

## OPTIMAL CONTROL OF A BIOREACTOR WITH MODAL SWITCHING\*

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**ABSTRACT:** We consider a model for bioremediation of a pollutant by bacteria in a well-stirred bioreactor. A key feature is the inclusion of dormancy for bacteria, which occurs when the critical nutrient level falls below a critical threshold. This feature gives a discrete component to the system due to the change in dynamics (governed by a system of ordinary differential equations between transitions) at switches to/from dormancy. After setting the problem in an appropriate state space, the control is the rate of injection of the critical nutrient and the functional to be minimized is the pollutant level at the final time and the amount of nutrient added. The existence of an optimal control and a discussion of the transitions between dormant and active states are given.

**Key Words:** Optimal control, existence, bioremediation, modal switching, hybrid system.

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### 1. Introduction

The concept of bioremediation has become increasingly significant in the treatment of environmental pollutant problems. The particular class of problems we now have in mind fits the following scenario:

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Some quantity of an undesirable pollutant is initially located with difficult access — e.g., an oil spill has soaked into the subsoil. Already present in this location is a bacterial population which can potentially metabolize<sup>†</sup> the pollutant, but which initially is in a dormant state, lacking some critical nutrient. The introduction of this nutrient is expected to re-activate the bacteria and so to induce the desired removal of the pollutant.

For a mathematical model, the relevant continuous state variables here are:

$\alpha$  = available amount of critical nutrient  
 $\beta$  = bacterial population (biomass)  
 $\pi$  = amount of pollutant remaining.

Note that we must include the distinction between ‘dormant’ and ‘active’ states for the biomass, say, by a (Boolean) indicator  $\chi = \{1 \text{ if ‘active’}; 0 \text{ if ‘dormant’}\}$ . This produces a hybrid state space: involving both discrete and continuous components. This discontinuous alteration of states is the ‘modal switching’ of our title.

We defer consideration of a Distributed Parameter System (DPS) model, treating the spatial variation which is likely to be significant for a realistic *in situ* bioremediation problem. See the related work of Butera, Fitzpatrick and Wypasek [3], in contaminant transport. Instead, for our present analysis, we treat a situation that is closer to a well-stirred tank bioreactor, in which the pollutant and the bacteria are present in spatially uniform, time-varying concentrations. Thus,  $\alpha, \beta, \pi$  will be functions of  $t$  only and the dynamics will involve ordinary differential equations between the state transitions. We note three recent papers involving optimal control of bioreactors, [8], [16], [9]. Two of these papers treat DPS models; the interesting feature of “well clogging” is treated in [16]. None of these papers considers the feature of dormancy which is a principal concern here. A paper by Bruni and Koch [2] treats a related idea of modeling the cell cycle with quiescent compartments. See the work by Bellomo and De Angelis [1] and Firmani, Guerri, and Preziosi [5] for modeling of the cellular interactions involved in immunology models involving tumor dynamics. The work by Fister and Panetta [6] involves optimal control and cell interaction in cancer chemotherapy.

It is clear from the above that determination of the rate  $u(\cdot)$  of injection of the critical nutrient as a function of time may be viewed as an optimal control problem, namely, balancing the costs of this procedure against the benefit of reduced pollutant. Feedback mechanisms are not easily implemented and we consider the problem in ‘open loop’.

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<sup>†</sup>This phrasing suggests that bacteria are directly consuming the pollutant, which, in turn, might suggest that the growth rate of bacteria would depend on the availability of pollutant. While consistently using this language here, we note that in our scenario the bacteria do not actually consume the pollutant directly, but that the consumption of other nutrients by the bacteria leads to the transformation of the pollutant into a less hazardous substance; this type of bioremediation is more properly called “cometabolism.” We note, for example, that paraxylene is degraded by metabolism while trichloroxylene is degraded by cometabolism [14], [17].

One notes that the modelling is somewhat uncertain due to the complexity of the structures involved and parameters which are difficult to estimate. For our present purposes we will, somewhat artificially, select a plausible structure which exemplifies the rather interesting technical details which may arise. Chief among these are

- the nature of the relevant cost functional, which might plausibly be a weighted sum of the amount of pollutant remaining at the end of the time considered and the total amount of nutrient added — effectively, the  $L^1$ -norm of the control function  $u$ , so one is working in a context involving a non-reflexive Banach space and
- the discontinuous transitions between dormant and active states which dominate our treatment of the model under consideration.

Our primary goal in this paper is to prove existence of an optimal control in an application. In Section 2, we model the problem in greater detail, giving the precise definition of solutions to our control system; existence of solutions is shown in Section 3. Section 4 is devoted to a relevant compactness result; the existence of an optimal control is then a corollary to that analysis. We will proceed in Section 5 to comment on some characterization of such optimal controls. Finally, in Section 6 we comment on some results and conjectures for related problems.

## 2. Modelling the problem

We are considering a finite horizon problem with a specified time interval  $[0, T]$  and with a given initial amount of pollutant. We seek to minimize the cost functional

$$\mathcal{J} := \int_0^T u(t) dt + b\pi(T). \quad (2.1)$$

We assume that the biomass may, at any time, be in either of two conditions (modes): *active* or *dormant*. For present purposes we model the transitions as governed entirely by the nutrient level and occurring ‘instantly’:

- if  $\alpha$  drops below  $\alpha_*$ , then the bacteria become dormant,
- if  $\alpha$  rises above  $\alpha^*$ , then the bacteria are reactivated,

where  $\alpha_*$  and  $\alpha^*$  are given constants with  $0 < \alpha_* < \alpha^*$ . We will discuss in more detail later such limit cases as, e.g.,  $\alpha(t)$  falling to  $\alpha_*$  at  $t = \tau$  and then rising or, perhaps, remaining at  $\alpha_*$  on an interval  $[\tau, \tau']$ .

We note [13], [7] that exogenous dormancy is imposed by an unfavorable chemical or physical environment: the uptake and the metabolism of organic compounds sometimes stops in such settings. On the other hand, [4], [10] the resulting spores will then germinate immediately on change to a favorable environment. We are assuming here that the bioreactor is rich in all nutrients required for the bacteria

for growth with a single exception — whose supply we may then identify as ‘critical.’ The addition of quantities of this nutrient will then constitute the controllable aspect of the dynamics. Implicit in this is that the pollutant will never itself be a ‘critical nutrient’ for the bacteria.

We assume that in a ‘full transition cycle’ (active  $\mapsto$  dormant  $\mapsto$  active) a (fixed) fraction  $(1 - \rho)$  of the bacteria fail to survive: the new value  $\beta(\tau'+)$  right after reactivation at  $t = \tau' \geq \tau$  is  $\rho\beta(\tau-)$  where  $\beta(\tau-)$  is the original value before dormancy. For convenience, we will allocate this loss of biomass entirely to the moment of transition to dormancy and take  $\beta(\tau+) = \rho\beta(\tau-)$ , noting that the value  $\beta(\cdot)$  is irrelevant for our control problem during the dormant interval until reactivation.

We are assuming here that there is a flux through the bioreactor with given flow rate. We assume the bacteria and the pollutant are insoluble and remain spatially fixed, but that the additive nutrient is soluble and is introduced by this flux (nominally, at rate  $u = \text{control}$ ) and also is carried in the outflow. When the biomass is dormant, then the dynamics takes a very simple form:

$$\dot{\alpha} = -\lambda\alpha + u \quad \beta, \pi \text{ constant} \quad (2.2)$$

where  $\lambda > 0$  is a constant determined by the flow rate in relation to the volume of the bioreactor. The constancy of  $\beta$  and  $\pi$  here is, of course, the meaning of ‘dormant’. When the bacteria are active, on the other hand, we have bacterial growth with rate  $\Gamma$  and have metabolism at rates  $\varphi, \psi$  of the critical nutrient and of the pollutant, respectively, so

$$\dot{\alpha} = -\lambda\alpha - \varphi\beta + u \quad \dot{\beta} = \Gamma \quad \dot{\pi} = -\psi\beta. \quad (2.3)$$

The injection rate  $u(\cdot)$  is necessarily non-negative; we do note that it is quite plausible (certainly as an idealization) to permit consideration of the injection of a bolus of nutrient, so  $u(\cdot)$  may contain  $\delta$ -functions; this is very much a question of time scales in the modelling, as any model we consider is an idealization of reality and a fixed amount of injected nutrient will appear as a  $\delta$ -function if it is being injected in a time which is short compared to the time scale of normal interest. The instantaneity of the state transitions is to be viewed similarly.

The (net) bacterial growth rate  $\Gamma$  which we consider is actually a balance between multiplication and death. It is plausibly an increasing function of the concentration of the critical nutrient: negative for  $\alpha$  near the transition level  $\alpha_*$ , but positive when  $\alpha$  would reach  $\alpha^*$ . For simplicity we take  $\Gamma = \gamma\beta$  with  $\gamma = \gamma(\alpha)$ . This is consistent with our underlying assumption that all other requirements for bacterial growth are abundantly met (biomass much smaller than the environmental carrying capacity), but we note that the alternative, saturation of  $\Gamma$  as the biomass might get closer to the carrying capacity, would not significantly affect our arguments. The dormancy phenomenon ensures that we need consider  $\Gamma$  only for  $\alpha \geq \alpha_* > 0$  and it will follow from the equations for  $\alpha$  that  $\alpha \leq \alpha_0 + \int_0^t u$ , implying boundedness. One then easily

gets a bound for  $\beta$  for finite  $T$ ; a Monod model [11] or term  $\Gamma = \gamma(\alpha)\beta/(k_1 + \beta)$  would, of course, bound  $\beta$  *a priori*.

It is plausible to take  $\varphi = \varphi(\alpha)$  and  $\psi = \psi(\alpha)$  — notationally suppressing the implicit specification that the bacteria cannot consume nonexistent pollutant:  $\psi = 0$  when  $\pi = 0$  — with  $\varphi(\alpha), \psi(\alpha) > 0$  for  $\alpha \geq \alpha_*$ ,  $\pi > 0$ . For simplicity, we assume the functions  $\Gamma$ ,  $\varphi$  and  $\psi$  are smooth on the relevant domains (except for the treatment of  $\psi$  at  $\pi = 0$ ), although simple continuity (or weaker!) would be sufficient for the existence arguments of Section 3. Note that with  $\alpha(0) \geq 0$ ,  $\beta(0) > 0$ ,  $\pi(0) > 0$  and  $\lambda, v, u \geq 0$ , we can expect to have  $\alpha(t), \beta(t), \pi(t) \geq 0$  for all time and will show this as a mathematical consequence below.

### 3. The State Equation

Our goal in this section is to investigate our state equation, proving the existence of a solution and establishing some properties of the solutions. First, let us note that the cost functional we consider is coercive for the  $L^1$ -norm — and, as has already been mentioned, we intend to admit  $\delta$ -functions in  $u(\cdot)$ . It is therefore convenient to reformulate the dynamics in terms of

$$\begin{aligned} U(t) &= \text{cumulative nutrient provided} = \alpha_0 + \int_0^t u(s)ds, \\ c(t) &= \text{nutrient consumed} = U(t) - \alpha(t), \\ \chi &= \{0 \text{ during dormancy; } 1 \text{ when active}\}. \end{aligned}$$

Thus, an admissible control function  $U(\cdot)$  will be any monotone (nondecreasing) function on  $[0, T]$ . We denote the set of all admissible controls by  $\mathcal{U}$ . For any given  $U(\cdot) \in \mathcal{U}$ , we expect  $c(\cdot)$ ,  $\beta(\cdot)$ ,  $\pi(\cdot)$  to be continuous — except for the jumps in  $\beta(\cdot)$  at times of transition from active to dormant.

In terms of the new variables, we now have, between modal transitions:

$$\dot{c} = \lambda(U - c) + \chi\varphi\beta, \quad \dot{\beta} = \chi\Gamma, \quad \dot{\pi} = -\chi\psi\beta \quad (3.1)$$

with  $\chi$  indicating the mode as above and with

$$\Gamma = \Gamma(U - c, \beta), \quad \varphi = \varphi(U - c), \quad \psi = \psi(U - c). \quad (3.2)$$

We assume that  $\Gamma(\cdot, \cdot)$ ,  $\varphi(\cdot)$ ,  $\psi(\cdot)$  are positive smooth (e.g.,  $C^1$ ) functions of their arguments for  $\alpha \geq \alpha_* > 0$  and  $\pi > 0$ . It would be reasonable to expect that the constitutive functions  $\Gamma, \varphi, \psi$  would each be nondecreasing in  $\alpha$ , since this only means that consumption and reproduction would not become more difficult (per bacterium) if more of the critical nutrient is available; we will find it convenient later to require for Theorem 6 that  $\Gamma, \psi$  be increasing functions of  $\alpha$ , but will not impose this as a requirement for the time being. We require that

$$\hat{\gamma} := \sup\{|\Gamma(\alpha, \beta)/\beta| : \alpha_* \leq \alpha \leq U(T)\} \quad (3.3)$$

is finite and set  $\hat{\varphi} := \max\{\varphi(\alpha) : \alpha_* \leq \alpha \leq \alpha^*\}$ . One might also permit some other dependencies, but we will simply take these as in (3.2).

We must now be somewhat more precise about the transition rules; compare [15]. Consider, first, the case when we are in the ‘dormant’ mode at time  $\tau-$ , so  $U(\tau-) - c(\tau) \equiv \alpha(\tau-) \leq \alpha_*$ . We will see that  $c(\cdot)$  is nondecreasing with  $0 = c(0) \leq c(t) \leq U(t)$ , so  $\alpha$  can increase only by control action: increasing  $U$  faster than  $c$ , perhaps admitting a jump in  $U$ . If  $\tau$  is such that  $\alpha(\tau+) \geq \alpha^*$ , then we *permit* a transition to the active mode (with  $(c, \beta, \pi)$  continuous at  $\tau$ ) with the transition *mandatory* if  $\alpha(\tau+) > \alpha^*$  or if continuation of the dormant mode would have  $\alpha(\cdot)$  increasing above  $\alpha^*$  for  $t$  near  $\tau+$ . Although we will later see that, for the cost functionals we consider, it can never be desirable to do so, it is admissible in principle that we might have  $\alpha(\tau+) = \alpha^*$  and then control so that  $\alpha(t) \equiv \alpha^*$  on some interval by taking  $u(t) = \dot{U}(t) = \lambda\alpha^*$  on this interval: we then consider it a further aspect of control to determine the length of this interval before either making the transition or having  $\alpha$  again drop strictly below  $\alpha^*$ .

Next, consider the case in which we are in the ‘active’ mode at time  $\tau-$ , with  $U(\tau-) - c(\tau) \equiv \alpha(\tau-) = \alpha_*$ . We now *permit* a transition to dormancy and *require* this transition if  $\alpha$  would fall below  $\alpha_*$  as  $t$  increases past  $\tau$ . At such a transition time,  $c, \pi$  are to remain continuous, but  $\beta(\tau+) = \rho\beta(\tau-)$  with  $\rho \in (0, 1)$  given. Again, it would be admissible to control so that  $\alpha(t) \equiv \alpha_*$  on some interval, by taking the control

$$u(t) = \dot{U}(t) = \lambda\alpha_* + \varphi(\alpha_*)\beta_*(t)$$

on this interval — obtaining  $\beta_*$  by solving:  $\dot{\beta} = \Gamma(\alpha_*, \beta)$  — and, as above, we then take the possibility and timing of a transition to dormancy to be a further element of control. Note that if we have a transition to dormancy at time  $\tau$ , then it is admissible (if one were to have a jump of at least  $(\alpha^* - \alpha_*)$  in  $U$ ) to have an immediate reactivation — this transition ‘back’ to the active mode is formally taken as subsequent to the transition to dormancy (although the transition times are numerically the same) and we do impose the loss of biomass associated with the full cycle.

Summarizing, a ‘solution’ of the system for a given admissible control function  $U \in \mathcal{U}$  consists of a (possibly empty, but finite) set of transition times  $(0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_N \leq T)$  alternately to active and dormant modes (for definiteness, we always assume that the system is dormant at  $t = 0$  so  $\tau_1$  is a transition time to the active mode) and the functions  $c, \beta, \pi$  subject to the defining conditions:

**[C1]** On each open interval  $(0, \tau_1), \dots, (\tau_{N-1}, \tau_N), (\tau_N, T)$ , the functions  $c, \beta, \pi$  satisfy the differential system (3.1).

**[C2]** The functions  $c, \beta, \pi$  are continuous with  $c(0) = 0, \beta(0) = \beta_0 > 0, \pi(0) = \pi_0 > 0$  — except that for transition from active to dormant we have  $\beta(\tau_{2k}+) = \rho\beta(\tau_{2k}-)$ .

[C3] At a transition time  $\tau$ , we have

$$\begin{aligned} U(\tau) - c(\tau) &=: \alpha(\tau) \geq \alpha^* & (\text{activation: } \tau = \tau_{2k+1}), \\ U(\tau-) - c(\tau) &=: \alpha(\tau_-) = \alpha_* & (\text{transition to dormancy: } \tau = \tau_{2k}). \end{aligned}$$

[C4] While active we have  $\alpha(t) \geq \alpha_*$ ; during dormancy,  $\alpha(t) \leq \alpha^*$ .

Our main result of this section is the following.

**THEOREM 1:** *Let  $U(\cdot) \in \mathcal{U}$  be any admissible control. Then the system has a solution in the sense above — which, moreover, satisfies*

$$0 \leq c(t) \leq U(t), \quad \forall t \in [0, T]. \quad (3.4)$$

$$c(t) \uparrow, \quad \pi(t) \downarrow; \quad \beta(t), \pi(t) > 0, \quad t \in [0, T], \quad (3.5)$$

and

$$\tau_{2k+2} - \tau_{2k+1} \geq \Delta := \frac{\alpha^* - \alpha_*}{\lambda U(T) + \hat{\varphi} \beta_0 e^{\hat{\gamma} T}}, \quad (3.6)$$

implying  $N \leq 1 + [T/\Delta]$ .

PROOF: For a given admissible control  $U(\cdot)$  and some  $T' \leq T$ , suppose  $(c, \beta, \pi)$  with transition times  $0 \equiv \tau_0 \leq \tau_1 \leq \dots \leq \tau_k$  is a ‘restricted solution’ of the dynamic system, i.e., satisfying the differential equations (3.1) and switching rules, on the restricted segment  $[0, T']$ . We first prove the *a priori* properties (3.4), (3.5), and (3.6) for such a solution. The existence of a solution on  $[0, T]$  will then follow easily by continuation in  $t$  and induction on  $k$ .

When the mode is ‘active’, we have  $\alpha(t) := U(t) - c(t) \geq \alpha_* > 0$ . Thus,  $c(t) \leq U(t)$  in any active intervals and at the beginning  $t = \tau_*$  of any interval of dormancy (noting also that  $c(0) = 0 \leq U(0)$  for the case  $\tau_* = 0$ ). Suppose there were some  $\bar{t}$  at which  $c(\bar{t}) > U(\bar{t})$ . Then there would be some  $\hat{t} < \bar{t}$  with  $c(\hat{t}) = U(\hat{t})$  and with  $c(t) > U(t)$  for  $t \in (\hat{t}, \bar{t}]$ . Since  $U(\cdot)$  is nondecreasing, we must then have  $c(\bar{t}) > U(\bar{t}) \geq U(\hat{t}) = c(\hat{t})$  while, on the other hand,

$$c(\bar{t}) - c(\hat{t}) = \int_{\hat{t}}^{\bar{t}} \dot{c}(t) dt = -\lambda \int_{\hat{t}}^{\bar{t}} [U(t) - c(t)] dt < 0.$$

This contradiction proves (3.4). This result is hardly surprising: it merely says that one cannot consume more nutrient than has already been put in.

Next, observe that  $\beta > 0$  always: we have  $\beta(0) > 0$  and  $|\dot{\beta}/\beta| \leq \hat{\gamma}$  between modal transitions (while, at the finitely many transitions to dormancy, we have a jump reduction in  $\beta$  but always by a fixed fraction so  $\beta$  remains positive there as well). It follows that  $c$  is nondecreasing (as  $\dot{c} = \lambda(U - c) + \chi\varphi\beta \geq 0$ ) so,  $c(t) \geq c(0) = 0$  and it then follows that

$$0 \leq U(t) - c(t) =: \alpha(t) \leq U(t) \leq U(T).$$

By the definition of  $\hat{\gamma}$ , we have  $\chi\Gamma \leq \hat{\gamma}\beta$  so  $\beta \leq \beta_0 e^{\hat{\gamma}T}$ . If we are to have an activation at time  $\tau_n$  and a transition back to dormancy at  $\tau_{n+1}$ , then (since  $U$  is increasing and  $\alpha(\tau_n+) \geq \alpha^*$ ,  $\alpha(\tau_{n+1}) \leq \alpha_*$ ), we must have

$$\begin{aligned} \alpha^* - \alpha_* &\leq c(\tau_{n+1}) - c(\tau_n) \\ &= \int_{\tau_n}^{\tau_{n+1}} \{\lambda[U(s) - c(s)] + \varphi(U(s) - c(s))\beta(s)\} ds \\ &\leq [\lambda U(T) + \hat{\varphi}\beta_0 e^{\hat{\gamma}T}](\tau_{n+1} - \tau_n). \end{aligned}$$

Condition (3.6) follows, bounding  $N$ , as noted; the monotonicity of  $\pi(\cdot)$  is clear.

With these observations in hand, we obtain existence of a solution by proceeding stepwise, terminating when  $t = T$ . To show existence it is sufficient — and simplest — to describe the particular solution, which is obtained by always making a modal transition as soon as possible. We may have  $U(\cdot)$  beginning with an immediate jump to (at least)  $\alpha^*$ , giving  $\tau_1 = 0$ ; otherwise there is an initial interval of dormancy until the (first) moment  $\tau_1$  at which — whether by a jump or by having the increase of  $U$  be more rapid than the increase of  $c$  — one has  $\alpha = U - c \geq \alpha^*$ , if this ever does occur. The system then satisfies the ‘active’ equation in (3.1) until a time  $\tau_2$  at which  $U - c = \alpha_*$  and one switches to ‘dormant’ mode with a jump decrease in  $\beta$ , and so on. We have already seen that the number of transitions is necessarily finite so this provides a solution in the sense defined earlier.  $\square$

From Theorem 1 we see that for any  $U(\cdot) \in \mathcal{U}$ , if  $(c, \beta, \pi)$  is a corresponding solution, then any active intervals (except possibly the last) are nontrivial: the length of such an interval is at least  $\Delta > 0$ . Note if  $\alpha^*$  were equal to  $\alpha_*$ , we would lose the significance of (3.6) providing a finite bound for  $N$ .

A key feature of our treatment of the transitions is that the rules allow some non-uniqueness of the solutions so one would not have well-posedness in the usual sense. Indeed, in this section we might have taken the constitutive functions  $\Gamma$ ,  $\varphi$  and  $\psi$  only to be continuous and one would then have the additional possibility of non-uniqueness through bifurcation within the modal intervals. One does, however, have a closure property: the limit of solutions is a solution as one might vary the data or the control function; compare the more detailed discussion in [15]. Aspects of this dominate the discussion in [15], etc., and this property is essential to our principal result on the existence of optimal controls. We will discuss this issue in the following section.

#### 4. Existence of Optimal Controls

The standard argument for existence of optima is to restrict consideration to a set of controls for which one has compactness in some sense (such that some minimizing sequence ‘converges’) and then, from this, to deduce adequate convergence of the state and the cost functional. Here, the compactness will be given by Helly’s Theorem (cf., e.g., [12]), but the nature of our dynamics (specifically, the possibil-



ity of discontinuous modal transitions) makes the subsequent treatment somewhat delicate.

**THEOREM 2:** *Let  $U^\nu \in \mathcal{U}$  be a pointwise convergent sequence of controls such that  $U^\nu(t) \rightarrow \bar{U}(t)$  for each  $t \in [0, T]$  with  $\bar{U} \in \mathcal{U}$ . Let  $(c^\nu, \beta^\nu, \pi^\nu)$  be corresponding solutions of the system with transition times  $0 \equiv \tau_0^\nu \leq \tau_1^\nu \leq \dots \leq \tau_{N_\nu}^\nu \leq T$ .*

*Then there exists a subsequence, again indexed simply by  $\nu$ , and limits  $(\bar{c}, \bar{\beta}, \bar{\pi})$  with transition times  $0 = \bar{\tau}_0 \leq \dots \leq \bar{\tau}_N \leq T$  such that, as  $\nu \rightarrow \infty$ ,*

$$\begin{cases} N_\nu = N & \text{for some fixed } N \geq 0 \\ \tau_n^\nu \rightarrow \bar{\tau}_n & \text{for each } n = 1, \dots, N \\ (c^\nu, \pi^\nu) \rightarrow (c, \pi) & \text{uniformly on } [0, T] \\ \beta^\nu \rightarrow \bar{\beta} & \text{uniformly on compact sets in each } (\tau_n, \tau_{n+1}), \end{cases} \quad (4.1)$$

[If desired, this subsequence can be chosen so the convergence:  $\tau_n^\nu \rightarrow \bar{\tau}_n$  will be monotone for each  $n$ .] Finally,  $(\bar{c}, \bar{\beta}, \bar{\pi})$  is a solution corresponding to the control  $\bar{U}(\cdot)$ .

PROOF: By the Dominated Convergence Theorem, we also have  $L^1$  convergence of  $U^\nu$  to  $\bar{U}$ ; as  $U^\nu(T) \rightarrow \bar{U}(T)$ , these are uniformly bounded so all the estimates obtained in the proof of Theorem 1 will hold uniformly in  $\nu$  — in particular, one has a bound on  $N = N_\nu = (\text{number of transitions})$  so we may again extract a subsequence with each  $N_\nu = N$  for some fixed  $N$ . Since  $[0, T]$  is compact, we may extract a further subsequence such that, for each given  $n = 1, 2, \dots, N$ , either  $\tau_n^\nu \downarrow \bar{\tau}_n$  or  $\tau_n^\nu \uparrow \bar{\tau}_n$ . Next, note that we have bounds (uniformly in  $\nu, t$ ) for  $\dot{c}^\nu$ ,  $\dot{\beta}^\nu$  and  $\dot{\pi}^\nu$  on the modal intervals. Thus  $\{c^\nu, \beta^\nu, \pi^\nu\}$  is equicontinuous there so, for some further subsequence, we have uniform convergence of  $c^\nu \rightarrow \bar{c}$ , and  $\pi^\nu \rightarrow \bar{\pi}$  on  $[0, T]$ ; and the convergence of  $\beta^\nu \rightarrow \bar{\beta}$  uniformly on each compact sub-interval of  $(\bar{\tau}_n, \bar{\tau}_{n+1})$ . [We have abused notation slightly by continuing to index simply by  $\nu$  through all the subsequence extractions.]

We must now verify that  $\{\bar{\tau}_1, \dots, \bar{\tau}_N\}$ , together with the limit functions  $(\bar{c}, \bar{\beta}, \bar{\pi})$  satisfy the set of conditions defining a solution of the system associated with the limit control function  $\bar{U}$ .

To verify [C1], we note that in integrated form we have — e.g., for  $[s, t]$  within any active interval as  $(\tau_1^\nu, \tau_2^\nu)$  —

$$c^\nu(t) = c^\nu(s) + \int_s^t \{\lambda[U^\nu(r) - c^\nu(r)] + \varphi(U^\nu(r) - c^\nu(r))\beta^\nu(r)\} dr \quad (4.2)$$

for  $\bar{\tau}_1 < s < t < \bar{\tau}_2$  for  $\nu \geq \bar{\nu}$  taking  $\bar{\nu}$  large enough to ensure that this implies also that  $\tau_1^\nu < s < t < \tau_2^\nu$ . Observe that, by our construction,  $c^\nu, \beta^\nu$  converge uniformly to  $\bar{c}, \bar{\beta}$  on  $[s, t]$  and  $U^\nu \rightarrow \bar{U}$  (pointwise, hence  $L^1$ ) with the integrand bounded uniformly in  $\nu$ . Thus, (4.2) gives, in the limit,

$$\bar{c}(t) = \bar{c}(s) + \int_s^t \{\lambda[\bar{U}(r) - \bar{c}(r)] + \varphi(\bar{U}(r) - \bar{c}(r))\bar{\beta}(r)\} dr \quad (4.3)$$

which, in view of the arbitrariness of  $s, t$ , gives the appropriate differential equation for  $\bar{c}$  on  $(\bar{\tau}_1, \bar{\tau}_2)$ . Essentially the same argument can be used to verify the equations for  $\bar{\beta}$  and for  $\bar{\pi}$  on this interval — and again for each of these functions on each of the relevant intervals (where nonempty) as appropriate.

Condition [C2] concerning the initial conditions is immediate, as the initial conditions are independent of  $\nu$ . To look at the correct jumps for  $\bar{\beta}$  at  $t = \bar{\tau}_{2k}$ , we introduce

$$\hat{\beta}^\nu(t) = \rho^{-k} \beta^\nu(t), \quad \tau_{2k}^\nu < t \leq \tau_{2k+1}^\nu$$

so  $\hat{\beta}^\nu$  is continuous across the transitions to dormancy. Thus, for some subsequence and some  $\hat{\beta}$ , we have  $\hat{\beta}^\nu \rightarrow \hat{\beta}$  uniformly in any closed interval  $[\bar{\tau}_{2k} - \varepsilon, \bar{\tau}_{2k} + \varepsilon]$  with  $\varepsilon > 0$  small enough that  $\bar{\tau}_{2k-1} < \bar{\tau}_{2k} - \varepsilon$ . Note that

$$\hat{\beta}(t) = \rho^{-k} \bar{\beta}(t) \quad \text{for } \tau_{2k}^\nu < t \leq \tau_{2k+1}^\nu$$

so the jumps in  $\bar{\beta}$  at  $\bar{\tau}_{2k}$  are correct.

The most delicate (and original) aspect of this analysis is the treatment of [C3], [C4]. To verify [C3] for  $n$  odd, note that for arbitrary  $\varepsilon > 0$  we have

$$U^\nu(\tau_n^\nu) \geq c^\nu(\tau_n^\nu) + \alpha^* \geq \bar{c}(\bar{\tau}_n) + \alpha^* - \varepsilon,$$

for large enough  $\nu$  since the uniform convergence  $c^\nu \rightarrow \bar{c}$  with  $\tau_n^\nu \rightarrow \bar{\tau}_n$  gives  $c^\nu(\tau_n^\nu) \rightarrow \bar{c}(\bar{\tau}_n)$ . By assumption,  $\bar{U}$  is continuous to the right so we also have

$$\bar{U}(\bar{\tau}_n) \geq \bar{U}(\bar{\tau}_n + \delta) - \varepsilon \quad \text{for } 0 < \delta \leq \delta(\varepsilon)$$

and  $U^\nu(\tau_n^\nu) \leq U^\nu(\bar{\tau}_n + \delta)$  by the monotonicity of each  $U^\nu$  and taking  $\nu$  large enough that  $\tau_n^\nu \leq \bar{\tau}_n + \delta$ . Combining these gives

$$\bar{c}(\bar{\tau}_n) + \alpha^* - \varepsilon \leq U^\nu(\tau_n^\nu) \leq U^\nu(\bar{\tau}_n + \delta) \rightarrow \bar{U}(\bar{\tau}_n + \delta) \leq \bar{U}(\bar{\tau}_n) + \varepsilon,$$

so  $\bar{U}(\bar{\tau}_n) \geq \bar{c}(\bar{\tau}_n) + \alpha^* - 2\varepsilon$  for arbitrarily small  $\varepsilon > 0$ , whence  $\bar{U}(\bar{\tau}_n) - \bar{c}(\bar{\tau}_n) \geq \alpha^*$  as desired. The argument for  $n$  even is similar, but slightly more delicate as we need equality. Given  $\varepsilon > 0$ , we have, for large  $\nu$ ,

$$U^\nu(\tau_n^\nu -) = c^\nu(\tau_n^\nu) + \alpha_* \leq \bar{c}(\bar{\tau}_n) + \alpha_* + \varepsilon.$$

By monotonicity,  $U^\nu(\bar{\tau}_n - \delta) \leq U^\nu(\tau_n^\nu -)$  for any  $\delta > 0$  with  $\nu$  so large that  $\tau_n^\nu > \bar{\tau}_n - \delta$ . Thus we have  $\bar{U}(\bar{\tau}_n - \delta) \leq \bar{c}(\bar{\tau}_n) + \varepsilon$  for  $\delta > 0$  so as  $\delta \rightarrow 0$  we have  $\bar{U}(\bar{\tau}_n -) \leq \bar{c}(\bar{\tau}_n) + \alpha_* + \varepsilon$  for arbitrary  $\varepsilon > 0$ , whence  $\bar{U}(\bar{\tau}_n -) - \bar{c}(\bar{\tau}_n) \leq \alpha_*$ . To show equality, we choose  $\delta > 0$  so that for large  $\nu$  one has  $\bar{\tau}_n - \delta < \tau_n^\nu$  so the system <sup>$\nu$</sup>  is ‘active’ at  $\bar{\tau}_n - \delta$  whence by [C4] <sup>$\nu$</sup>  we have

$$U^\nu(\bar{\tau}_n - \delta) - c^\nu(\bar{\tau}_n - \delta) \geq \alpha_*.$$

Now first taking  $\nu \rightarrow \infty$  and then  $\delta \rightarrow 0+$  (noting the continuity of  $\bar{c}$ ) we obtain  $\bar{U}(\bar{\tau}_n -) - \bar{c}(\bar{\tau}_n) \geq \alpha_*$  and so [C3].

The argument for [C4] is much like the final part of the argument above. For any (fixed)  $t$  during what is to be an ‘active’ interval for the limit problem (so  $\bar{\tau}_n < t < \bar{\tau}_{n+1}$ , with  $n$  odd), we will have  $\tau_n^\nu < t < \tau_{n+1}^\nu$  (so system $^\nu$  is also active) for large enough  $\nu$ . By [C4] $^\nu$  we then have  $U^\nu(t) - c^\nu(t) \geq \alpha_*$  and letting  $\nu \rightarrow \infty$  gives [C4] here. Similarly, for  $t$  in the interior of a dormant interval, we have system $^\nu$  dormant at  $t$  for large  $\nu$  whence  $U^\nu(t) - c^\nu(t) \leq \alpha^*$  and we get [C4] here as well when  $\nu \rightarrow \infty$ .

This completes our verification of [C1]–[C4] in the limit.  $\square$

This theorem provides the requisite form of ‘well-posedness’ under pointwise convergence of control functions and the existence of an optimizer for  $\mathcal{J}$  is then immediate.

**THEOREM 3:** *There is an optimal control, i.e.,  $\mathcal{J}$  attains its minimum  $\bar{\mathcal{J}}$  over admissible control functions  $U(\cdot)$  and corresponding solutions, subject to the system dynamics as described above.*

PROOF: Take a minimizing sequence  $\{U^\nu\}$  with associated solutions  $\{(c^\nu, \beta^\nu, \pi^\nu)\}$  (and transition times  $\{\tau_n^\nu\}$ ). Since  $U^\nu(T) \leq \mathcal{J} = \mathcal{J}^\nu \rightarrow \bar{\mathcal{J}}$  and each  $U^\nu(\cdot)$  is nondecreasing, by Helly’s Theorem we have existence of a subsequence such that  $U^\nu(t) \rightarrow \bar{U}(t)$  pointwise everywhere on  $[0, T]$  with  $\bar{U}$  admissible, nondecreasing, and  $\bar{U}(0) = \alpha_0$ . By Theorem 2, we can extract a subsequence for which  $U^\nu, c^\nu, \beta^\nu, \pi^\nu$  are suitably convergent to  $\bar{U}, \bar{c}, \bar{\beta}, \bar{\pi}$ , in such a way that  $(\bar{c}, \bar{\beta}, \bar{\pi})$  (and  $\{\bar{\tau}_n\}$ ) provide a solution for the admissible control  $\bar{U}$ . Clearly, the convergence of  $\pi^\nu(T) \rightarrow \bar{\pi}(T)$  and of  $U^\nu(T) \rightarrow \bar{U}(T)$  ensure that, in the limit,  $\mathcal{J} = \bar{\mathcal{J}}$ , i.e., that  $\bar{U}$  (together with auxiliary choices, if any, used to get this solution) is an optimal control.  $\square$

## 5. Characterization

In this section, we derive necessary conditions satisfied by the controls we consider. These are not intended be the full set of ‘first order (necessary) conditions for optimality’, which we hope to discuss in more detail in a subsequent paper, but only some preliminary comments on the characterization of optimal controls and optimally controlled solutions, especially with regard to the ‘switching structure’. In what follows, we let  $(\bar{U}, \bar{c}, \bar{\beta}, \bar{\pi})$  be optimal with transition times satisfying

$$\bar{\tau}_0 \equiv 0 \leq \bar{\tau}_1 \leq \bar{\tau}_2 \leq \cdots \leq \bar{\tau}_N \leq T.$$

[By our convention, the bacteria are dormant in  $[\bar{\tau}_{2k}, \bar{\tau}_{2k+1})$  and are active in  $[\bar{\tau}_{2k+1}, \bar{\tau}_{2k+2})$  for each  $k \geq 0$ .]; we also denote  $\bar{\alpha} = \bar{U} - \bar{c}$ .

Our first result is that the optimal control  $\bar{U}$  is continuous at  $t = T$ , i.e.,  $\bar{U}(T-) = \bar{U}(T)$ . [To see this, suppose  $\bar{U}(T-) < \bar{U}(T)$ . We then define  $\tilde{U}(t)$  to be the same as  $\bar{U}(T)$ , except redefined at  $t = T$  so  $\tilde{U}(T) = \bar{U}(T-)$ . Then  $\tilde{U} \in \mathcal{U}$  and  $(\bar{c}, \bar{\beta}, \bar{\pi})$  is a solution of the state equation under the control  $\tilde{U}$ . Clearly,

$$\mathcal{J}(\tilde{U}) = \mathcal{J}(\bar{U}) - [\bar{U}(T) - \bar{U}(T-)] < \mathcal{J}(\bar{U}),$$

which contradicts the assumed optimality of  $\bar{U}$ .] This result is not very surprising, since the possible jump at time  $t = T$  in the control  $U$  does not play any role in reducing the pollutant  $\pi$  at  $t = T$  so an optimal control  $\bar{U}$  will not incur the cost of such a jump. The following result is more interesting.

**THEOREM 4:** *Let the situation be as described above. Then (except possibly at the beginning or end) there are no non-trivial dormancy sub-intervals, i.e.,*

$$\bar{\tau}_{2k} = \bar{\tau}_{2k+1}, \quad \forall k \geq 1 \text{ with } 2k < N. \quad (5.1)$$

*In addition, if  $\bar{\tau}_1 > \bar{\tau}_0 := 0$  then  $\bar{U}(t) = 0$  on  $t \in [0, \bar{\tau}_1)$  and if  $N = 2M$  with  $\bar{\tau}_N < T$  then  $\bar{U}(t) = \bar{U}(\bar{\tau}_N)$  on  $t \in [\bar{\tau}_N, T]$ .*

PROOF: Suppose for some  $k \geq 1$  with  $2k < N$ , we have  $\bar{\tau}_{2k} < \bar{\tau}_{2k+1}$ . Note that the bacteria are dormant in  $[\bar{\tau}_{2k}, \bar{\tau}_{2k+1})$  and active on  $[\bar{\tau}_{2k-1}, \bar{\tau}_{2k})$  and  $[\bar{\tau}_{2k+1}, \bar{\tau}_{2k+2})$ . Pick any  $\hat{\tau} \in (\bar{\tau}_{2k}, \bar{\tau}_{2k+1})$ . One has

$$\bar{c}(\hat{\tau}) < \bar{c}(\bar{\tau}_{2k+1}), \quad \bar{\beta}(\hat{\tau}) = \bar{\beta}(\bar{\tau}_{2k}), \quad \bar{\pi}(\hat{\tau}) = \bar{\pi}(\bar{\tau}_{2k}).$$

Now define

$$\begin{aligned} \tilde{U}(t) &= \begin{cases} \bar{U}(t) & \text{on } [0, \hat{\tau}), \\ \bar{U}(t - \hat{\tau} + \bar{\tau}_{2k+1}) - \bar{c}(\bar{\tau}_{2k+1}) + \bar{c}(\hat{\tau}) & \text{on } [\hat{\tau}, T + \hat{\tau} - \bar{\tau}_{2k+1}), \\ \bar{U}(T) - \bar{c}(\bar{\tau}_{2k+1}) + \bar{c}(\hat{\tau}) & \text{on } [T + \hat{\tau} - \bar{\tau}_{2k+1}, T], \end{cases} \\ \tilde{c}(t) &= \begin{cases} \bar{c}(t) & \text{on } [0, \hat{\tau}), \\ \bar{c}(t - \hat{\tau} + \bar{\tau}_{2k+1}) - \bar{c}(\bar{\tau}_{2k+1}) + \bar{c}(\hat{\tau}) & \text{on } [\hat{\tau}, T + \hat{\tau} - \bar{\tau}_{2k+1}), \end{cases} \\ \tilde{\beta}(t) &= \begin{cases} \bar{\beta}(t) & \text{on } [0, \hat{\tau}), \\ \bar{\beta}(t - \hat{\tau} + \bar{\tau}_{2k+1}) & \text{on } [\hat{\tau}, T + \hat{\tau} - \bar{\tau}_{2k+1}), \end{cases} \\ \tilde{\pi}(t) &= \begin{cases} \bar{\pi}(t) & \text{on } [0, \hat{\tau}), \\ \bar{\pi}(t - \hat{\tau} + \bar{\tau}_{2k+1}) & \text{on } [\hat{\tau}, T + \hat{\tau} - \bar{\tau}_{2k+1}). \end{cases} \end{aligned}$$

Note that on  $[T + \hat{\tau} - \bar{\tau}_{2k+1}, T]$ , we could solve the differential equations with  $U = \tilde{U}$  to obtain  $\tilde{c}(\cdot)$ ,  $\tilde{\beta}(\cdot)$  and  $\tilde{\pi}(\cdot)$ . Thus  $(\tilde{U}, \tilde{c}, \tilde{\beta}, \tilde{\pi})$  is admissible and

$$\begin{aligned} \tilde{U}(T) &= \bar{U}(T) - \bar{c}(\bar{\tau}_{2k+1}) + \bar{c}(\hat{\tau}) < \bar{U}(T), \\ \tilde{\pi}(T) &\leq \tilde{\pi}(T + \hat{\tau} - \bar{\tau}_{2k+1}) = \bar{\pi}(T), \end{aligned}$$

which contradicts the optimality of  $(\bar{U}, \bar{c}, \bar{\beta}, \bar{\pi})$ :

$$\mathcal{J}(\tilde{U}) = \tilde{U}(T) + \omega(\tilde{\pi}(T)) < \bar{U}(T) + \omega(\bar{\pi}(T)) = \mathcal{J}(\bar{U}).$$

In the case  $\bar{\tau}_1 > \bar{\tau}_0 = 0$ , if  $\bar{U}(\bar{t}) > 0$  for some  $\bar{t} \in [0, \bar{\tau}_1)$ , we take  $\hat{\tau} \in (\bar{t}, \bar{\tau}_1)$  and repeat this argument with similar contradiction. In the case  $N = 2M$  and  $\bar{\tau}_N < T$ , if  $\bar{U}(\bar{\tau}_N) < \bar{U}(\bar{t})$  for some  $\bar{t} \in (\bar{\tau}_N, T]$  (as  $\bar{U}(T-) = \bar{U}(T)$ , we may assume  $\bar{t} < T$ ), then we take  $\hat{\tau} \in (\bar{t}, T)$  and repeat the argument to get a contradiction again. This completes our proof.  $\square$

**THEOREM 5:** *Suppose we additionally assume that  $\Gamma$  is nondecreasing and  $\psi$  increasing as functions of the argument  $\alpha$ . Then, under optimal control, transition to dormancy can only (possibly) occur to give a terminal dormant interval: an ‘internal’ transition to dormancy can never occur.*

PROOF: Suppose, to the contrary, we were to have an optimal controlled solution with one or more internal transitions to dormancy — by the previous theorem, necessarily with immediate transition back to activity by an impulsive control. Let  $\tau$  be the last such pair of transition times:  $\tau = \bar{\tau}_{2k-1} = \bar{\tau}_{2k}$  (with  $k > 1$ , so this is preceded by an active interval) and let  $T_1$  be the end of the active sub-interval initiated at  $\tau$ , i.e.,  $T_1 = T$  if the system remains active until then ( $N = 2k$ ), but  $T_1 = \bar{\tau}_{2k+1}$  if there would be a final interval of dormancy.

We wish to compare the presumed optimal  $(\bar{U}, \bar{c}, \bar{\beta}, \bar{\pi})$  with what would occur if we were to use ‘the same control’ (in a sense to be described below) on  $[\tau+, T_1]$  *without* the transition cycle to dormancy at  $t_{2k-1}$  with immediate re-activation; by our definition of the control structure, this alternative would also constitute an admissible control. As in the earlier arguments, if we can show that the alternative would lead to a decrease of  $\mathcal{J}$ , then we will have shown that this last transition pair could not have occurred under optimal control — and so, recursively, that there are *no* internal transition pairs.

It is easiest to make the comparison after making a change of variable for time during the active sub-interval  $[\tau+, T_1]$ , introducing a new ‘time variable’  $s = s(t)$  by setting

$$\dot{s} = \beta(t) \quad \text{so } s(t) = \int_{\tau}^t \beta(\hat{t}) d\hat{t} \quad (5.2)$$

for this sub-interval. We set  $s_* = s(T_1)$ , using  $\bar{\beta}$  in (5.2), so the sub-interval has become  $[0, s_*]$ ; abusing notation slightly, we continue to write  $(U, \alpha, \dots)$  for the control and state variables, now considered as functions of  $s \in [0, s_*]$  — and, similarly,  $(\bar{U}, \bar{\alpha}, \dots)$  for the optimal set.

We now use a subscript  $_s$  to indicate differentiation with respect to  $s$ . In this context, on the active sub-interval  $(0, s_*]$  the dynamics is given by

$$(\alpha - \bar{U})_s = -\varphi(\alpha) - \lambda/\beta, \quad \beta_s = \Gamma(\alpha, \beta)/\beta, \quad \pi_s = -\psi(\alpha) \quad (5.3)$$

both for the optimal controlled solution and for our comparison solution, which we denote by  $(\tilde{\alpha}, \tilde{\beta}, \tilde{\pi})$ . Note that we are taking the control function  $\bar{U}$  to be the same in each case as a function of  $s$  — although, since  $\tilde{\beta} \neq \bar{\beta}$  in (5.2), the relation of  $s$  to  $t$  will be different and the control function  $\tilde{U}$  will not generally be the same as the original putative optimal control  $\bar{U}$  as a function of  $t$ . What is really different in the comparison case, however, is the initial condition at  $s = 0$ , corresponding to the optimal controlled solution at  $t = \tau+$ . If we set

$$\alpha_{\dagger} = \alpha \Big|_{t=\tau} \quad \beta_{\dagger} = \beta \Big|_{t=\tau-}, \quad \pi_{\dagger} = \pi \Big|_{t=\tau},$$

then the initial conditions for (5.3) will be:

$$\begin{aligned} \bar{\alpha}(0) &= \alpha_{\dagger} & \bar{\beta}(0) &= \rho\beta_{\dagger} & \bar{\pi}(0) &= \pi_{\dagger} \\ \tilde{\alpha}(0) &= \alpha_{\dagger} & \tilde{\beta}(0) &= \beta_{\dagger} & \tilde{\pi}(0) &= \pi_{\dagger}. \end{aligned} \tag{5.4}$$

From (5.3) with (5.4) we now observe that  $\tilde{\alpha} > \bar{\alpha}$  and  $\tilde{\beta} > \bar{\beta}$  on  $(0, s_*]$  whence  $\tilde{\pi} < \bar{\pi}$  on  $(0, s_*]$ . To see this, we first note that  $\bar{\beta}(0) < \tilde{\beta}(0)$  because of the assumed failure of a fraction  $(1 - \rho)$  of the bacteria to survive the switching cycle (supposedly) involved in the optimal controlled solution, so there is a maximal  $\hat{s}$  in  $[0, s_*]$  with  $\tilde{\beta} > \bar{\beta}$  (and  $\lambda/\tilde{\beta} < \lambda/\bar{\beta}$ ) on  $[0, \hat{s})$ . Using this in the first equation of (5.3) then gives<sup>‡</sup>  $\tilde{\alpha} > \bar{\alpha}$  on  $(0, \hat{s}]$ . It follows that  $\gamma(\tilde{\alpha}) \geq \gamma(\bar{\alpha})$  on  $[0, \hat{s}]$  so  $\tilde{\beta} > \bar{\beta}$  there — whence, by the maximality of  $\hat{s}$ , we must have  $\hat{s} = s_*$ . Since we have assumed  $\psi(\cdot)$  is (strictly) increasing, we then have  $-\psi(\tilde{\alpha}) < -\psi(\bar{\alpha})$  on  $(0, s_*]$  so  $\tilde{\pi} < \bar{\pi}$  there; in particular,  $\tilde{\pi} < \bar{\pi}$  at the end of the interval.

We can now obtain the ‘actual’ alternative control function  $\tilde{U}$  (as a function of  $t$ ) by solving  $d\tilde{s}/dt = \tilde{\beta}(\tilde{s})$  with  $\tilde{s}(\tau) = 0$  and then setting

$$\tilde{U}(t) = \begin{cases} \bar{U}(t) & \text{for } 0 \leq t \leq \tau \\ \bar{U}(\tau + \tilde{s}(t)) & \text{for } \tau \leq t \leq T_2 \end{cases}$$

where  $\tilde{s}(T_2) = s_*$ . Since  $\tilde{\beta} > \bar{\beta}$  on the interval, a comparison with (5.2) shows that  $T_2 < T_1 \leq T$ . We extend  $\tilde{U}$  as constant (equal to  $\bar{U}(T)$ , of course) on  $[T_2, T]$  and note that

$$\tilde{\pi} \Big|_{t=T} \leq \tilde{\pi} \Big|_{t=T_2} < \bar{\pi} \Big|_{s=s_*} = \bar{\pi} \Big|_{t=T}$$

so (2.1) gives  $\tilde{\mathcal{J}} < \bar{\mathcal{J}}$ . This shows that the original scenario could not have been optimal, as assumed.  $\square$

We have shown that, for this particular problem, the switching structure under optimal control is fairly trivial and one might well ask why it was important to consider at all the full set of complications attendant on treatment of this as a hybrid system. We may note that more complicated structures seem likely to occur (as noted in the next section) for related problems, but a more direct answer would be that our treatment is strongly affected by the *possibility* of repeated transitions, which could not be ruled out even here without an argument embedded in this general hybrid setting.

## 6. Further remarks

We begin by noting that several variants of (2.1) might also be of interest. For example, if one would consider termination of nutrient injection (so  $u(t) \equiv 0$  for

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<sup>‡</sup>This would be easy if  $\bar{U}$  were smooth. One could set  $\Delta(s) := \tilde{\beta}(s) - \bar{\beta}(s) > 0$ , set  $\beta(s; \eta) := \bar{\beta}(s) + \eta\Delta(s)$  for  $0 \leq s \leq 1$ , solve the differential equation using  $\beta = \beta(\cdot; \eta)$  to get  $\alpha(s; \eta)$ , and differentiate with respect to  $\eta$  to see that  $z = \partial\alpha/\partial\eta$  satisfies  $z_s = -\varphi'(\alpha)z + (\lambda/\beta^2)\Delta$  ( $z(0) = 0$ ) so  $z > 0$  for  $s > 0$  — with a uniform positive lower bound since we consider  $\varphi'(\alpha)$  on a compact set. We can then get the present case by a limit argument.

$t > 0$ ) but permit the system to continue until, at some time  $T' \geq T$ , one had transition to permanent dormancy, one might use  $\pi(T')$  rather than  $\pi(T)$  in (2.1). A further variant would be the use of an increasing function  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of the amount of residual pollutant as a generalization of the term  $b\pi$ .

Somewhat more interesting for consideration is the time-optimal problem:

Fix an ‘acceptable pollutant level’  $\pi_* > 0$  and, defining  $T$  as the time at which  $\pi(T) \leq \pi_*$  (for some specified ‘acceptable pollutant level’  $\pi_*$ ), determine the control so as to minimize

$$\mathcal{J} = \mathcal{J}_2 := \int_0^T u(t) + aT + b \int_0^T \pi(t), \quad \text{subject to } \pi(T) \leq \pi_* \quad (6.1)$$

with  $a, b \geq 0$  given.

Note that the first term in  $\mathcal{J}$  is, in this case as for (2.1), the cost of adding nutrient — taken as simply proportional to the total amount added.

For this case we can again prove existence of optimal controls by an argument rather similar to that given for the Corollary following Theorem 2.

**COROLLARY:** *There is an optimal control, i.e.,  $\mathcal{J}_2$  attains its minimum over admissible functions  $U(\cdot)$  and corresponding solutions, subject to the system dynamics as described above.*

**PROOF:** For the cost functional  $\mathcal{J}_2$  we must modify the previous argument slightly, since we then have  $T^\nu$  instead of  $T$ . The simplest approach is to fix  $T^*$  bounding  $T^\nu$  for some minimizing sequence  $\{[U^\nu, T^\nu], \dots\}$  and define  $U^\nu(t) := U^\nu(T^\nu)$  for  $T^\nu < t \leq T^*$  such that  $\{U^\nu(T^*)\}$  is bounded. By Theorem 1 we may extend the solution  $\{\dots, c^\nu, \beta^\nu, \pi^\nu\}$  to  $[0, T^*]$  and then proceed with our previous argument — now with the prior extraction of a subsequence so  $T^\nu \rightarrow \bar{T}$ . The argument ensures that we have a solution in the limit — as a solution on  $[0, T^*]$  but this certainly means that the restriction to  $[0, \bar{T}]$  is a solution there. It is then clear that

$$\mathcal{J}_2(U^\nu) \rightarrow \mathcal{J}_2(\bar{U}).$$

Hence  $\mathcal{J}_2$  attains its minimum using this limit control.

We must still verify satisfaction of the constraint. The uniform convergence and uniform continuity imply  $\pi^\nu(T^\nu) \rightarrow \bar{\pi}(\bar{T})$  so the constraint:  $\bar{\pi}(\bar{T}) \leq \pi_*$  will also be satisfied in the limit, provided it is satisfied for each  $\nu$ . We conclude by returning to show that it was, indeed, possible in this setting to have a minimizing sequence as assumed. To see this, a single exemplar satisfying the constraint will be sufficient. We proceed by choosing some fixed  $\bar{\alpha} > \alpha^*$  and defining the control  $U$  as starting with an initial jump at  $t = 0$  to  $\bar{\alpha}$  (so  $\tau_1 = 0$ ) and then keeping  $\alpha(t) \equiv \bar{\alpha}$  (i.e.,  $U(t) = c(t) + \bar{\alpha}$ ) while solving the ‘active’ equation in (3.1) forward ‘as long as necessary’. Note that this gives  $\gamma \equiv \gamma(\bar{\alpha})$  so  $\beta(t) = \beta_0 e^{\gamma(\bar{\alpha})t}$  and we have  $\dot{c} = \lambda\bar{\alpha} + \varphi(\bar{\alpha})\beta$  so there is no difficulty with this construction. As this gives  $\psi(\bar{\alpha}, \pi) \geq \psi_* > 0$ , we have  $\dot{\pi}$  bounded from above strictly below 0 here and so

are assured that  $\pi$  must decrease to  $\pi_*$  at some stopping time. This completes the proof.  $\square$

It would also be of interest to consider an infinite horizon problem with the continued introduction of further pollutant (at some deterministic or stochastic rate) and corresponding modification of the equations. We conjecture that optimal control for that problem would be asymptotically periodic with repeated transitions between the dormant and the active states.

Also of possible future interest would be the consideration of a class of models in which the flow rate through the reactor (determining  $\lambda$  in the system dynamics) would also be controllable. One might also have coupling with a holding tank for incoming pollutant prior to the bioreactor — with mobility (some solubility,...) for the pollutant and then a capacity limit for the holding tank and a cost for any pollutant in the outflow. We defer consideration of these and other possibilities to a future time.

Finally, we remark on the possibility of constructing a discretization scheme to compute approximate solutions of the system. The difficulty in this is closely related to the possibility of bifurcation in the solution — there are situations which can occur in which arbitrarily small perturbations of the dynamics (comparable, e.g., to discretization error) can lead to the possibility of a significant change in some transition and so to significant changes in the subsequent evolution. The best one can hope for is that limits of solutions for the discretization will be solutions of our system and that, conversely, every solution of the original system can be obtained in this way. It is, in fact, possible to construct such a scheme, but consideration of the details will also be deferred.

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