) = \alpha_+$ and $\tilde{v} = \hat{v}$ on $[\underline{s}, \hat{\sigma}] \supset [\underline{s}, \hat{\varphi}_-]$ so $\tilde{v}(\hat{\varphi}_-) = \hat{v}(\hat{\varphi}_-)$ and $\varphi_- = \hat{\varphi}_- \leq \hat{\sigma}$; if $\hat{\sigma} = \hat{\varphi}_-$, then $\tilde{v}(\hat{\sigma}) = \hat{v}(\hat{\sigma}) = \alpha_+$ so $\varphi_- = \hat{\sigma} = \hat{\varphi}_-$. Also, $\hat{v}(\hat{\varphi}_+) = \alpha_-$ so, as $\hat{\sigma} \leq \hat{\varphi}_+$ with $\tilde{v}$ decreasing, we must have $\tilde{v}(\hat{\sigma}) = \hat{v}(\hat{\sigma}) \geq \hat{v}(\hat{\varphi}_+) = \alpha_-$ so $\varphi_+ \geq \hat{\sigma}$; if $\hat{\sigma} = \hat{\varphi}_+$, then $\tilde{v}(\hat{\sigma}) = \hat{v}(\hat{\sigma}) = \alpha_-$ so $\varphi_+ = \hat{\sigma} = \hat{\varphi}_+$.

[Treatment of each of the endpoint cases is comparably straightforward — for example, if $\hat{\varphi}_- = \underline{s}$ one must have $\hat{v}(\underline{s}) = \tilde{v}(\underline{\tau}) \leq \alpha_+$ so $\tilde{v} \leq \alpha_+$ on $[\underline{s}, \bar{s}]$ whence also $\varphi_- = 0 \leq \hat{\sigma}$, $\ldots$— and we leave these cases as an exercise.] This, for each $\tau \in [\underline{\tau}, \bar{\tau}]$, gives applicability of Lemma 3.3 and so completes the justification of (4.1-*ii*).

To show uniqueness[5] we partially reverse the argument we have used: Given a solution $\tilde{v}, \tilde{\jmath}$ for (4.1), we may define $\hat{v}, \hat{\varphi}_\pm, \hat{\sigma}$ as earlier and essentially the same argument we have just used shows that Lemma 3.3 is applicable to give $\hat{\sigma} \equiv \sigma$ so the original construction recovers the same $\tilde{v}, \tilde{\jmath}$ with which we started — i.e., that construction provides the *only* solution.

---

[5]We note that (4.5) left $\tilde{\jmath}(\tau, s)$ undefined on the interface $s = \tilde{\sigma}(\tau)$. This is not really a problem, however, as it does not at all affect the differential equation (4.1-*i*) in the final step of the construction —and afterwards one can use (3.1) directly from $\tilde{v}$ to obtain $\tilde{\jmath}$ for those points on the interface itself.

At this point we remark that "anomalous points" for the switching rules (3.1), as mentioned in Section 2, can only occur only in $\{t, \sigma(t)\}$ (and, indeed, only within intervals of local constancy for σ); since these can actually occur, there may remain a slight bit of nonuniqueness in our specification of $\tilde{\jmath}$. We do note, however, that $\tilde{v}$ is entirely unaffected by this and that $\tilde{\jmath}$ is uniquely defined ae, so we treat this technical difficulty as irrelevant to our concerns.

We turn now to obtaining a continuity estimate for the dependence of solutions of (4.1) on the data and the constitutive functions. Thus, in addition to the problem above with constitutive functions $\tilde{f}_j$ and data $\tilde{v}^*$, s_* we consider another problem with corresponding constitutive functions $\tilde{f}'_j$ and data $\tilde{v}^{*\prime}$, s'_*. From our hypotheses on the constitutive functions, we note the existence of constants $\beta, \lambda, K > 0$ such that

$$\begin{array}{lll} i. & \tilde{f}_1(\tau,s,r) \geq \beta > 0 & \text{when } r \geq \alpha \\ ii. & \left|\tilde{f}_j(\tau,s,r) - \tilde{f}_j(\tau,s,r')\right| & \leq \ \lambda\,|r - r'| \\ iii. & \left|\tilde{f}_0(\tau,s,r) - \tilde{f}_1(\tau,s,r)\right| & \leq K \end{array} \tag{4.6}$$

We measure the difference between the problems (4.1) and (4.1)' by

$$\begin{gathered} |s_* - s'_*| \leq \varepsilon_0, \qquad |\tilde{v}^*(\tau) - \tilde{v}^{*\prime}(\tau)| \leq \varepsilon_1, \\ \left|\tilde{f}_j(\tau,s,r) - \tilde{f}'_j(\tau,s,r)\right| \leq \hat{\varepsilon}(\tau,s) \ \text{ with } \int_{\underline{s}}^{\bar{s}} \hat{\varepsilon}\,ds \leq \varepsilon_2 = \varepsilon_2(\tau) \end{gathered} \tag{4.7}$$

and, following the steps of the construction above, now proceed to estimate the difference between the corresponding solution pairs $[\tilde{v}, \sigma]$ and $[\tilde{v}', \sigma']$.

Subtract (4.3)' from (4.3), integrate, and then use (4.6-*ii*) and (4.7): the Gronwall Inequality now gives

$$|\hat{v}(\tau,s) - \hat{v}'(\tau,s)| \leq (\varepsilon_1 + \varepsilon_2)e^{\lambda[\bar{s}-\underline{s}]} =: \varepsilon_3. \tag{4.8}$$

In view of (4.6-*i*) and this, we use (3.5) to see that

$$|\varphi_\pm(\tau) - \varphi'_\pm(\tau)| \leq \varepsilon_3/\beta \tag{4.9}$$

and then use (3.8) to see that

$$|\sigma(\tau) - \sigma'(\tau)| \leq \max\{\varepsilon_0, \varepsilon_3/\beta\} =: \varepsilon_4. \tag{4.10}$$

Now, using (4.6-*i,iii*) and (4.7), we note that

$$\begin{aligned} \left|\tilde{f}_{\tilde{\jmath}}(\cdot,\tilde{v}) - \tilde{f}'_{\tilde{\jmath}'}(\cdot,\tilde{v}')\right| \ &\leq |\tilde{f}_{\tilde{\jmath}}(\cdot,\tilde{v}) - \tilde{f}_{\tilde{\jmath}}(\cdot,\tilde{v}')| + |\tilde{f}_{\tilde{\jmath}}(\cdot,\tilde{v}') - \tilde{f}_{\tilde{\jmath}'}(\cdot,\tilde{v}')| \\ &\qquad\qquad + |\tilde{f}_{\tilde{\jmath}'}(\cdot,\tilde{v}') - \tilde{f}'_{\tilde{\jmath}'}(\cdot,\tilde{v}')| \\ &\leq \lambda|\tilde{v} - \tilde{v}'| + K|\Delta| + \hat{\varepsilon}, \end{aligned} \tag{4.11}$$

with $\Delta = \{1 \text{ where } j \neq j';\ 0 \text{ where } j = j'\}$. Since $\int_{\underline{s}}^{\bar{s}} |\Delta|\, ds = |\sigma(\tau) - \sigma'(\tau)|$, applying the Gronwall Inequality to the difference of (4.1-*i*) and (4.1-*i*)' gives the final estimate

$$|\tilde{v}(\tau, s) - \tilde{v}'(\tau, s)| \leq (\varepsilon_1 + \varepsilon_2 + K\varepsilon_4)e^{\lambda[\bar{s} - \underline{s}]}, \tag{4.12}$$

showing uniform convergence $[\tilde{v}', \sigma'] \to [\tilde{v}, \sigma]$ as $\varepsilon_0, \varepsilon_1, \varepsilon_2 \to 0$ in (4.7). ■

Remark 4.2. There is no difficulty in using the chain rule to pass, by (1.4), from (1.1) to (1.6) and so to the equations (4.1-*i,ii*), but a comment is needed as to how we obtain the data for (4.1) from (1.2) — the constitutive functions $\tilde{f}_j$ and the input data $\tilde{v}^*$ are given on the image $\tilde{\mathcal{Q}}_0$ by (1.5) but we must construct $\tilde{f}_j$ on $\tilde{\mathcal{Q}} \setminus \hat{\mathcal{Q}}$ and $\tilde{v}^*$ for $\underline{\tau} = -\bar{s} \leq \tau < -\underline{s}$ to consider (4.1) on $\tilde{\mathcal{Q}}$ — and must do this in such a way as to match (1.2) on the segment $\{\tau = -s\}$ corresponding to the initial segment $\{t = 0\}$ for $\mathcal{Q}_0$.

We first observe that if $\tilde{\jmath}$ and the constitutive functions are defined on the triangle $\Delta = \{(\tau, s) \in \tilde{\mathcal{Q}} : \tau \leq -s\} \subset \tilde{\mathcal{Q}} \setminus \tilde{\mathcal{Q}}_0$, then the ordinary differential equation

$$d\tilde{v}/ds = -\tilde{f}_{\tilde{\jmath}}(\cdot, \tilde{v}) \qquad \tilde{v}(\tau, -\tau) = \overset{\circ}{v}(-\tau) \tag{4.13}$$

can be solved on $[\underline{s}, -\tau]$ for each $\tau \in [\underline{\tau}, -\underline{s})$ to obtain $\tilde{v}$ on Δ and so the required $\tilde{v}^*(\tau) = \tilde{v}(\tau, \underline{s})$ on $[\underline{\tau}, -\underline{s})$.

To this end, we begin by taking s_* as in (1.3) and then defining $\tilde{\jmath}$ on Δ as 1 for $s < s_*$ and as 0 for $s > s_*$, so consistent with (1.3) on $\{\tau = -s\}$. With $\tilde{f}_j > 0$ on Δ, as we will assume, this ensures that

$$\tilde{v}(\tau, s) > \tilde{v}(\tau, -\tau) = \tilde{v}^*(-\tau) \geq \overset{\circ}{v}(s_*) \geq \alpha_-$$

(so $\tilde{\varphi}_-(\tau) \geq s_*$) for $-s_* \leq \tau \leq -\underline{s}$, consistent with the choice of $\tilde{\jmath} = 1$ in this part of Δ. For $\underline{\tau} \leq \tau < -s_*$ we wish to choose $\tilde{f}_0(\tau, s, r) > 0$ so the solution of (4.13) satisfies $\alpha_- \leq \tilde{v}(\tau, s_*) \leq \alpha_+$ — possible for $\overset{\circ}{v}(-\tau) < \alpha_+$ when $-tau > s_*$. [It is not too difficult to see that this extension of $\tilde{f}_j$ to Δ can be done so as to maintain continuity.] Since we have arranged that $\alpha_+ \leq \tilde{v}(\tau, s_*)$ here, we will have $\sigma(\tau) \equiv s_*$ on $[\underline{\tau}, -s_*]$ from (3.7) so our choice of $\tilde{\jmath}$ on Δ is consistent with (4.1-*ii*).

It is now clear that the solution of (4.1) resulting from this construction will give the desired solution of the original problem (1.1)-(1.2) when restricted to $\tilde{\mathcal{Q}}_0$ and (1.4) used in reverse. On reversing (1.4) we see that Theorem 4.1 also shows well-posedness for the convection/reaction/switching problem (1.1)-(1.2). ∎

5 Solving the fully coupled system

We now turn to consideration of our final result: well-posedness for the fully coupled system discussed in Remark 1.1. Following that discussion, we take the system — after the substitution (1.4) — to have the form (1.12), i.e.,

$$
\begin{array}{lrll}
i. & \dfrac{\partial v}{\partial s} & = -f_j(t,x,v,w), & v\Big|_{s=\underline{s}(y)} = v^*(t,y) \\
ii. & j(\cdot,x) & = W[v(\cdot,x)], & \\
iii. & \dfrac{\partial w}{\partial t} - \mathbf{L}w & = g_j(t,x,v,w). &
\end{array}
\tag{5.1}
$$

on $\mathcal{Q} = [\underline{\tau},\bar{\tau}] \times \Omega$ with Ω as in (1.9) so $x = [s,y]$ with $\underline{s}(y) \le s \le \bar{s}(y)$ for $y \in Y \subset \mathbb{R}^{m-1}$. To the system (5.1) we then adjoin initial conditions on Ω at $t = \underline{\tau}$:

$$
w(\underline{\tau},x) = \mathring{w}(x) \qquad j(\underline{\tau},x) = \left\{ \begin{array}{ll} 1 & \text{if } \underline{s}(y) \le s < \sigma_0(y) \\ 0 & \text{if } \sigma_0(y) < s \le \bar{s}(y). \end{array} \right. \tag{5.2}
$$

Remark 5.1. We introduce the Banach space $\mathcal{W}$ of $\mathbb{R}^K$-valued functions on Ω determined by the norm

$$
\|w\|_{\mathcal{W}} = \sup_{y\in Y} \left\{ \int_{\underline{s}(y)}^{\bar{s}(y)} |w(s,y)|\, ds \right\} \tag{5.3}
$$

(where $|\cdot|$ denotes any convenient norm on $\mathbb{R}^K$). We then assume that $\mathbf{L}$ in (5.1-*iii*) is a densely defined linear operator on this space $\mathcal{W}$ (with any relevant homogeneous boundary conditions included in this definition) which is the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$ on $\mathcal{W}$. Note that standard semigroup theory then gives existence of constants M,δ such that

$$
\|\mathbf{S}(t)w_0\|_{\mathcal{W}} \le Me^{\delta t}\, \|w_0\|_{\mathcal{W}} \,. \tag{5.4}
$$

We will further assume that the function $\overset{\circ}{w}$ of (5.2) is in $\mathcal{W}$.

It is unnecessary to specify initial data $\overset{\circ}{v}$ for v since, given (5.2), we can obtain this by solving (5.1-*i*) at $t = \underline{\tau}$. We will assume that this induced $\overset{\circ}{v}$ and the σ_0 of (5.2) are consistent with (3.1-*i*), i.e., that $\alpha_- \leq \overset{\circ}{v} (\sigma_0(y), y) \leq \alpha_+$ for each y. Of course, the specification in (5.2) of j at $t = \underline{\tau}$ has really meant specification of the initial data $\sigma(\underline{\tau}, y) = \sigma_0(y)$ for the family of double obstacle problems we encounter from consideration of the family — parametrized by $y \in Y$ — of simple problems (5.1-*i,ii*) as in Section 4. As in the hypotheses for Theorem 4.1, we ask that the input boundary data v^* should be continuous in t for each y and now also ask that v^* be uniformly bounded in y for each t.

For the constitutive functions $f_j : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^K \longrightarrow (0, \infty)$ and $g_j : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^K \longrightarrow \mathbb{R}^K$ $(j = 0, 1)$, we assume continuity and — compare (4.6) — the existence of positive constants β, λ, K such that, uniformly,[6]

$$\begin{array}{lll} i. & f_1(\cdot, r, \omega) \geq \beta > 0 \qquad \text{when } r \geq \alpha_- & \\ ii. & \begin{array}{ll} |f_j(\cdot, r, \omega) - f_j(\cdot, r', \omega')| & \leq \lambda[|r - r'| + |\omega - \omega'|] \\ |g_j(\cdot, r, \omega) - g_j(\cdot, r', \omega')| & \leq \lambda[|r - r'| + |\omega - \omega'|] \end{array} & \\ iii. & \begin{array}{ll} |f_1(\cdot, r, \omega) - f_0(\cdot, r, \omega)| & \leq K \\ |g_1(\cdot, r, \omega) - g_0(\cdot, r, \omega)| & \leq K \end{array} & \end{array} \tag{5.5}$$

■

Theorem 5.2. *Assume the data* $v^*, \sigma_0, \overset{\circ}{w}$ *and the constitutive functions* f_j, g_j *are given as in Remark 5.1 above. Then the problem* (5.1)-(5.2) *has a unique solution* $[v, w, j]$, *depending continuously on the data.*

PROOF: We obtain the solution as the unique fixpoint of a contractive mapping $\mathcal{F}$ on the Banach space $\mathcal{C}_{\mathcal{W}} = \mathcal{C}([0, T] \to \mathcal{W})$ of continuous $\mathcal{W}$-valued functions on $[\underline{\tau}, \bar{\tau}]$ for which, with our choice of the parameter γ to be made later, we will use the exponentially weighted norm

$$\|w\|_{\mathcal{C}} = \sup_{0 \leq t \leq T} \left\{ e^{-\gamma(t - \underline{\tau})} \|w(t, \cdot)\|_{\mathcal{W}} \right\}. \tag{5.6}$$

[6] The *a priori* boundedness in (5.5-*iii*) might, for example, be an inherent property of f_j, g_j as functions but might alternatively be deduced — e.g., using suitable estimation to restrict the arguments v, w to compact sets.

To construct $\mathcal{F}$ we proceed as follows

- Given $\hat{w} \in \mathcal{C}_{\mathcal{W}}$, find $\hat{v}, \hat{\sigma}, \hat{\jmath}$ by solving (5.1-*i,ii*) as in Theorem 4.1, independently for each y — using $v^*(\cdot, y)$ as boundary data, $\sigma_0(y)$ for s_*, and using $f_j(t, s, y, r, \hat{w}(t, s, y))$ for $\tilde{f}_j(t, s, r)$ in (4.1-*i*).
- Having obtained $\hat{v}, \hat{\jmath}$ as above, solve $w_t = \mathbf{L}w + g(\cdot, w)$ with the inital data $\overset{\circ}{w}$, taking $g(\cdot, \omega) = g_{\hat{\jmath}(\cdot)}(\cdot, \hat{v}(\cdot), \omega)$, to obtain $w =: \mathcal{F}(\hat{w})$.

It is clear that a fixpoint of this map will provide a solution of (5.1)-(5.2) as desired so contractivity of $\mathcal{F} : \mathcal{C}_{\mathcal{W}} \to \mathcal{C}_{\mathcal{W}}$ (with respect to the metric $\|\cdot\|_{\mathcal{C}}$ for any suitable choice of γ) will imply existence of a unique solution and, by standard perturbation results for contractive mappings, will also show the continuous dependence on the data. We proceed, then, to estimate $\|\mathcal{F}(\hat{w}) - \mathcal{F}(\hat{w}')\|_{\mathcal{C}}$ for $\hat{w}, \hat{w}'$ in $\mathcal{C}_{\mathcal{W}}$.

At our first step we obtained $\hat{v}, \hat{\sigma}$ and $\hat{v}', \hat{\sigma}'$ (independently for each $y \in Y$) as in Theorem 4.1 — having conceptually replaced (5.1-*ii*) by (3.7) with (4.4). We now follow the well-posedness estimation there; note that we have assumed (5.5) to give (4.6). In comparing the problems, we have $\varepsilon_0 = 0$ and $\varepsilon_1 = 0$ for (4.7) since we are keeping the data fixed and have

$$\begin{aligned} |\tilde{f}_j(\cdot, r) - \tilde{f}_j'(\cdot, r)| &= |f_j(\cdot, r, \hat{w}(\cdot)) - f_j(\cdot, r, \hat{w}'(\cdot))| \\ &\le \lambda |\hat{w}(\cdot) - \hat{w}'(\cdot)| =: \hat{\varepsilon}(\cdot) \qquad \text{so} \end{aligned}$$
$$\int_{\underline{s}(y)}^{\bar{s}(y)} \hat{\varepsilon}(\cdot)\, ds \le \varepsilon_2(t) := \lambda \|\hat{w}(t, \cdot) - \hat{w}'(t, \cdot)\|_{\mathcal{W}} \le \lambda e^{\gamma(t - \underline{\tau})} \|\hat{w} - \hat{w}'\|_{\mathcal{C}}$$

for each t, y by the definitions of $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{C}}$. By (4.8) and (4.10) we then conclude that

$$\begin{aligned} |\hat{\sigma}(t, y) - \hat{\sigma}'(t, y)| &\le C\, e^{\gamma(t - \underline{\tau})} \|\hat{w} - \hat{w}'\|_{\mathcal{C}} \\ |\hat{v}(t, s, y) - \hat{v}'(t, s, y)| &\le C\, e^{\gamma(t - \underline{\tau})} \|\hat{w} - \hat{w}'\|_{\mathcal{C}} \end{aligned} \tag{5.7}$$

for a constant C independent of $\hat{w}, \hat{w}'$ and γ — indeed, C depends only on the constants β, λ, K of (5.5) and a bound ℓ on $[\bar{s}(y) - \underline{s}(y)]$. From (5.7) we have $\|\hat{v} - \hat{v}'\|_{\mathcal{C}} \le \ell C \|\hat{w} - \hat{w}'\|_{\mathcal{C}}$.

We now continue to the second step of the construction of $\mathcal{F}$ and wish to apply (5.4) to the representation

$$[w - w'](t) = \int_{\underline{\tau}}^{t} \mathbf{S}(t - \tau)\, [g(\tau) - g'(\tau)]\, d\tau,$$

obtaining

$$\|w(t) - w'(t)\|_{\mathcal{W}} \leq \int_{\underline{\tau}}^{t} Me^{\delta(t-\tau)} \, \|g(\tau) - g'(\tau)\|_{\mathcal{W}} \, d\tau. \tag{5.8}$$

Here, of course, $g(\tau) = g_{\hat{\jmath}(\tau,\cdot)}(\tau, \cdot, \hat{v}(\tau\, d), w(\tau, \cdot))$ and correspondingly for g' so we proceed to use (5.7) to estimate $g - g'$ — much as for (4.11) so similarly taking $\Delta = \{1 \text{ where } \hat{\jmath} \neq \hat{\jmath}';\ 0 \text{ where } \hat{\jmath} = \hat{\jmath}'\}$, etc. We then have

$$\begin{aligned} |g(\cdot) - g'(\cdot)| \ &= \left| g_{\hat{\jmath}}(\cdot, \hat{v}, w) - g'_{\hat{\jmath}'}(\cdot, \hat{v}', w') \right| \\ &\leq |g_{\hat{\jmath}}(\cdot, \hat{v}, w) - g_{\hat{\jmath}}(\cdot, \hat{v}', w')| \\ &\qquad + |g_{\hat{\jmath}}(\cdot, \hat{v}', w') - g_{\hat{\jmath}'}(\cdot, \tilde{v}', w')| \\ &\leq \lambda(|\hat{v} - \hat{v}'| + |w - w'|) + K|\Delta| \end{aligned}$$

whence, by the definition of $\|\cdot\|_c$ and (5.7),

$$\begin{aligned} \|g(\tau) - g'(\tau)\|_{\mathcal{W}} \ &\leq \sup_{y \in Y} \left\{ \int_{\underline{s}}^{\bar{s}} [\lambda(|\hat{v} - \hat{v}'| + |w - w'|) + K|\Delta|] \ ds \right\} \\ &\leq \lambda \|\hat{v} - \hat{v}'\|_{\mathcal{W}} + \lambda \|w - w'\|_{\mathcal{W}} \\ &\qquad\qquad + K \sup_{y \in Y} \{ |\hat{\sigma}(\tau, y) - \hat{\sigma}'(\tau, y)| \} \\ &\leq e^{\gamma(\tau - \underline{\tau})} \left[C' \|\hat{w} - \hat{w}'\|_c + \lambda \|w - w'\|_c \right] \end{aligned}$$

with $C' = (\lambda \ell + K)C$. Inserting this in (5.8) gives

$$\begin{aligned} &e^{-\gamma(t-\underline{\tau})} \|w(t) - w'(t)\|_{\mathcal{W}} \\ &\qquad \leq Me^{-\gamma(t-\underline{\tau})} \int_{\underline{\tau}}^{t} e^{\delta(t-\tau)} \, e^{\gamma(\tau-\underline{\tau})} \left[C'\|\hat{w} - \hat{w}'\|_c + \lambda\|w - w'\|_c \right] \, d\tau \\ &\qquad = \int_{\underline{\tau}}^{t} e^{-(\gamma-\delta)(t-\tau)} \, d\tau \, M \left[C'\|\hat{w} - \hat{w}'\|_c + \lambda\|w - w'\|_c \right] \\ \|w - w'\|_c \ &\leq \ \frac{MC'}{\gamma - \delta - M\lambda} \, \|\hat{w} - \hat{w}'\|_c \end{aligned}$$

so $\|\mathcal{F}(\hat{w}) - \mathcal{F}(\hat{w}')\|_c \leq (1/2)\|\hat{w} - \hat{w}'\|_c$ if one takes $\gamma \geq \delta + M\lambda + 2MC'$.

From our earlier observations this completes the desired well-posedness argument for (5.1). ■

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