Some aspects of modeling with discontinuities

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Abstract

Despite the aphorism, "Nature does not make jumps," it is frequently useful to work, either descriptively or prescriptively, with simplified models which involve switching between different modes of evolution. We describe a variety of examples of such modeling with particular attention to some situations in which the interpretation of the reduced model is a matter of concern.

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1 Introduction

Mathematical models are always created, selected, and analyzed with a purpose and we keep this functionality at the forefront of our present concern: convenience is one of the major desiderata in the selection of appropriate models and our present theme is the frequent convenience of involving discontinuities in our modeling despite the well-known aphorism, *"Natura non facit saltus."*[1]

To this end, we begin by noting an important distinction between *descriptive*[2] and *prescriptive* modes of modeling: the first is what a scientist does in trying to understand the various patterns arising in the world; the second

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[1] *"Nature does not make jumps"* (attributed to Newton, Leibniz, Linnaeus,. . .)

[2] Within this category we could further have distinguished between phenomenological models and those ostensibly based on first principles. Noting that the latter still involve

is what a composer or an engineer does in designing artificial patterns for various purposes. E.g., in viewing leonine behavior the first is the modality of the naturalist while the second is the approach of a lion tamer.

We are led to ask, in each of these contexts, *What are the characteristics of a "good model"?* For descriptive settings, a model is "good" if it simplifies the real world enough to make analysis/computation feasible while retaining enough of the complexity of the world for its predictions to give an acceptable approximation of the real events one might observe within the range of concern. For prescriptive settings, a model is "good" if the desired behavior is attained within the formal framework of the model and if also the model may feasibly be implemented in the real world. It is this last which links the two contexts:

The prescriptive model must be a descriptive model of its implementation.
so, effectively, its interpretation must be much the same.

All this is very much a matter of scale. At the quotidian scale we may think of our cycle of wakefulness and sleep as an alternation of characteristic modes with discontinuous transitions so the precise time of awakening is a significant and more-or-less predictable event. On a yearly scale, however, these discrete events blur into the texture of our lives and we might work with averages; on a scale of many years we might consider how these averages slowly change. On the longer scales the individual discontinuities have disappeared, although they remain part of the underlying pattern. This alteration of viewpoint to consider long term averages will be considered more technically later.

Conversely, in the natural world (neglecting quantum physics) discontinuities typically arise similarly in a context of multiscaling: the apparent discontinuity often can be resolved as a transition on a time scale finer than the 'normal' time scale. One version of this multiscale consideration arises in a singularly perturbed system of the form

$$\dot{x} = f(x, y) \qquad \varepsilon\dot{y} = g(x, y) \tag{1.1}$$

where we take y to be scalar-valued for expository simplification with $x \in \mathbb{R}^k$. If we can assume that the second (fast-scale) equation is everywhere stable

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significant assumptions (that certain things can be neglected, that parameters remain effectively constant, ...), we treat all descriptive model as convenient approximations after neglecting measurement errors, unmodeled details, and uncertain small perturbations.

(i.e., $\partial g/\partial y < 0$) — so we have the approximation

$$0 = g(x,y) \tag{1.2}$$

for very small $\varepsilon > 0$ — and that we can solve (1.2) for y to get $y = Y(x)$ as an effective constitutive relation, then a standard singular perturbation analysis shows that (1.1) tracks the reduced ordinary differential equation

$$\dot{x} = F(x) := f(x, Y(x)) \tag{1.3}$$

and this is what one would see on the normal time scale. Note that y disappears here, becoming a hidden variable which may conveniently be ignored completely in our modeling effort.

It is possible, of course, that the solution set $\Gamma = \{(x,y) : (1.2)\}$ could be both the graph of a function Y and also a smooth manifold in the xy-space, yet contain a vertical segment — say, along the surface $\mathcal{S} = \{x : \xi(x) = a\}$ — so Y becomes discontinuous there. This discontinuity, of course, appears only in the limit $\varepsilon \searrow 0$ so the model (1.3) is an idealization of (1.1): for very small $\varepsilon > 0$ the transition across the apparent jump in Y is continuously resolved, typically taking time $\mathcal{O}(\varepsilon)$. Now, writing the continuous branches of Y separately as Y_1 and Y_2, the idealized limit model (1.3) becomes

$$\dot{x} = F_j(x) \qquad j = j(x) = \begin{cases} 1 & \text{for } \xi(x) < a \\ 2 & \text{for } \xi(x) > a \end{cases} \tag{1.4}$$

with $F_j(x) := f(x, Y(x)) = f(x, Y_j(x))$. Thus on the normal time scale what one would see is a discontinuous modal switch $1 \leftrightsquigarrow 2$ for j at the switching surface $\mathcal{S}$ given by the threshold $\xi = a$; cf., e.g., [5] for an analysis of such models.

Note that some nonuniqueness can arise here since (1.4) leaves j undefined on the switching surface $\Gamma = \{x : \xi(x) = a\}$. If the mode $j = 2$ would be directed away from the switching surface and the trajectory (in mode $j = 1$) comes to the switching surface tangentially at a time t and then veers away, then there could be two quite distinct 'solutions' of the reduced equation (1.4): one with the trajectory continuing past t in mode $j = 1$ and the other with the trajectory continuing past t after a modal switch $J : 1 \curvearrowright 2$ at t. The appropriate selection of 'solution' is not inherent in the reduced model and we refer to any values of t giving a possibility of this kind of behavior as *anomalous points*.

The possible existence of anomalous points is one of the characteristic technical difficulties of discontinuity modeling involving threshold-based switching.

We discuss this further in Section 3.

A still more interesting possibility occurs when, for example, (1.1) might be something like

$$\dot{x} = f(x,y) \qquad \varepsilon\dot{y} = g(x,y) = \xi + y - y|y| \qquad \xi = \xi(x). \tag{1.5}$$

[We are here keeping y scalar while allowing $x \in \mathbb{R}^k$; we assume the functional ξ is then 'nice', perhaps linear.] For this example, solving (1.2) is not possible globally, but we obtain two stable branches[3]

$$\begin{aligned} y = Y_1(x) &= \frac{-1-\sqrt{1-4\xi}}{2} \quad \text{for } \xi = \xi(x) \le 1/4 \\ y = Y_2(x) &= \frac{1+\sqrt{1+4\xi}}{2} \quad \text{for } \xi \ge -1/4 \end{aligned} \tag{1.6}$$

What we would now see (on the normal time scale) is tracking of each reduced ordinary differential equation

$$\dot{x} = F_j(x) := f(x, Y_j(x)) \tag{1.7}$$

to the extent possible, so F_1 and F_2 are distinct modes of the system. Of course Y_1 and so F_1 are undefined for $\xi(x) > 1/4$: something must happen if (1.7) with $j = 1$ would make ξ increase above $1/4$. Noting that $y = -1/2$ at that moment and $g(x,y) > 0$ for $y < 0$, $\xi > 1/4$, what happens is that y increases 'immediately' to $Y_2(x)$. Thus, when the system is in the mode $j = 1$ the surface $\{x : \xi(x) = 1/4\}$ is a switching surface: one has a (discontinuous!) jump $1 \curvearrowright 2$ of the modal index j as the trajectory would cross this surface making ξ exceed the threshold value. Similarly, one has another switching surface when in the mode $j = 2$ as j jumps $2 \curvearrowright 1$ when ξ drops below $-1/4$. In this[4] reduced model we see that (1.3) becomes

$$\dot{x} = F_j(x) \qquad j = W[\xi(x)] \tag{1.8}$$

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[3]Joining these two branches is another branch: $y = Y_*(x) = \frac{1}{2}\,\mathrm{sgn}(\xi)\,\left(1 - \sqrt{1+4|\xi|}\right)$. Since this is unstable in forward time ($\partial g/\partial y > 0$ here), it would never actually be seen on the normal time scale and we therefore disregard it.

[4]Of course there are more complicated (and quite interesting) possibilities when y also is no longer scalar-valued — e.g., canards (perhaps leading to mixed mode oscillations) and the more general constructions of René Thom's Catastrophe Theory (cf., e.g., [20]) — but we do not discuss these here.

with $F_j(x) := f(x, Y_j(x))$ as in (1.7). We might then recognize that this relation W is just the elementary hysteron of [8] with thresholds $\omega_\pm = \pm 1/4$: for an input function: $t \mapsto r(t)$ we construct the consequent[5] output function: $t \mapsto W$ by

$$W\Big|_t = \begin{cases} 1 & \text{if } r(t) < \omega_- \\ 2 & \text{if } r(t) > \omega_+ \\ \text{switching rules} & \text{when } \omega_- \le r \le \omega_+ \end{cases} \tag{1.9}$$

where — at least for continuous inputs $r(\cdot)$ — these 'switching rules' require that

$$\begin{gathered} t \mapsto W[r(\cdot)] \text{ is piecewise constant —} \\ \text{changing } 1 \curvearrowright 2 \text{ only when } r = \omega_+ \text{ and } 2 \curvearrowright 1 \text{ only when } r = \omega_- \end{gathered} \tag{1.10}$$

Note that we now do *not* have a 'differential equation with discontinuous righthand side' as in [5] because the relevant switching surface is now history dependent (alternatively, because evolution of the state component j is here given discretely by (1.10) rather than by a differential equation). The system (1.8), (1.9), (1.10) is a 'switching system' in the sense of [15, 16].

Observe that the state of the system has been augmented from x to the pair $[x, j]$ — the modal index j must now be taken as a component of the state since without it one cannot determine the source term in the heat equation when $\omega_- < u(t, P) < \omega_+$ without recalling the past history: the elementary hysteron $W[\cdot]$ is not actually a function, but an *input/output relation*.[6]

It is important to recognize that our switching rules (1.10) are indeterminate when, as certainly seems possible, the input $r(\cdot)$ might reach a threshold value non-transversely, i.e., r might rise to the upper threshold so $r(t) = \omega_+$ but perhaps then drop without going above ω_+ (or might drop to ω_- but then rise without going below). In such a context it is not clear whether the model should require, permit, or forbid the corresponding transition. As earlier with single switching surfaces, we class these values of t as *anomalous points.*

[5]The initial data for x, j should, of course, be consistent with (1.9).

[6]It is interesting to note that $r(\cdot) \mapsto W[](\cdot)$ can be meaningfully extended to classes of discontinuous functions. Here, of course, where $r = \xi \circ x$ in (1.8), we always do have $r(\cdot)$ continuous, so (1.10) applies. We may also note [8] that this map is rate-independent and nondecreasing, but those properties are not relevant to our present concerns.

2 Some examples

Once one raises the issue, the ubiquity of discontinuities is evident: in nature one sees the Earth's surface, solidification interfaces, crack propagation, flame fronts, shock waves, austenite-martensite transitions, nerve impulses, active/dormant transitions, births and deaths, etc.; in design one sees switches, fuses, relays, valves opening/closing, stopping times, OS interrupts, A/D conversion, etc. These are both spatial and temporal, both fixed (as traffic lanes or sampling times or grid geometries) and variable (as free boundaries or thermostatic control). We explore some general considerations in the context of a few examples. For some expository consistency, we will concentrate on *event-driven temporal discontinuities* and particularly modal switching induced by reaching thresholds.

Example 2.1. We begin with consideration of a (trivial?) Calculus problem: dropping a ball. Every freshman knows that the relevant ordinary differential equation is $\ddot{h} = -g$, but the range of validity of this standard model ends at the ball's collision with the floor. The usual treatment of the bouncing ball consists of repeatedly restarting the ODE with an *ad hoc* determination of the new (upward) velocity following each impact: e.g., Newton's *Law of Restitution* posits that this velocity[7] is simply proportional to the pre-impact velocity so the interimpact time intervals decrease geometrically and the bouncing terminates in finite time. This situation, in which one has a limit of discontinuity times, is called a *Zeno point* and must again be resolved phenomenologically.

The possible existence of Zeno points is one of the characteristic technical difficulties of discontinuity modeling.

We note that more general problems of *impact dynamics,* modeling systems of colliding rigid bodies, have now become a very active area of research. Typically, these formulations (involving, e.g., complementarity and quasivariational inequalities) permit proof of existence using available technical tools, but uniqueness often remains open and one must be quite careful in formulating concepts of well-posedness: consider the case of a grazing impact. [Related, but slightly different in not involving inertia, are *sweeping processes;* cf., e.g., [10] and the *play operator* of [8].]

[7]Note that this also corresponds to energy absorption per impact proportional to the ball's pre-impact kinetic energy.

Example 2.2. We next turn to a biological example involving dormant bacteria already present in the soil being resuscitated by the renewed availability of some critical nutrient; presumably an earlier event was the transition from an ACTIVE to this DORMANT state when supply of this nutrient was interrupted. These transitions are far from instantaneous, but may be treated that way in a longer term ecological context. Quite standard models in Population Biology consider the bacteria in the ACTIVE state, consuming nutrients, reproducing, etc., but for these bacteria the range of validity of this standard model does not include population dynamics when the nutrient availability fluctuates below a threshold and including this thus constitutes an extension of the model to settings in which such transitions occur.

Note that it is possible to have anomalous points where, e.g., the bacteria are in their ACTIVE state and the nutrient concentration drops to the threshold value but perhaps then rises without going below it. In the context of such non-transversal behavior it is not clear whether the model should require, permit, or forbid an ACTIVE $\curvearrowright$ DORMANT transition.

This is a descriptive model, as described above. Of course, in the context of some localized pollutant the existence of such a population of (dormant) bacteria might suggest a possibility of using them for bioremediation with some of the critical nutrient supplied to resuscitate the bacteria and to maintain their activity while they break down the pollutant. If the pollutant is undesirable but the nutrient expensive, this leads to an optimal control problem [11] balancing these considerations and this must involve in its dynamics the alternating states of the bacteria, especially if one might anticipate future re-occurrences of the pollution. The model, discontinuity and all, has now become a prescriptive model.

A quite different biological problem involving switching is the consideration of such population control policies as 'one family, one child'. Having such a law in force or not constitutes a choice of modes and the passing/repeal of such a law is a control discontinuity by modal selection. Whether or not this prescriptive model corresponds with useful accuracy to a descriptive model would then depend on the significance of the implementation transient, including anticipation of the transition, delayed effects of existing pregnancies, etc.

Example 2.3. For a thermostat to turn the furnace ON involves a course of comparatively rapid changes within the thermostat and within the furnace. Nevertheless, considering all this as 'switching instantaneously' is a convenience in keeping our attention focused on the normal time scale. It is not immediately obvious how best to model a thermostat in its interaction with the ambient temperature distribution.

The simplest model takes the furnace operation ($j = 0$ for OFF, $j = 1$ for ON) as $j = w(u(t, P))$ where u is the temperature distribution — here evaluated at the thermostat location P — and w is a convexified (set valued) step function

$$w(r) = \begin{cases} 1 & \text{if } r < \omega_* \\ 0 & \text{if } r > \omega_* \\ [0, 1] & \text{if } r = \omega_* \end{cases} \tag{2.1}$$

with ω_* the setpoint for the thermostat. Indeed, the author's initial interest in problems of this kind was stimulated by [6], which used this model. Their numerical simulations seemed to show that one always would settle down to a periodic ON/OFF cycle. However, while one can easily show that a 'periodic solution' always exists for that model, the mathematical system supports a constant solution with $u(t, P) \equiv \omega_*$ and $w \in (0, 1)$ so the furnace would be (nonphysically) 'partly ON'; it remains open whether, using this model, there always exists a *nontrivial* periodic solution with $\{0, 1\}$-valued w.

Refining the modeling slightly, one notes that actual thermostats have a slight separation between the switching thresholds $\omega_\pm$ for ON and OFF and we take $j = 2 - W[u(\cdot, P)]$ where W is the elementary hysteron defined by (1.9), (1.10). Since this model is threshold-based, one must again deal with the possibility of anomalous points. On the other hand, with appropriate regularity one can bound $u_t(\cdot, P)$ and so bound from below the length of each interswitching interval in terms of the threshold separation $(\omega_+ - \omega_-)$, ensuring that Zeno points could not arise here.

An interesting thermostat model in [3] involves temperature control for an automobile engine: coolant is circulated within the engine (mode $j = 0$) but is diverted by a valve to flow also through the radiator (mode $j = 1$) when the sensor temperature exceeds a threshold. A crude caricature of such a model (not that of [3]) might be

$$T'(t) = a(t) - \lambda T(t) - \mu_{\mathbf{j}(t)} T(t - \delta), \qquad \mathbf{j}(\cdot) = W[T(\cdot)] \tag{2.2}$$

with $\mu_0 = 0$. [Here a gives engine-generated heat; for the radiator we are using a quasi-steady state heat transfer model for simplification, avoiding a diffusion/conduction partial differential equation.] Note that the lower-temperature mode ($j = 0$) is modeled in (2.2) by an ordinary differential equation, while the radiator mode requires a delay differential equation with δ the circulation time through the radiator: a relevant descriptor of the continuum component of the state thus switches between finite dimensional and infinite dimensional spaces.

As prescriptive models we have many systems of such a general form: a sensor controlling modality, typically by a relay-operated switch. For many of these it would be wasteful to have rapid switching back and forth between the modes[8] and the slight separation between the switching thresholds is deliberately introduced for just this reason; compare the discussion in [18] of optimizing this. A related strategy for ensuring separation of the switching times is simply to introduce a 'dead time' so as to enforce a lower bound τ_* on the length of interswitching intervals. As a prescriptive model we might formulate this by introducing a new mode (say, $j = *$) involving a new variable τ. Suppose, for example, we would have had an index j switching: $1 \curvearrowright 2$ at a time t_* — e.g., as a sensor value $r(\cdot)$ reaches its threshold ω. We would now have a jump $j : 1 \curvearrowright *$ at t_* and would also reset $\tau : \cdot \curvearrowright 0$ then. In this mode $j = *$ the continuous part of the system evolves as in the mode $j = 2$ (adjoining $d\tau/dt = 1$ to the dynamics), but with a different switching rule: the rule now is simply that (regardless of $r(\cdot)$) we switch from $j = *$ when τ rises (necessarily transversely) to its threshold τ_*. At that time it would, of course, be a design decision whether we should switch j from $*$ to 1 or to 2, somehow depending on $r(\cdot)$ over $[t_*, t_* + \tau_*]$. We do note that this is a bit different from most[9] of our previous treatments in that we not only have a discontinuous change of mode, but also a discontinuity of the continuous component of the state itself, not just its derivative, on resetting

[8] E.g., for a thermostatically controlled oil furnace one has, on the more rapid time scale, an inevitable loss of fuel in the transient as the furnace is turned ON; somewhat less so as it is turned OFF. Similar considerations apply to a thermostatically controlled air conditioning compressor and *a fortiori* to the compressors of Example 2.5.

Wasteful or not, we also note the occurrence of these "chattering modes" in the implementation of *sliding mode* control; cf., e.g., [21]. Of course, in some applications such a rapid alternation may be precisely the point of the design, as with a buzzer.

[9] The exceptions being the bouncing ball of Example 2.1 (for which the velocity — a component of the continuous part of the state — jumps discontinuously when the ball bounces, reversing its direction) and the model (2.2) above (with the continuous part of the state switching between $\mathbb{R}$ and $\mathbb{R} \times \mathcal{X}$ with $\mathcal{X} = C([-\delta, 0])$ an infinite dimensional space of 'histories' — assuming the interswitching times will be long enough to permit resetting this to 0 as an acceptable approximation on switching: $0 \curvearrowright 1$ so we need not remember histories through an interval $\mathbf{j} \equiv 0$).

the ‘clock variable’ τ.

Example 2.4. Quite different is the treatment (in [14], etc.) of magnetic hysteresis. We might consider a ferromagnetic material with magnetic domains of varying alignment, each reversing its orientation according to the external field when the relevant component of that field reaches an appropriate threshold. For a 1-dimensional model, the magnetization of each of these is effectively given by an elementary hysteron with thresholds corresponding to the alignment, but this is at a finer spatial scale than desired. Making the transition to a macroscopic model, we then have the *Preisach model* of magnetic hysteresis in which observable magnetization is given by averaging over the ensemble of elementary hysterons parametrized by $\omega = (\omega_-, \omega_+)$ —

$$W_*[r(\cdot)] = \int_H W_\omega[r(\cdot)]\,\mu(d\omega), \tag{2.3}$$

integrating with respect to the relevant ensemble distribution (measure μ over the Preisach half-plane $H = \{\omega : \omega_- < \omega_+\}$).

Essentially similar models are used to describe a variety of disparate situations (cf., e.g., [13]). For example such a model has been developed independently by hydrologists to describe the hysteresis in soil wetting by groundwater: in that setting the underlying fine structure is the network of interstitial pores which behave differently when wet or dry. In these models (provided the measure μ is non-atomic) the hysteresis remains but the discontinuity appearing in the fine structure disappears from the macroscopic model — while remaining essential to understanding (2.3) and its rationale.

Example 2.5. Next we consider a network of gas pipelines for which there are several interesting modeling considerations. A comparatively simple model for the gas flow dynamics is given by the isothermal Euler equations

$$\rho_t + (\rho v)_s = 0 \qquad (\rho v)_t + (\rho v^2 + a\rho)_s = f \tag{2.4}$$

where, with suitable normalization, s is distance along the segment (a single pipe) and the unknowns ρ, v are gas pressure and velocity; here $f = f(\rho, v)$ is a friction term from pipeline roughness. Besides initial data, (2.4) reqires input data where u is directed into the segment. [A primary source of technical difficulty here is the generation of shocks — discontinuities in the pressure — but these are not the discontinuities with which we are concerned here.] Of course, the pipeline network lives on a graph — so one has many copies of (2.4) coupled through the attendant nodal conditions for input/output, even assuming we may take the flow direction along each edge as known. We avoid formulating the full system of equations here — especially as we immediately proceed to complicate it further.[10]

While this system would be formidable enough, we are concerned that the friction may cause an unacceptably great loss of pressure so, in practice, compressors are introduced at some of the nodes. While necessary, the compressors are fueled by using some of the gas at considerable cost — so, prescriptively, each of the compressors must be switched ON and OFF 'as needed'. This switching is the discontinuity under consideration. One would then have a difficult optimization problem to determine when to switch, presumably preceded by a well-posedness argument for the model. Of particular interest is the possibility of decentralized feedback control, in which each compressor has access to one or more pressure sensors and switches ON and OFF depending on the vector of sensor values entering some regions; for a single sensor one expects this action to be given by thresholds so the state of each compressor would be determined from this sensor value by an elementary hysteron. [Note that we now have *several* of these hysterons coupled through the system, so we must be concerned for their interaction.]

We also note that very similar considerations are involved in the use of signal lights to control traffic flow, etc.

Example 2.6. An *optimal stopping time* problem has the following form: monitor some — possibly stochastic — situation (state x) with the option of stopping at any time and obtaining a final value $V^* = V^*(t,x)$ evaluated at the stopping time. The value V at earlier times is then taken to be the expectation of V^* (conditioned on the current t,x), assuming one follows the optimal causal policy — then implicitly determined as: stopping when $V(t,x) = V^*(t,x)$. In operation one has a discontinuous change of mode at this stopping time, but the analysis simply consists of determining the deterministic function V from the stochastic evolution of x for comparison with the given function V^*.

While this stopping is certainly a switching from one mode to another, one may have a more general opportunity of *modal control,* selecting from a

[10] ...recalling the dictum (attributed to the philosopher Hannah Arendt) that, *"There is no situation, however complicated, which cannot, by looking at it correctly, be made even more complicated."*

discrete set of available modes[11] with some switching cost c_{jk}. Then the value function V^*further depends on the current modal index j as a component of the state. Thus we would write $V^* = V_j^*(t,x)$ and optimal switching would mean making the transition $j \curvearrowright k$ when $V_j(t,x) = V_k^*(t,x) - c_{jk}$. Once these switching surfaces are specified, the controlled evolution is again a switching system in the sense of [15], [16]. Due to the switching cost the switching surfaces for $j \curvearrowright k$ and for $k \curvearrowright j$ are separated; compare the analysis of [4].

One variant of this is the familiar 'change point problem' of Statistics and we consider an example of this. Suppose a machine tool rapidly produces widgets with some small (acceptable) rate of random defects, e.g., observed by sampling the product output and testing. At some point the tool may itself become defective (by wear or a drill bit chipping or ...) and the defect rate jumps to an unacceptable level. Without modeling any causes for such a change, the problem is to detect this as soon as possible so as to stop the machine for repair. Sounding such an 'alarm' is our control action and we must do this balancing the cost of delay against the cost of a false alarm (since the observed product defects themselves arise randomly). The solution to this problem is to keep a running estimate of the probability that the change has already occurred and sound the alarm when this reaches[12] some threshold, whose optimal setting would depend on the level and variance of the observation process as well as on the relative costs. Since the sampling also has a cost, a possible variant of this (compare [12]) might be to 'declare an ALERT' at a lower threshold with an intermediate control action of sampling more frequently, with subsequent switching actions either to ALARM or back to the less frequent sampling. Again we have a threshold-based discontinuity.

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[11] A once-familiar example might have been shifting gears in driving an automobile. Some other examples of this are the population control policy of Example 2.2, the use of signals in traffic control, shifting a shared resource from one task to another with a loss of time for setup (as, e.g., in [7]), etc.

[12] Again one would have the possibility of anomalous points. Although the the prescriptive nature of this together with the discreteness both of 'observed defects' and of the sampling times might seem to permit us to resolve this somewhat arbitrarily here, we note that this would be at the cost of returning from the continuous-time model to a *discrete event system*: the sequential decision procedure of which it is a somewhat simplified reduction.

3 Discussion

We said in the Introduction that the predictions of a good prescriptive model must, for some implementation, give an acceptable approximation of the real events one might then observe within the range of concern, i.e.,

The prescriptive model must be a descriptive model of its implementation.

The converse of this, especially to the extent that the fine structure (e.g., action on a faster time scale) is unmodeled, is that

Every descriptive model may be viewed as a prescriptive model

whose prescriptive objective is precisely to approximate real events acceptably.

We will proceed somewhat anecdotally[13] in considering the appropriate interpretation of 'solution' for models such as those we have been discussing — systems with threshold-based event-driven discontinuities. Since the evolution is determined pointwise in t, it is sufficient to consider local analyses of possible scenarios in the neighborhood of a single solution trajectory; since interpretation is clear during interswitching intervals,[14] we need only consider the occurrence and nature of 'jumps' (modal transitions). Note that, as occurred in several of the examples in Section 2, a modal transition may involve not only a jump in the mode index $\mathbf{j} : j \curvearrowright k$, but also a jump in the continuum component:[15]

$$x_* := \mathbf{x}(t_*-) \quad \curvearrowright \quad \mathbf{x}(t_*+) := x^* = F_{j,k}(x_*). \tag{3.1}$$

Suppose we have a sensor ξ (i.e., $\xi : \mathcal{X} \to \mathbb{R}$) with critical threshold a so $\mathcal{S} = \{x \in \mathcal{X} : \xi(x) = a\}$ is a switching surface. In this case, that is to mean that a solution $[\mathbf{x}, \mathbf{j}]$ may not have $\mathbf{j} = 1$ when $\xi(\mathbf{x}) > a$ and may be

[13] This is to avoid the flavor of footnote [10] involved in introducing the technical detail required by a model (e.g., more-or-less following [15, 16]) general enough to handle the variety of examples of the previous section,

[14] We assume the evolution is there given by some well-posed system: ordinary differential equation, integrodifferential equation, delay differential equation, or partial differential equation as appropriate to the model.

[15] Here (with $\mathcal{S}_{j,k}$ the switching surface $\{x \in \mathcal{X}_j : \xi_{j,k}(x) = a_j\}$ where switching $\mathbf{j} : j \curvearrowright k$ is permitted), we assume a continuous function: $F_{j,k} : \mathcal{X}_j \supset \mathcal{S}_{j,k} \to \mathcal{X}_k : x_* \mapsto x^*$ such that each x^* can be used as initial data in mode$_k$.

Note that we are permitting the continuum state spaces to be different for different modes, as occurs, e.g., for (2.2) in Example 2.3. Even if $\mathcal{X}_j = \mathcal{X}_k$ we note that, as in the case of the bouncing ball of Example 2.1, we need not have continuity at t_* of the continuum component $\mathbf{x}(\cdot)$ which would require that $F_{j,k}(x_*) = x_*$ for $x_* \in \mathcal{S}_{j,k}$.

permitted to jump $\mathbf{j} : 1 \curvearrowright 2$ when $\mathbf{x} = x_* \in \mathcal{S}$; we then follow footnote[15] in taking $x^* = F_{1,2}(x_*)$ as initial value at t_* in mode$_2$, obtaining a solution $\tilde{\mathbf{x}}(\cdot)$ on some nonempty interval $[t_*, t_1)$. Now suppose we have a solution $[\mathbf{x}(\cdot), 1]$ on a time interval $[t_0, t_*)$ giving $\mathbf{x}(t_*) = x_*$ so $\sigma(t) = \xi(\mathbf{x}(t)) \leq a$ on $[t_0, t_*)$, rising to a at t_*; further, we assume that the differential equation defining mode$_1$ permits use of x_* as initial data at t_* for a solution $\hat{\mathbf{x}}$ (so we might view $\hat{\mathbf{x}}$ as a potential extension of $\mathbf{x}$). If σ now crosses[16] the threshold, then our switching rules require that we *must* have the jump $\mathbf{j} : 1 \curvearrowright 2$ at t_*.

If, however, $\sigma = \xi(\hat{\mathbf{x}}(\cdot)) \leq a$ on some nonempty interval $[t_*, t_1)$, then we have an anomalous point with two distinct apparently acceptable extensions of $\mathbf{x}$ past t_* so we have two candidate solutions:

$$[\mathbf{x}, \mathbf{j}]_1 = \begin{cases} [\mathbf{x}, 1] & \text{on } [t_0, t_*) \\ [\tilde{\mathbf{x}}, 2] & \text{on } [t_*, t_1) \end{cases} \quad \text{and} \quad [\mathbf{x}, \mathbf{j}]_2 = \begin{cases} [\mathbf{x}, 1] & \text{on } [t_0, t_*) \\ [\hat{\mathbf{x}}, 1] & \text{on } [t_*, t_1) \end{cases} \tag{3.2}$$

To resolve the question of which is actually to be accepted means determining what we are to mean in speaking of "a solution" in the model.

Although the considerations are intertwined, the appropriate interpretation of 'solution' for the idealized model is a modeling decision rather than a question decidable purely by mathematical analysis of the idealized model in isolation and we argue that one should accept *both.* This resolution of the problem comes from our goal for a "good model" — to be useful by predicting "what one would see on our normal time scale in the real world." Our insight then comes from footnote[2] — we must allow for small perturbations of the idealized (reduced or prescriptive) model so our uncertainty is real: even if we were to restrict consideration to a small perturbation of the threshold value alone, *each* of $[\mathbf{x}, \mathbf{j}]_1$ and $[\mathbf{x}, \mathbf{j}]_2$ is a limit of potentially observable real-world evolutions; consideration, e.g., of small uncertainty in the initial data would lead to similar conclusions. We see that our uncertainty is inherent in the sensitivity of the situation at the anomalous point and reliable approximate prediction of the actual outcome is here impossible from the information available. Given this, we may ask: How can the model continue to be useful? Unable to say, "This is what *will* happen," we nevertheless *can* say usefully:

"Here is the set of alternatives: each of these might *happen and we can neglect all other scenarios."*

It would perhaps be interesting to treat the uncertainty as random, much

[16] By this we mean only that $\xi(\hat{\mathbf{x}}(t_n)) > a$ for some sequence $t_n \searrow t_*$.

as a coin toss, with some attempt to assign meaningful nonzero probabilities to the alternatives, but we do not pursue that possibility here. We do, however, regard these considerations as reason to choose an inclusive interpretation of 'solution'. Note that in the context of forced switching (as, e.g., in footnote[16]) one has the unique solution $[\mathbf{x},\mathbf{j}]_1$ of (3.2) since the condition of footnote[16] makes it impossible for $[\mathbf{x},\mathbf{j}]_2$ to be a limit of suitable approximants so the rejection of $[\mathbf{x},\mathbf{j}]_2$ as a solution is again consistent with our principle of interpretation.

We may refer to a solution of the model as a *regular solution* if it involves neither Zeno points nor anomalous points. Such a solution on $[0,T]$ is then characterized by

- the set of switching times $0 < t_1 < \ldots < t_N < T$ ($t_0 = 0$, $t_{N+1} = T$)
- the index values $\mathbf{j} = j^0, \ldots, j^N$ on the $N+1$ interswitching intervals: $\mathcal{I}_k = [t_k, t_{k+1}]$ for $k = 0, \ldots, N$
- the trajectories $\mathbf{x}_k : \mathcal{I}_k \to \mathcal{X}_{j^k}$ on these interswitching intervals — for comparisons it is convenient to rescale each $\mathcal{I}_k$ to $[0,1]$.

With N and (j^k) locally fixed, we then topologize these solutions $\{[\mathbf{x},\mathbf{j}]\}$ by

$$(t_k)_1^N \in \mathbb{R}^N \qquad \mathbf{x} \leftrightarrow (\mathbf{x}_k)_0^N \in \prod_{k=0}^{N} C\left([0,1] \to \mathcal{X}_{j^k}\right). \tag{3.3}$$

We then have local[17] well-posedness regular solution under such reasonable hypotheses as in footnotes[14, 15]:

Theorem 3.1. *Near a regular solution, one has unique regular solutions (with N and (j^k) locally fixed) for sufficiently close data with convergence of the data giving corresponding convergence of the solutions in the sense of* (3.3). *More generally (still excluding Zeno points, but now admitting anomalous points), the solution set depends upper semicontinuously on the data, in that every limit of solutions is a solution and every solution without Zeno points is such a limit of regular solutions.*

[17] We can take perturbation of the 'data' to include not only change in the initial $\mathbf{x}(0) \in \mathcal{X}_{j^0}$ but also small structural change in the dynamics within each mode, changes of the switching surfaces (perturbing the relevant sensor functions and corresponding thresholds), and change of the transition functions $F_{j^k,j^{k+1}}$

This analysis excludes a quite interesting situation arising for problems of forms related to (1.4): the hypothesis in footnote[15] may fail in that one may be unable continue after a modal jump, using the available component $x^* = x_*$ as initial data in the new mode. We are discussing here a differential equation $\dot{\mathbf{x}} = F(\mathbf{x})$ with $F : \mathcal{X} \to \mathcal{X}$ discontinuous across a switching surface $\mathcal{S}$. Writing this as in (1.4), we assume that each F_j is continuous up to the common boundary $\mathcal{S}$ and that

$$\mathbf{n} \cdot F_1(x_*) \ > \ 0 \ > \ \mathbf{n} \cdot F_2(x_*)$$

— which just means that each of these direction fields points towards the switching surface so, starting at x_*, neither of the modal equations has a solution in the appropriate halfspace for that mode: one cannot leave $\mathcal{S}$. For interpretation we think of this as a prescriptive model and consider how it might be implemented. The simplest possibility corresponds roughly to (1.1) so y is re-introduced, giving an interpolation between F_1 and F_2 within a fuzzy 'thickening' (width $\mathcal{O}(\varepsilon)$) of the switching surface and one moves within this fuzzy 'surface'. We refer to this interpolatory implementation as *blending* F_1, F_2. It is easy to see what the limit motion will be in this case as $\varepsilon \to 0$: since one cannot leave $\mathcal{S}$, the components normal to $\mathcal{S}$ must cancel when averaged on the normal time scale and this determines the unique convex combination of F_1, F_2 tangential to $\mathcal{S}$. Indeed, one would expect that *any* plausible implementation[18] here would give a velocity in the convex hull of $\{F_j\}$ and then necessarily be tangential to $\mathcal{S}$ (so normal to $\mathbf{n}$). This *sliding mode* is thus uniquely determined and is the idealization of the approximating *chattering mode* given by anyimplementation; although (1.4) does not specify any dynamics when on $\mathcal{S}$, the idealized approximation to "what one would see" is clearly[19] given by the sliding mode so our interpretation principle makes this 'the solution' in such situations. Cf., e.g., [5, 21].

Suppose, however, in this type of problem we might have two sensor functions ξ, η, each taken with threshold 0, and so switching surfaces $\mathcal{S}_\xi, \mathcal{S}_\eta$

[18] One implementation of particular interest to us might be a splitting of $\mathcal{S}$ into two slightly separated switching surfaces, using the elementary hysteron $W[\xi(\mathbf{x})]$ with thresholds $\omega_\pm = a \pm \varepsilon$. Because of the $\mathcal{O}(\varepsilon)$ separation, we then have a sequence of switchings with $\mathcal{O}(\varepsilon)$ interswitching intervals. Our earlier analysis of W assumed we had occasional switching, but here we have frequent switching on the 'normal time scale' which then involves 'long term averages' when compared to the time scale of the individual switchings.

[19] While mathematically correct, it is precisely an apparent failure of this in an experimental setting [9] which interested the author in the analysis below of intersecting switching surfaces as a plausible explication [17, 18].

intersecting to partition $\mathcal{X}$ into four regions ($j = 1, 2, 3, 4$ corresponding to the quadrants of $\mathbb{R}^2$ with ξ, η used as coordinates); one would have vector fields $\{F_j\}$ defined on each of the regions and we now consider the case in which these all point 'inward'. We would then seek a sliding mode within the codimension 2 intersection $\mathcal{S}_* = \mathcal{S}_\xi \cap \mathcal{S}_\eta$ and, by essentially the same logic as above, we would expect this to be a convex combination of the four fields $\{F_j\}$ and tangential to $\mathcal{S}_*$, i.e., both to $\mathcal{S}_\xi$ and to $\mathcal{S}_\eta$. Unfortunately, this gives only three linear conditions and the recipe no longer suffices to determine the four needed coefficients — although the sliding mode may still be uniquely determined if the fields $\{F_j\}$ are suitably related.

What now happens is that the selection of sliding mode (when this exists at all) is no longer determinable from the reduced model alone, but needs some additional information about the fine structure: a residue of model reduction as in [18]. [It is just this ambiguity which seems to permit the apparent recipe failure of footnote [19].] We know [17] that if one of the sensors gives much more rapid switching than the other, then time scale separation permits analysis as sequential consideration of two single-sensor problems. If they operate on comparable time scales, we know [1] how to compute the sliding mode if we know that the implementation is done by some form of 'blending'.

Perhaps the most interesting implementation possibility is the use of a pair of elementary hysterons to split each of the switching surfaces so, on the rapid 'switching time scale, one has a dynamical system alternating among the four modes. The desired coefficients for the convex combination are easily seen to be the (long-term average) fractions of total time which the system spends in each of the modes. It is by no means obvious that the long-term dynamical behavior should ensure the existence of such fractions, but it is shown in [2] that such existence does hold generically for these situations — although there seems no easy recipe to compute the coefficients.

[Similar questions might be raised in a setting with more than two switching surfaces intersecting in this way, but the results of [1, 2] noted here use essentially 2-dimensional arguments and this generalization remains open. Further, no such results are presently available for other than the blending and hysteretic implementations.]

While we are speaking of such things, we recall our interest in the existence of periodic solutions for systems with threshold-based discontinuities.

For stable 'linear' systems

$$\dot{\mathbf{x}} = A\mathbf{x} \pm u \qquad \pm = W[\langle \zeta, \mathbf{x} \rangle] \tag{3.4}$$

with $\sigma := \langle \zeta, A^{-1}u \rangle > \omega > 0$ we know [16] that there is a compact attractor and that a periodic solution exists for the thresholds $\omega_{\pm} = \pm\omega$ when ω is close enough to σ; existence may be conjectured for the more interesting situation of small ω, but this seems open except for special cases (e.g., if $\dim \mathcal{X} = 2$). On the other hand, an example is given in [16] of a more nonlinear system with $\mathcal{X} = \mathbb{R}^2$ for which the trajectories always alternate modes, going back and forth between two disks, but for which there cannot be any periodic solution. Clearly there are many open problems in looking at the long-term dynamical behavior of systems with threshold-based discontinuities

We have seen that in considering models involving discontinuities in their temporal fine structure (as the sliding modes above) the discontinuities disappear as such, but determine the macroscopic behavior. As a final remark, we note another setting, related to Example 2.2, involving a continuum of switching. Here we again have a bacterial population with hysteretic switching between ACTIVE and DORMANT states governed by the concentration α of a critical nutrient. Now, however, we take this population as spatially distributed with the nutrient carried by a known groundwater flow across a 1-dimensional interval. The switching of the bacteria is given by our elementary hysteron W independently at each spatial point — with bacteria growing and metabolizing nutrient when ACTIVE. The resulting convection/reaction/switching system can then be analyzed [19] as a free boundary problem for the space-time region in which the bacteria are ACTIVE. Interestingly, the question of treating anomalous points becomes irrelevant here (as these can only occur within the boundary of that region, without affecting the region itself) so one obtains well-posedness in the usual sense for this problem.

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