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The Parabolic-Hyperbolic System Governing the Spatial Motion of Nonlinearly Viscoelastic Rods

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This paper is dedicated to Jerry Marsden on the occasion of his sixtieth birthday.

Abstract

This paper treats initial-boundary-value problems governing the motion in space of nonlinearly viscoelastic rods of strain-rate type. It introduces and exploits a set of physically natural constitutive hypotheses to prove that solutions exist for all time and depend continuously on the data. The equations are those for a very general properly invariant theory of rods that can suffer flexure, torsion, extension, and shear. In this theory, the contact forces and couples depend on strains measuring these effects and on the time derivatives of these strains.

The governing equations form an eighteenth-order quasilinear parabolic-hyperbolic system of partial differential equations in one space variable (the system consisting of two vectorial equations in Euclidean 3-space corresponding to the linear and angular momentum principles, each equation involving third-order derivatives). The existence theory for this system or even for its restricted version governing planar motions has never been studied. Our work represents a major generalization of the treatment of purely longitudinal motions of [12], governed by a scalar quasilinear third-order parabolic-hyperbolic equation. The paper [12] in turn generalizes an extensive body of work, which it cites.

Our system has a strong mechanism of internal friction embodied in the requirement that the constitutive function taking the strain rates to the contact forces and couples be uniformly monotone. As in [12], our system is singular in the sense that certain constitutive functions appearing in the principal part of the differential operator blow up as the strain variables approach a surface corresponding to a "total compression".

We devote special attention to those inherent technical difficulties that follow from the underlying geometrical significance of the governing equa-

tions, from the requirement that the material properties be invariant under rigid motions, and from the consequent dependence on space and time of the natural vectorial basis for all geometrical and mechanical vector-valued functions. (None of these difficulties arises in [12].) In particular, for our model, the variables defining a configuration lie on a manifold, rather than merely in a vector space. These kinematical difficulties and the singular nature of the equations prevent our analysis from being a routine application of available techniques.

The foundation of our paper is the introduction of reasonable constitutive hypotheses that produce an a priori pointwise bound preventing a total compression and a priori pointwise bounds on the strains and strain rates. These bounds on the arguments of our constitutive functions allow us to use recent results on the extension of monotone operators to replace the original singular problem with an equivalent regular problem. This we analyze by using a modification of the Faedo-Galerkin method, suitably adapted to the peculiarities of our parabolic-hyperbolic system, which stem from the underlying mechanics. Our constitutive hypotheses support bounds and consequent compactness properties for the Galerkin approximations so strong that these approximations are shown to converge to the solution of the initial-boundary-value problem without appeal to the theory of monotone operators to handle the weak convergence of composite functions.

Contents

I. Introduction

1. Background

There is an extensive literature (some of which is cited in [12, 20, 40]) devoted to the analysis of the nonlinear scalar third-order quasilinear parabolic-hyperbolic equation

$$w_{tt} = n(w_s, w_{st}, s)_s \tag{1.1}$$

in which n is a strictly increasing function of its second argument. All but a handful of these studies take n to be an affine function of its second argument. Under different assumptions on the form of n, this equation can describe longitudinal motions of a viscoelastic rod, shearing motions of a viscoelastic layer, and longitudinal motions of a viscous gas [9, 11, 12].

The simplest example of a linear scalar parabolic-hyperbolic equation is $w_{tt} = \alpha w_{ss} + \beta w_{sst}$ where α and β are positive constants. If β were 0, this equation would reduce to the (hyperbolic) wave equation, and if α were 0, this equation would reduce to the (parabolic) heat equation for w_t. Hence the terminology.

In the present paper we study the generalization of (1.1) that describes the spatial deformation of a nonlinearly viscoelastic rod. We employ a geometrically exact and properly invariant theory, which is the most general theory of rods in which the stress resultants are the familiar contact force and couple. Here the unknowns are a vector-valued function $\boldsymbol{r}$ and an orthonormal triple of vector-valued functions $\boldsymbol{d}_k$, $k = 1, 2, 3$. These quantities satisfy the eighteenth-order system

$$\boldsymbol{n}_s = (\rho A)(s)\boldsymbol{r}_{tt}, \tag{1.2}$$

$$\boldsymbol{m}_s + \boldsymbol{r}_s \times \boldsymbol{n} = \partial_t[(\rho J_{pq}(s))w_q\,\boldsymbol{d}_p] \tag{1.3}$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ are vector-valued functions of $\boldsymbol{r}_s$, $\boldsymbol{r}_{st}$, $\boldsymbol{d}_k$, $\partial_s\boldsymbol{d}_k$, $\partial_{st}\boldsymbol{d}_k$, and s, prescribed in a properly invariant form, where $\partial_t\boldsymbol{d}_k = \boldsymbol{w} \times \boldsymbol{d}_k$, where ρA is a prescribed positive-valued function, where the ρJ_{pq} are the prescribed components of a positive-definite symmetric matrix-valued function, and where the summation convention is used. Heretofore there has been no existence theory for such a system or for its planar version or for any constrained version with nonlinear constitutive functions, obtained, e.g., by requiring the rod to be inextensible or unshearable. Caflisch & Maddocks [15] did however treat the planar motion of an inextensible unshearable elastic rod with the bending couple depending linearly upon the curvature. It is governed by a sort of semilinear hyperbolic system.

Our analysis consists in showing that some of the techniques developed in [12] for (1.1) carry over without difficulty to our system, while the far richer Euclidean geometry of our problem, with its concomitant requirements of invariance, provides new technical obstacles. In this connection it is important to note that some techniques used in the analyses of simpler versions of (1.1) are not applicable to systems. E.g., the maximum principle for scalar parabolic equations (used in [18,38] to construct a priori estimates) is not directly applicable to systems of parabolic equations related to our systems. Likewise, the clever transformation of Andrews [2], which was further exploited in [3,28], relies on the assumption that $n(s, w_s, \cdot)$ be affine, and so is available for neither the general form of (1.1) nor the system we treat here.

When a scalar-valued function $n(y, \cdot, x)$ is affine, there is a scalar-valued partial derivative g_y such that $n(y, \cdot, x)$ has the form

$$n(y, z, x) - n(y, 0, x) = g_y(y, x)z, \tag{1.4}$$

so that

$$g_y(w_s(s,t), s)w_{st}(s,t) = \partial_t g(w_s(s,t), s). \tag{1.5}$$

This identity played a crucial role in the analysis of Andrews, and related identities were central to the studies of Kanel' [22] and MacCamy [27]. But the analog of (1.5) is not generally available for systems generalizing (1.1) in which n is replaced with a vector-valued function $\mathbf{n}$, depending on s and on the vectors $\mathbf{y}$ and $\mathbf{z}$, that is affine in $\mathbf{z}$. Such a function has the form

$$\mathbf{n}(\mathbf{y}, \mathbf{z}, x) - \mathbf{n}(\mathbf{y}, 0, x) = \mathbf{A}(\mathbf{y}, x) \cdot \mathbf{z}, \tag{1.6}$$

where $\mathbf{A}$ is a matrix-valued function, and $\mathbf{A}(\mathbf{y}, x) \cdot \mathbf{z}$ is the image of $\mathbf{z}$ under $\mathbf{A}(\mathbf{y}, x)$. (See the discussion of notation at the end of this section.) In general, $\mathbf{A}$ need not be the Fréchet derivative of any vector-valued function $\mathbf{g}$, i.e., there need not be a $\mathbf{g}$ such that

$$\mathbf{A}(\mathbf{y}, x) = \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{y}, x), \tag{1.7}$$

so that there need not be a $\mathbf{g}$ such that

$$\mathbf{n}(\mathbf{u}(s,t), \mathbf{u}_t(s,t), s) - \mathbf{n}(\mathbf{u}(s,t), 0, s) = \partial_t \mathbf{g}(\mathbf{u}(s,t), s). \tag{1.8}$$

Moreover, there is no compelling physical warrant for assumption (1.7) (although one may wish to study such systems and may find that such assumptions are physically useful; cf. [41,42]). Thus, for systems, we would have to restrict our material response significantly to get an analog of (1.5). In our study, we do not restrict our attention to constitutive functions affine in the strain rate, and so have no expectation of such a generalization of (1.5). Nevertheless, by using suitable constitutive *inequalities* we can retain for our systems many of the advantages of (1.4) and (1.5) for the scalar equation (1.1).

Some of the difficulties we face are analogs of those that arise in the mechanics of rigid bodies: We employ a fundamental set of unknown vectors, the directors, that form an orthonormal basis depending on position and time. We could represent this basis by a matrix with respect to a fixed orthonormal basis, i.e., by a proper-orthogonal tensor field, which has the disadvantage that its nine components are subject to six constraints. We could represent the basis by Euler angles, which have the disadvantage that they have a unpleasant polar singularity. We could use alternative representations that have the disadvantage that they are subject to constraints and they have singularities. Instead, we use a coordinate-free formulation involving the axial angular velocity vector $\boldsymbol{w}$ and its analog $\boldsymbol{u}$ for the derivative with respect to the independent spatial variable. We thus follow the lead of the formulation of rigid-body mechanics in terms of the angular velocity vector [37].

It is worth noting that much of the mathematical and mechanical structure of our equations is lost in the transition to equilibrium equations: The dynamical equations, even for elastic rods, cannot be constructed by slapping acceleration terms onto various legitimate equilibrium equations.

To avoid obscuring the central ideas of our approach, we limit our attention in the main part of this paper to a specific class of boundary conditions and to a reasonable set of constitutive restrictions, not seeking the utmost generality. In Secs. 19 and 20, we indicate how our methods can readily be extended to far more general circumstances. In Secs. 21 and 22 we discuss the analytical consequences of alternative formulations. The main developments of purely mechanical interest are the discussion of constitutive assumptions in Secs. 6 and 19, the treatment of the preclusion of total compression in Sec. 10, and the treatment of the rotation of the directors throughout the paper, but especially in Secs. 4 and 13. The technical aspects of the analysis that we employ are confined almost entirely to Part IV.

2. Notation

We employ Gibbs notation for vectors and tensors: Vectors, which are elements of Euclidean 3-space $\mathbb{E}^3$, and vector-valued functions are denoted by lower-case, italic, bold-face symbols. The dot and cross product of (vectors) $\boldsymbol{u}$ and $\boldsymbol{v}$ are denoted $\boldsymbol{u}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\times\boldsymbol{v}$. A tensor is a linear transformation of $\mathbb{E}^3$ to itself. The value of a tensor $\boldsymbol{A}$ at a vector $\boldsymbol{v}$ is denoted $\boldsymbol{A}\cdot\boldsymbol{v}$ (in place of the more usual $\boldsymbol{A}\boldsymbol{v}$) and the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted $\boldsymbol{A}\cdot\boldsymbol{B}$ (in place of the more usual $\boldsymbol{A}\boldsymbol{B}$). The transpose of $\boldsymbol{A}$ is denoted $\boldsymbol{A}^*$. We write $\boldsymbol{v}\cdot\boldsymbol{A}=\boldsymbol{A}^*\cdot\boldsymbol{v}$. The dyadic product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (used but rarely here) is the tensor denoted $\boldsymbol{a}\boldsymbol{b}$ (in place of the more usual $\boldsymbol{a}\otimes\boldsymbol{b}$), which is defined by $(\boldsymbol{a}\boldsymbol{b})\cdot\boldsymbol{v}=(\boldsymbol{b}\cdot\boldsymbol{v})\boldsymbol{a}$ for all $\boldsymbol{v}$. Thus $(\boldsymbol{a}\boldsymbol{b})^*=\boldsymbol{b}\boldsymbol{a}$. Since the space of tensors has bases consisting of dyadic products, we define the cross product $\boldsymbol{u} \times \boldsymbol{A}$ of a vector and a tensor by defining it for $\boldsymbol{A} = \boldsymbol{ab}$ by $\boldsymbol{u} \times (\boldsymbol{ab}) := (\boldsymbol{u} \times \boldsymbol{a})\boldsymbol{b}$.

Lower-case Latin indices, except for s and t, range over 1,2,3, and such twice-repeated indices are summed from 1 to 3. Lower-case Greek indices range from over 1,2, and such twice-repeated indices are summed from 1 to 2.

Triples of real numbers are denoted by lower-case, sans-serif, bold-face symbols. E.g., the triple (u_1, u_2, u_3) of components of a vector $\boldsymbol{u}$ with respect to a certain nonconstant basis is denoted u. We set $\mathsf{u} \cdot \mathsf{v} := u_i v_i$, $|\mathsf{u}| = \sqrt{u_k u_k}$, $\mathsf{u} \times \mathsf{v} := (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$. The matrix of a tensor $\boldsymbol{A}$ with respect to a specified basis is denoted A. Its action on a triple u is denoted $\mathsf{A} \cdot \mathsf{u}$. The dot product and norm for other n-tuples are treated analogously.

The (Gâteaux) differential of $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ at $\boldsymbol{v}$ in the direction $\boldsymbol{h}$ is $\frac{d}{dt}\boldsymbol{f}(\boldsymbol{v} + t\boldsymbol{h})\big|_{t=0}$. When it is linear in $\boldsymbol{h}$, we denote this differential by $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{v}) \cdot \boldsymbol{h}$ or $\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{v}) \cdot \boldsymbol{h}$. We often denote the function $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ by $\boldsymbol{f}(\cdot)$. The partial derivative of a function f with respect to a scalar argument t is denoted by either f_t or $\partial_t f$. Obvious analogs of these notations will also be used. The operator ∂_t is assumed to apply only to the term immediately following it. The ordinary derivative of a function f with respect to its single argument t is denoted by either f_t or $d_t f$.

For any vector-valued functions $\boldsymbol{f}$ and $\boldsymbol{g}$ and scalar-valued functions x and y we define

$$
\begin{aligned}
&\langle x, y \rangle := \int_0^1 x(s) y(s)\, ds, \\
&\langle \boldsymbol{f}, \boldsymbol{g} \rangle := \int_0^1 \boldsymbol{f}(s) \cdot \boldsymbol{g}(s)\, ds, \qquad \langle \boldsymbol{f}, y \rangle \equiv \langle y, \boldsymbol{f} \rangle := \int_0^1 \boldsymbol{f}(s) y(s)\, ds.
\end{aligned}
\tag{2.1}
$$

We use analogous notation for n-tuples. We use Fubini's Theorem without comment.

We let c, ε, and C denote typical positive constants that are supplied as data or that can be estimated in terms of data. Their meanings usually change with each appearance (even in the same equation or inequality. C may be regarded as increasing and c and ε as decreasing with each appearance). Similarly, $t \mapsto \gamma(t)$ and $t \mapsto \Gamma(t)$ denote typical positive-valued continuous functions depending on the data. Tacit in the statement of an inequality of the form $\|u\| \leq C$ is an assertion that there exists a positive number C such that this estimate holds.

Throughout our exposition we use without comment the Hölder inequality, the Cauchy-Bunyakovskiĭ-Schwarz inequality, and the inequality for arithmetic and geometric means: $2|ab| \leq \frac{\eta^p |a|^p}{p} + \frac{|b|^q}{q \eta^q}$ for real a, b and for positive ε, p, q with $p^{-1} + q^{-1} = 1$. If we take η to be small, then we can replace η^p with ε, and use the convention just discussed to write this last estimate as $|ab| \leq \varepsilon |a|^p + C|b|^q$.

We use only real function spaces. For each nonnegative integer k, $C^k[0,T]$ denotes the space of functions that are k times continuously differentiable on the interval $[0,T]$, $C^{0,\alpha}[0,T]$ denotes the space of functions that are Hölder continuous with exponent $\alpha \in (0,1]$ on the interval $[0,T]$, $H^k(0,1)$ denotes the Sobolev space of functions defined on the interval $(0,1)$ whose distributional derivatives of order k are integrable to the pth power, $H^0(0,1) = L^2(0,1)$, and $L^\infty(0,T)$ denotes the space of essentially bounded functions on $[0,T]$. We use the same notation for spaces of vector-valued functions; the interpretation will be clear from the context.

We denote the norm on a Banach space $\mathfrak{X}$ by $\|\cdot,\mathfrak{X}\|$, but omit $\mathfrak{X}$ when it is a Cartesian product of $L^2(0,1)$. If $\mathfrak{X}$ is a Banach space of functions on the interval $(0,1)$ and if $\mathfrak{Y}$ is a Banach space of real-valued functions on the interval $[0,T]$, then as usual $\mathfrak{Y}(0,T,\mathfrak{X})$ denotes the Banach space of mappings $[0,T] \ni t \mapsto w(\cdot,t) \in \mathfrak{X}$ with norm $\big\| [t \mapsto \|w(\cdot,t),\mathfrak{X}\|\,],\mathfrak{Y}\big\|$. In particular, the square of a norm of w in $H^1(0,T,H^1(0,1))$ is

$$
\begin{aligned}
(2.2) \qquad & \|w, H^1(0,T,H^1(0,1))\|^2 \\
&= \int_0^T \{\|w(\cdot,t),H^1(0,1)\|^2 + \|w_t(\cdot,t),H^1(0,1)\|^2\}\,dt \\
&= \int_0^T\int_0^1 [w(s,t)^2 + w_s(s,t)^2 + w_t(s,t)^2 + w_{st}(s,t)^2]\,ds\,dt.
\end{aligned}
$$

3. Symbols

The following table lists the principal symbols used, their meanings, and where they are defined.

A	See ρA.
C	Positive constant depending only on data. Sec. 2.
c	Small positive constant depending only on data. Sec. 2.
$\boldsymbol{d}_k$	Orthonormal basis giving the orientation of cross sections. (4.1), (4.2).
$\boldsymbol{d}_k^\circ$	Initial function for $\boldsymbol{d}_k$. (7.3).
$\boldsymbol{d}_k^{\mathsf{N}}$	Galerkin approximation of $\boldsymbol{d}_k$. (13.8).
$\mathbf{e}_k$	Standard basis for $\mathbb{R}^3$. Sec. 2.
$\boldsymbol{f}$	Body force per unit reference length of the base curve. (5.1).
H	Energy-like form based on accelerations and strain rates. (11.2).
J_{kl}	See ρJ_{kl}.
$\boldsymbol{J}$	See $\rho\boldsymbol{J}$.
$\mathbf{J}$	See $\rho\mathbf{J}$.
K	Kinetic energy. (9.1a).
K^{N}	Kinetic energy for the Galerkin approximation. (14.2).
$\boldsymbol{l}$	Body couple per unit reference length of the base curve. (5.2).
M	Square root of the dissipative stress power. (9.5a).
M^{N}	Galerkin approximation of M. (14.2).
m_k	Components of $\boldsymbol{m}$ with respect to the basis $\boldsymbol{d}_k$. (5.3).
$\boldsymbol{m}$	Contact couple vector. (4.3).
$\mathbf{m}$	$\equiv (m_1, m_2, m_3)$. Triple of components of $\boldsymbol{m}$ with respect to basis $\boldsymbol{d}_k$. (5.3).
$\hat{\mathbf{m}}$	Constitutive function for $\mathbf{m}$. (5.3), (5.5).

m^{D} Constitutive function for dissipative part of $\hat{\mathsf{m}}$. (5.3), (5.6).
$\bar{\boldsymbol{m}}$ Boundary value of $\boldsymbol{m}$. (7.1c).
m^{N} Galerkin approximation of m. (13.14).
$\mathsf{m}_{\mathrm{D}}^{\mathsf{N}}$ Galerkin approximation of m^{D}. (13.14).
n_k Components of $\boldsymbol{n}$ with respect to the basis $\boldsymbol{d}_k$. (5.3).
$\boldsymbol{n}$ Contact force vector. (4.3).
n $\equiv (n_1, n_2, n_3)$. Triple of components of $\boldsymbol{n}$ with respect to basis $\boldsymbol{d}_k$.(5.3).
$\hat{\mathsf{n}}$ Constitutive function for n. (5.3), (5.5).
n^{D} Constitutive function for dissipative part of $\hat{\mathsf{n}}$. (5.3), (5.6).
$\bar{\boldsymbol{n}}$ Boundary value of $\boldsymbol{n}$. (7.1c).
n^{N} Galerkin approximation of n. (13.14).
$\mathsf{n}_{\mathrm{D}}^{\mathsf{N}}$ Galerkin approximation of n^{D}. (13.14).
$\boldsymbol{p}$ $\equiv \boldsymbol{r}_t$. Velocity of material points on the base curve. (4.4).
p $\equiv (p_1, p_2, p_3)$. Triple of components of $\boldsymbol{p}$ with respect to basis $\boldsymbol{d}_k$. (4.4).
$\boldsymbol{p}_{\mathsf{a}}$ Coefficients of Galerkin approximate of $\boldsymbol{p}$. (13.1).
$\boldsymbol{p}^{\circ}$ Initial function for $\boldsymbol{p}$. (4.4).
$\boldsymbol{p}^{\mathsf{N}}$ Galerkin approximation of p. (13.1).
$q^{\pm}$ Constitutive functions. (19.18).
$\boldsymbol{R}$ $\equiv \boldsymbol{d}_k \mathbf{e}_k$. Orthogonal tensor taking triples z to vectors $\boldsymbol{z}$. (4.6).
$\boldsymbol{R}^{\mathsf{N}}$ Galerkin approximation of R. (13.13).
$\boldsymbol{r}$ Position of material point$s$ on the base curve. (4.1).
$\boldsymbol{r}^{\circ}$ Initial function for $\boldsymbol{r}$. (7.3).
s Arc-length parameter of base curve (4.1).
T Arbitrary fixed time. Theorem 8.1.
t Time. (4.1).
u_k Components of $\boldsymbol{u}$ with respect to the basis $\boldsymbol{d}_k$. (4.4).
$\boldsymbol{u}$ Strain vector accounting for flexure and torsion. (4.3).
u $\equiv (u_1, u_2, u_3)$. Triple of components of $\boldsymbol{u}$ with respect to basis $\boldsymbol{d}_k$. (4.4).
$\dot{\mathsf{u}}$ Argument of constitutive functions occupied by u_t. (5.5a)
u° Initial function for u. (7.5).
$\boldsymbol{u}^{\mathsf{N}}$ Galerkin approximation of $\boldsymbol{u}$. (13.10).
u^{N} Galerkin approximation of u. (13.12).
v_k Components of $\boldsymbol{r}_s$ with respect to the basis $\boldsymbol{d}_k$. (4.4).
$\boldsymbol{v}$ $\equiv \boldsymbol{r}_s$. Strain vector accounting for shear and extension. (4.4).
v $\equiv (v_1, v_2, v_3)$. Triple of components of $\boldsymbol{r}_s$ with respect to basis $\boldsymbol{d}_k$. (4.4).
$\dot{\mathsf{v}}$ Argument of constitutive functions occupied by v_t. (5.5a)
v° Initial function for v. (7.5).
$\boldsymbol{v}^{\mathsf{N}}$ Galerkin approximation of $\boldsymbol{v}$. (13.9).
v^{N} Galerkin approximation of v. (13.12).
$\mathfrak{W}$ A Hilbert space of H^1 functions. (7.9).
w_k Components of $\boldsymbol{w}$ with respect to the basis $\boldsymbol{d}_k$. (4.4).
$\boldsymbol{w}$ Angular velocity of the triad $\boldsymbol{d}_k$. (4.3).
$\boldsymbol{w}^{\circ}$ Initial function for $\boldsymbol{w}$. (7.3).
w $\equiv (w_1, w_2, w_3)$. Triple of components of $\boldsymbol{w}$ with respect to basis $\boldsymbol{d}_k$. (4.3).
$\boldsymbol{w}_{\mathsf{a}}$ Coefficients of Galerkin approximate of $\boldsymbol{w}$. (13.1).
$\boldsymbol{w}^{\mathsf{N}}$ Galerkin approximation of $\boldsymbol{w}$. (13.1).
$\boldsymbol{X}^{\mathsf{N}}$ Projector onto span$\{x_1, \ldots, x_{\mathsf{N}}\}$. (13.6).
$\boldsymbol{Y}^{\mathsf{N}}$ Projector onto span$\{y_1, \ldots, y_{\mathsf{N}}\}$. (13.6).
x Argument of constitutive functions occupied by s. (5.5a).
x_{a} Normalized derivatives of the shape functions y_{a}. (13.3).
y_{a} Shape functions for Galerkin approximation. (13.3).
β Constitutive function associated with coercivity. (19.3).

Γ	Positive-valued function of t depending only on the data, bounded on bounded intervals. Sec. 1.
γ	Small positive-valued function of t depending only on the data. Sec. 1.
δ	Difference operator. (18.1).
ε	Small constant depending on the data. Sec. 2.
$\boldsymbol{\zeta}$	Strain variables with $\zeta_3 := v_3 - \Upsilon(u_1, u_2, s)$. (6.5).
$\check{\boldsymbol{\zeta}}$	Function delivering $\boldsymbol{\zeta}$ from $\boldsymbol{\eta}$. (6.5).
$\boldsymbol{\eta}$	$\equiv (\mathsf{v}, \mathsf{u})$. (4.2).
$\dot{\boldsymbol{\eta}}$	Argument of constitutive functions occupied by $\boldsymbol{\eta}_t$. (5.5a)
$\boldsymbol{\eta}^\circ$	Initial value of $\boldsymbol{\eta}$. (7.5).
$\boldsymbol{\eta}^{\mathsf{N}}$	$\equiv (\mathsf{v}^{\mathsf{N}}, \mathsf{u}^{\mathsf{N}})$. Galerkin approximation of $\boldsymbol{\eta}$. (13.8).
$\boldsymbol{\eta}^\sharp$	Cut-off strain. (12.3).
$\boldsymbol{\Lambda}$	Positive-definite square root of the symmetric part of $\boldsymbol{\sigma}_{\dot{\boldsymbol{\eta}}}$. Hypothesis 6.11.
$\boldsymbol{\kappa}$	Inverse of $\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \cdot, x)$. (11.28).
ν_{a}	Eigenvalues associated with shape functions. (13.2).
ρA	Mass per unit reference length. (5.1).
ρJ_{kl}	Mass moments of inertia of a cross section with respect to $\boldsymbol{d}_k$. (5.2).
$\rho \boldsymbol{J}$	Tensor of mass moments of inertia of a cross section. (5.2).
$\rho \mathsf{J}$	Matrix of mass moments of inertia of a cross section. (5.2).
$\boldsymbol{\sigma}$	$\equiv (\mathsf{n}, \mathsf{m})$. (5.3).
$\hat{\boldsymbol{\sigma}}$	Constitutive function for $\boldsymbol{\sigma}$. (5.6).
$\boldsymbol{\sigma}^{\mathrm{D}}$	Constitutive function for dissipative part of $\hat{\boldsymbol{\sigma}}$. (5.6).
$\check{\boldsymbol{\sigma}}$	Constitutive function depending on $\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}$. (6.6).
$\boldsymbol{\sigma}^\sharp$	Cut-off constitutive function. (12.4).
Υ	Function associated with the Jacobian of deformation. (4.14).
Φ	Total stored energy. (9.1b).
Φ^{N}	Total stored energy for the Galerkin approximation. (14.2).
φ	Stored-energy function for elastic part of constitutive equations. (5.6).
ψ	Function describing strong dissipation near a total compression. (6.10).
$\boldsymbol{\chi}$	$\equiv (v_1, v_2, \mathsf{u})$. (12.2), (19.1).
Ω	Work of the dissipative part of the stresses. (9.1c).
Ω^{N}	Galerkin approximation of Ω. (14.2).

II. Governing Equations

We give a brief coordinate-free formulation of the classical form of geometrically exact equations of motion for a rod that can suffer flexure, extension, torsion, and shear. For full details and motivations, see [7, Chap.$\tilde{8}$].

4. Geometry of Deformation

The *motion of a rod* is defined here by three vector-valued functions

$$[0, 1] \times \mathbb{R} \ni (s, t) \mapsto \boldsymbol{r}(s, t),\ \ \boldsymbol{d}_1(s, t),\ \ \boldsymbol{d}_2(s, t) \in \mathbb{E}^3 \tag{4.1}$$

with $\{\boldsymbol{d}_1(s, t), \boldsymbol{d}_2(s, t)\}$ orthonormal. The function $\boldsymbol{r}(\cdot, t)$ may be interpreted as the configuration at time t of the curve of centroids of a slender 3-dimensional body. The vectors $\boldsymbol{d}_1(s,t)$ and $\boldsymbol{d}_2(s,t)$ may be interpreted as characterizing the orientation of the material section at s at time t. In particular, $\boldsymbol{d}_1(s,t)$ and $\boldsymbol{d}_2(s,t)$ may be regarded as characterizing the configurations at time t of a pair of orthogonal material lines of the section s. We assume that s is the arc-length parameter of the reference configuration of $\boldsymbol{r}$, and we scale the length so that $0 \le s \le 1$. We set

$$\boldsymbol{d}_3 := \boldsymbol{d}_1 \times \boldsymbol{d}_2. \tag{4.2}$$

Since $\{\boldsymbol{d}_k(s,t)\}$ is a right-handed orthonormal basis for $\mathbb{E}^3$ for each (s,t), there are vector-valued functions $\boldsymbol{u}$ and $\boldsymbol{w}$ such that

$$\partial_s \boldsymbol{d}_k = \boldsymbol{u} \times \boldsymbol{d}_k, \quad \partial_t \boldsymbol{d}_k = \boldsymbol{w} \times \boldsymbol{d}_k. \tag{4.3}$$

Since the basis $\{\boldsymbol{d}_k\}$ is natural for the intrinsic description of deformation, we decompose relevant vector-valued functions with respect to it:

$$\boldsymbol{v} := \boldsymbol{r}_s = v_k \boldsymbol{d}_k, \quad \boldsymbol{p} := \boldsymbol{r}_t = p_k \boldsymbol{d}_k, \quad \boldsymbol{u} = u_k \boldsymbol{d}_k, \quad \boldsymbol{w} = w_k \boldsymbol{d}_k \tag{4.4}$$

(so that $v_k := \boldsymbol{v} \cdot \boldsymbol{d}_k$, etc.). We take $\{\mathsf{e}_k\}$ to be the standard basis for $\mathbb{R}^3$:

$$\mathsf{e}_1 = (1,0,0), \quad \text{etc.} \tag{4.5}$$

We adopt the notation that the triple (z_1, z_2, z_3) of components of a vector $\boldsymbol{z}$ with respect to the orthonormal basis $\{\boldsymbol{d}_k\}$ is denoted by the corresponding bold sanserif symbol:

$$\begin{gathered}(z_1, z_2, z_3) \equiv \mathsf{z} \equiv z_k \mathsf{e}_k \equiv \boldsymbol{z} \cdot \boldsymbol{d}_k \mathsf{e}_k \equiv (\mathsf{e}_k \boldsymbol{d}_k) \cdot \boldsymbol{z} =: \boldsymbol{R}^* \cdot \boldsymbol{z}, \\ \boldsymbol{z} = \mathsf{z} \cdot \mathsf{e}_k \boldsymbol{d}_k \equiv (\boldsymbol{d}_k \mathsf{e}_k) \cdot \mathsf{z} =: \boldsymbol{R} \cdot \mathsf{z}.\end{gathered} \tag{4.6}$$

(Here we have used the dyadic notation described in the first paragraph of Sec. 1. Note that $\boldsymbol{R}$ is an orthogonal transformation (from $\mathbb{R}^3$ to $\mathbb{E}^3$), so that its inverse is its transpose $\boldsymbol{R}^*$. In view of (4.3), it satisfies

$$\boldsymbol{R}_t = \boldsymbol{w} \times \boldsymbol{R} \tag{4.7}$$

(see the first paragraph of Sec. 2 for notation), so that

$$\boldsymbol{z}_t = \boldsymbol{R} \cdot \mathsf{z}_t + \boldsymbol{w} \times \boldsymbol{z}, \qquad \boldsymbol{z}_{tt} = \boldsymbol{R} \cdot \mathsf{z}_{tt} + 2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathsf{z}_t) + \boldsymbol{w}_t \times \boldsymbol{v} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{z}) \tag{4.8}$$

for any function $\boldsymbol{z}$.

Of course, the specification of the $\boldsymbol{d}_k$ is equivalent to the specification of the proper-orthogonal (rotation) tensor $\boldsymbol{R}$ (cf. [33]). We suppress the role of $\boldsymbol{R}$, limiting its use to a notational device, because the form of the constitutive equations invariant under rigid motion is most easily expressed in terms of components of the stress resultants with respect to the base vectors $\boldsymbol{d}_k$, and because in more general theories of rods, the $\boldsymbol{d}_k$ need be neither orthonormal nor three in number.

The equality of mixed partial derivatives of $\boldsymbol{r}$ and of the $\boldsymbol{d}_k$, together with the use (4.8), yields the compatibility conditions

$$\boldsymbol{p}_s = \boldsymbol{v}_t = \boldsymbol{R} \cdot \mathsf{v}_t + \boldsymbol{w} \times \boldsymbol{v}, \quad \boldsymbol{w}_s = \boldsymbol{u}_t + \boldsymbol{u} \times \boldsymbol{w} = \boldsymbol{R} \cdot \mathsf{u}_t. \tag{4.9}$$

Equation$(4.3)_2$, equation of $(4.9)_2$, and identity $(4.8)_1$ each have the form

$$\boldsymbol{y}_t = \boldsymbol{w} \times \boldsymbol{y} + \boldsymbol{f}. \tag{4.10a}$$

We need some simple estimates for the solution of this equation. Let $\boldsymbol{\Psi}$ be the fundamental tensor solution of homogeneous version of (4.10a) with $\boldsymbol{\Psi}(0) = \boldsymbol{I}$. Equation (4.7) implies that $\boldsymbol{R}$ is a fundamental tensor solution, so that $\boldsymbol{\Psi}(t) = \boldsymbol{R}(t) \cdot \boldsymbol{R}(0)^{-1}$. The solution $\boldsymbol{y}$ of (4.10a) is given by

$$\boldsymbol{y}(t) = \boldsymbol{\Psi}(t) \cdot \boldsymbol{y}(0) + \boldsymbol{\Psi}(t) \cdot \int_0^t \boldsymbol{\Psi}(\tau)^{-1} \cdot \boldsymbol{f}(\tau)\, d\tau. \tag{4.10b}$$

Since $\boldsymbol{R}$ is orthogonal, so is $\boldsymbol{\Psi}$. From this formula for $\boldsymbol{y}$ then follows the bound

$$|\boldsymbol{y}(t)| \le |\boldsymbol{y}(0)| + \int_0^t |\boldsymbol{f}(\tau)|\, d\tau \tag{4.11}$$

(which is the same bound as that obtained by taking $\boldsymbol{w} = \boldsymbol{o}$ in (4.10a). It is sharper than the bound derivable from the differential inequality $d_t|\boldsymbol{y}|^2 \le |\boldsymbol{y}|^2 + |\boldsymbol{f}|^2$, which is obtained by taking the dot product of (4.10a) with $\boldsymbol{y}$. The sharper bound simplifies some of our formulas. Though not crucial here, it would be crucial in an analysis requiring detailed decay rates).

We set

$$\boldsymbol{\eta} := (\mathsf{v}, \mathsf{u}) \equiv (v_1, v_2, v_3, u_1, u_2, u_3). \tag{4.12}$$

The components of $\boldsymbol{\eta}$ are the *strain* variables corresponding to the motion (4.1). For each fixed t the function $\boldsymbol{\eta}(\cdot, t)$ determines $\boldsymbol{r}(\cdot, t)$, $\boldsymbol{d}_1(\cdot, t)$, $\boldsymbol{d}_2(\cdot, t)$ (the *configuration at time* t) to within a rigid motion and thus accounts for change of shape. The strains v_1 and v_2 measure shear, v_3 measures dilatation, u_1 and u_2 measure flexure, and u_3 measures torsion.

It follows from (4.9) that

$$\boldsymbol{\eta}_t \equiv (\mathsf{v}_t, \mathsf{u}_t) = (\mathsf{p}_s + \mathsf{u} \times \mathsf{p} - \mathsf{w} \times \mathsf{v},\ \mathsf{w}_s - \mathsf{w} \times \mathsf{u},). \tag{4.13}$$

We shall freely switch between the pair (v, u) and its single symbol $\boldsymbol{\eta}$, using whichever leads to a more compact or illuminating expression in each particular circumstance. We shall not need the second form of (4.13); it could be used to show that the governing equations could be cast with all the time derivatives on one side.

Let us interpret our kinematic variables as corresponding to generalized coordinates arising from the imposition of a fairly general family of constraints on the deformation of a 3-dimensional rod-like body. Then a rod-theoretic analog of the 3-dimensional requirement that the Jacobian of the deformation be positive (so that orientation is preserved) is that there be a function $(u_1, u_2, s) \mapsto \Upsilon(u_1, u_2, s)$ (depending only on the flexural strains and s) for which $\Upsilon(0, 0, s) = 0$, $\Upsilon(u_1, u_2, s) > 0$ for $u_\alpha u_\alpha > 0$, and such that

$$v_3 > \Upsilon(u_1, u_2, s). \tag{4.14}$$

(See [7, Secs. 8.6, 14.2]. If these constraints ensure that plane sections remain plane and undeformed, then the function $\Upsilon(\cdot,\cdot,s)$ is convex and homogeneous of degree 1, so that for each s, the surface $v_3 = \Upsilon(u_1,u_2,s)$ is a cone in (u_1,u_2,v_3)-space. E.g., for a rod with a circular cross section at s of radius $h(s)$, the function Υ reduces to $\Upsilon(u_1,u_2,s) = h(s)\sqrt{(u_1)^2+(u_2)^2}$ [7, Sec. 8.7].) We adopt (4.14) as an essential restriction on the deformation. A consequence of it is that

$$v_3 \equiv \boldsymbol{r}_s \cdot \boldsymbol{d}_3 > 0. \tag{4.15}$$

This condition implies that (i) $|\boldsymbol{r}_s| \equiv \sqrt{v_k v_k} > 0$, so that the local ratio of deformed to reference length of the axis cannot be reduced to zero, and (ii) a typical section s cannot undergo a total shear in which the plane determined by $\boldsymbol{d}_1(s,t)$ and $\boldsymbol{d}_2(s,t)$ is tangent to the curve $\boldsymbol{r}(\cdot,t)$ at $\boldsymbol{r}(s,t)$. The general condition (4.14) further ensures that two distinct material cross sections cannot intersect within a deformed configuration of the rod-like body. We say that a *total compression* occurs when (4.14) is violated.

5. Mechanics and Material Behavior

In the configuration at time t, the resultant contact force and contact couple exerted by the material of $(s,1]$ on the material of $[0,s]$ (for $0 < s \le 1$) are respectively denoted $\boldsymbol{n}(s,t)$ and $\boldsymbol{m}(s,t)$. At (s,t) the rod is subjected to a body force of intensity $\boldsymbol{f}(s,t)$ and body couple of intensity $\boldsymbol{l}(s,t)$ per unit reference length at (s,t). Then the classical equations of motion (under the interpretation that $\boldsymbol{r}$ is a suitably weighted material curve of centroids [7, Ex. 8.4.8]) have the form

$$\boldsymbol{n}_s + \boldsymbol{f} = \rho A \boldsymbol{r}_{tt}, \tag{5.1}$$

$$\begin{aligned} \boldsymbol{m}_s + \boldsymbol{r}_s \times \boldsymbol{n} + \boldsymbol{l} &= \partial_t(\rho \boldsymbol{J}\cdot \boldsymbol{w}) := \partial_t(\rho J_{pq} w_q \boldsymbol{d}_p) \\ &\equiv \rho \boldsymbol{J}\cdot \boldsymbol{w}_t + \boldsymbol{w}\times(\rho \boldsymbol{J}\cdot\boldsymbol{w}). \end{aligned} \tag{5.2}$$

Here $(\rho A)(s)$ is the prescribed positive mass density per reference length at s, the $(\rho J_{\gamma\delta})(s)$, $\gamma,\delta = 1,2$, are the prescribed components of the positive-definite symmetric 2×2 matrix of mass-moments of inertia of the section s. The positive-definite symmetric 3×3 matrix $\rho \mathbf{J} := (\rho J_{pq})$ is defined by $\rho J_{\gamma 3} = \rho J_{3\gamma} = 0$, $\rho J_{33} = \rho J_{\gamma\gamma}$, and $\rho \boldsymbol{J} := \rho J_{pq}\boldsymbol{d}_p\boldsymbol{d}_q$. (Thus $\rho\boldsymbol{J}$ depends on t, but $\rho\mathbf{J}$ does not.) It is reasonable to assume that $\rho A, \rho J_{pq}$ are piecewise continuous. (The jumps that these functions may suffer reflect abrupt changes in the geometry of the cross sections or in the density distribution of a rod interpreted as a 3-dimensional body.) For simplicity of exposition, however, we take $\rho A, \rho J_{pq}$ to be continuous; the straightforward adjustments for piecewise continuous functions are left to the interested reader. (If $\boldsymbol{r}$ is not the weighted curve of centroids, the resulting equations of motion are somewhat more complicated, but the analysis is only more difficult from the viewpoint of notation. Moreover, the general case can be reduced to that treated here by a suitable change of variables.)

Since the treatment of nonzero loads is standard, we take $\boldsymbol{f} = \boldsymbol{o} = \boldsymbol{l}$ to simplify our presentation. Thus (5.1), (5.2) reduce to (1.2), (1.3).

The reference values of the $\boldsymbol{d}_k$ can be chosen so that the matrix $\rho\mathsf{J}$ is diagonal, but if the geometry of the cross sections in the reference configuration were to change abruptly, then the reference values of the $\boldsymbol{d}_k$ that effect this diagonalization might be discontinuous, whence we could expect the $\boldsymbol{d}_k$ to be discontinuous. Even though we take $\rho\mathsf{J}$ to be continuous, for the purpose of generalizing our results to handle the more complicated case, we do not insist that $\rho\mathsf{J}$ be diagonal.

The complications caused by the acceleration terms on the right-hand side of (5.2) are even worse than those that arise in Euler's equations of motion for a rigid body.

Constitutive equations. Let

$$n_k := \boldsymbol{n}\cdot\boldsymbol{d}_k, \quad m_k := \boldsymbol{m}\cdot\boldsymbol{d}_k, \qquad (5.3)$$
$$\mathbf{n} := (n_1, n_2, n_3), \quad \mathbf{m} := (m_1, m_2, m_3), \quad \boldsymbol{\sigma} := (\mathbf{n}, \mathbf{m}).$$

n_1 and n_2 are the *shear forces*, $\boldsymbol{n}\cdot\boldsymbol{r}_s/|\boldsymbol{r}_s|$ is the *tension*, m_1 and m_2 are the *bending couples*, and m_3 is the *twisting couple*. $\boldsymbol{\sigma}$ is the set of *stress resultants*. We set

$$\boldsymbol{\sigma}\cdot\boldsymbol{\eta} := \mathbf{n}\cdot\mathbf{v} + \mathbf{m}\cdot\mathbf{u} = n_k v_k + m_k u_k, \quad \text{etc.} \qquad (5.4)$$

We limit our attention to rods that are *viscoelastic of strain-rate type* (*of complexity 1*), which have the defining property that there is a constitutive function

$$(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \mapsto \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \qquad (5.5a)$$

such that

$$\boldsymbol{\sigma}(s,t) = \hat{\boldsymbol{\sigma}}\big(\boldsymbol{\eta}(s,t), \boldsymbol{\eta}_t(s,t), s\big). \qquad (5.5b)$$

Throughout this paper, superposed dots, like that over $\boldsymbol{\eta}$ in (5.5a), have no operational significance; in (5.5a) the $\dot{\boldsymbol{\eta}}$ merely identifies the second argument of $\hat{\boldsymbol{\sigma}}$, which is typically occupied by the time derivative $\boldsymbol{\eta}_t$. The last argument of $\hat{\boldsymbol{\sigma}}$ in (5.5a) is denoted x so that we can distinguish between the partial derivative $\partial_x\hat{\boldsymbol{\sigma}}\big(\boldsymbol{\eta}(s,t), \boldsymbol{\eta}_t(s,t), s\big)$ with respect to the last argument and the total partial derivative $\partial_s\hat{\boldsymbol{\sigma}}\big(\boldsymbol{\eta}(s,t), \boldsymbol{\eta}_t(s,t), s\big)$, which must be computed by the chain rule. The domain of the constitutive function (5.5a) is defined by (4.14). This form of the constitutive equations ensures that the material response is unaffected by rigid motions. For simplicity of exposition we assume that the constitutive function $\hat{\boldsymbol{\sigma}}$ is continuously differentiable in $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$. It is reasonable to assume that this function is piecewise continuous in x. Possible jumps in the dependence of $\hat{\boldsymbol{\sigma}}$ on x would reflect abrupt changes in the geometry or in the material properties of the 3-dimensional body modelled by our theory. For simplicity of exposition, however, we take $\hat{\boldsymbol{\sigma}}$ to be continuously differentiable in x. (If this function were only uniformly continuously differentiable on a finite number of disjoint open intervals whose closures cover $[0,1]$, then we would have to relate our analyses on each such interval of continuity by connection formulas coming from a weak formulation of the governing equations.)

We assume that the *equilibrium* (or *elastic*) *response* $\hat{\boldsymbol{\sigma}}(\cdot,\mathbf{0},s)$ of $\hat{\boldsymbol{\sigma}}(\cdot,\cdot,s)$ is the derivative of a stored-energy function $\varphi(\cdot,s)$ with respect to $\boldsymbol{\eta}$, so that $\hat{\boldsymbol{\sigma}}$ admits the following decomposition into equilibrium and *dissipative parts*:

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},x) = \varphi_{\boldsymbol{\eta}}(\boldsymbol{\eta},x) + \boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},x) \quad \text{where} \quad \boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta},\mathbf{0},x) = \mathbf{0}. \tag{5.6}$$

Of course, the constitutive functions depend on the choice of a reference configuration. Since our analysis is global, in particular, not limited to a neighborhood of an equilibrium configuration (which could be taken to be a reference configuration), we do not make the dependence on the reference configuration explicit.

The governing partial differential equations. We now recast our governing partial differential equations as a vectorial system of first order in the time derivative. Equations (4.3)–(4.9), (5.1), (5.2) imply that

$$\partial_t \boldsymbol{d}_k = \boldsymbol{w} \times \boldsymbol{d}_k, \tag{5.7a}$$

$$\boldsymbol{v}_t = \boldsymbol{p}_s, \tag{5.7b}$$

$$\boldsymbol{u}_t = \boldsymbol{w}_s - \boldsymbol{u} \times \boldsymbol{w}, \tag{5.7c}$$

$$\rho A \boldsymbol{p}_t = \partial_s(\hat{n}_k \boldsymbol{d}_k), \tag{5.7d}$$

$$\partial_t(\rho \boldsymbol{J} \cdot \boldsymbol{w}) = \partial_s(\hat{m}_k \boldsymbol{d}_k) + \boldsymbol{v} \times \hat{n}_k \boldsymbol{d}_k, \tag{5.7e}$$

where the arguments of $\hat{m}_k$ and $\hat{n}_k$ are $\boldsymbol{\eta}, \boldsymbol{\eta}_t, s$. If we take these arguments to be

$$\boldsymbol{v} \cdot \boldsymbol{d}_l, \quad \boldsymbol{u} \cdot \boldsymbol{d}_l, \quad \boldsymbol{p}_s \cdot \boldsymbol{d}_l - (\boldsymbol{w} \times \boldsymbol{v}) \cdot \boldsymbol{d}_l, \quad \boldsymbol{w}_s \cdot \boldsymbol{d}_l, \quad s, \tag{5.8}$$

then the equations have a general conservation form with all the time derivatives on the left-hand side. Note that the ordinary differential equation (5.7a) preserves the dot products $\boldsymbol{d}_k \cdot \boldsymbol{d}_l$ and therefore ensures that $\{\boldsymbol{d}_k(s,t)\}$ is an orthonormal basis for all s,t if $\{\boldsymbol{d}_k(s,0)\}$ is an orthonormal basis for all s.

6. Constitutive Restrictions

We impose the very mild requirement that the stored-energy function be bounded below (without loss of generality by 0) and become infinite at infinite strains:

6.1. Hypothesis.

$$\varphi(\boldsymbol{\eta},x) \geq 0, \qquad \varphi(\boldsymbol{\eta},x) \to \infty \quad \textit{as} \quad |\boldsymbol{\eta}| \to \infty \quad \textit{uniformly in } x. \tag{6.1}$$

We require that effects of internal friction grow with the strain rates:

6.2. Hypothesis. *There is a positive number c such that*

$$[\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}_1, x) - \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}_2, x)] \cdot [\dot{\boldsymbol{\eta}}_1 - \dot{\boldsymbol{\eta}}_2] \geq c|\dot{\boldsymbol{\eta}}_1 - \dot{\boldsymbol{\eta}}_2|^2 \tag{6.2}$$

for all values of the variables that appear.

This monotonicity condition ensures that the response be truly dissipative (uniformly in $\boldsymbol{\eta}$) and that the governing equations of motion have a parabolic character. Inequality (6.2) is responsible for the regularity of solutions, and, in particular, for the absence of shocks, which are typically present in analogous problems for elastic rods (in which $\hat{\boldsymbol{\sigma}}$ depends only on $\boldsymbol{\eta}$). Of course, we can replace $\hat{\boldsymbol{\sigma}}$ in (6.2) with $\boldsymbol{\sigma}^{\mathrm{D}}$. Since the constitutive function $\hat{\boldsymbol{\sigma}}$ is assumed to be differentiable in $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$, (6.2) is equivalent to

$$\boldsymbol{\xi} \cdot \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi} \cdot \boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \cdot \boldsymbol{\xi} \geq c|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi}. \tag{6.3}$$

Condition (6.2) clearly implies that

$$\boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \cdot \dot{\boldsymbol{\eta}} \geq c|\dot{\boldsymbol{\eta}}|^2. \tag{6.4}$$

In the 3-dimensional nonlinear theories of elasticity and viscoelasticity, the Strong Ellipticity Condition, which is the generalization of the monotonicity condition, is most easily represented in terms of constitutive functions in which the invariance under rigid motions is suppressed. A virtue of our formulation of rod theory is that the invariant constitutive equations (5.5) admit the elegant form (6.2) of the monotonicity condition.

To describe the behavior of the constitutive functions for strains near those for a total compression, we introduce new strain variables by the change of variables

$$\boldsymbol{\zeta} = \check{\boldsymbol{\zeta}}(\boldsymbol{\eta}, x), \quad \check{\zeta}_3(\boldsymbol{\eta}, x) := v_3 - \Upsilon(u_1, u_2, s), \quad \check{\zeta}_j(\boldsymbol{\eta}, x) := \eta_j,\ j = 1, 2, 4, 5, 6. \tag{6.5}$$

(Υ was defined in (4.14).) Note that for each x the mapping $\check{\boldsymbol{\zeta}}(\cdot, x)$ has an inverse, which is denoted by $\check{\boldsymbol{\eta}}(\cdot, x)$ and that (4.14) confines $\boldsymbol{\zeta}$ to the region $\zeta_3 > 0$. Furthermore, $\check{\zeta}_3(\mathbf{0}, x) = 0$, so that bounding ζ_3 away from zero bounds $|\boldsymbol{\eta}|$ away from zero. We define

$$\check{\boldsymbol{\sigma}}(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x) := \hat{\boldsymbol{\sigma}}(\check{\boldsymbol{\eta}}(\boldsymbol{\zeta}, x), \check{\boldsymbol{\eta}}_{\boldsymbol{\zeta}}(\boldsymbol{\zeta}, x) \cdot \dot{\boldsymbol{\zeta}}, x) \tag{6.6}$$

and define the dissipative part $\check{\boldsymbol{\sigma}}^{\mathrm{D}}$ of $\check{\boldsymbol{\sigma}}$ as in (5.6).

We require that the resultants become infinite at a total compression:

$$|\check{\boldsymbol{\sigma}}(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x)| \to \infty \quad \text{as} \quad \zeta_3 \searrow 0. \tag{6.7}$$

A reasonable specialization of (6.7) is that

$$\check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x) \to -\infty \quad \text{as} \quad \zeta_3 \searrow 0. \tag{6.8}$$

A much stronger restriction than these is the requirement that

$$\varphi(\boldsymbol{\eta}, x) \to \infty \quad \text{as} \quad \zeta_3 \searrow 0. \tag{6.9}$$

This condition can be used to show that total compression cannot occur for reasonable *equilibrium* problems (cf. [4,7,32]). The energy estimate (9.14) below immediately shows that for any fixed t, the set of s on which there is a total compression must have measure 0. But (6.9) has never been shown capable by itself of preventing total compression everywhere for dynamical problems. For this purpose we require here that viscous effects become infinitely large in a suitable way at a total compression. We do not impose (6.9).

There are several versions of constitutive hypotheses that enable us to give a precise expression to this requirement and to complementary requirements at large strains. We face two difficulties:
(i) The weakest hypotheses, which accommodate the richest variety of constitutive response, are not easy to state and provide the most challenges to analysis.
(ii) There are several closely related systems of constitutive hypotheses that support the analysis. They differ in whether constants that enter the inequalities are defined for all of certain collections of strain variables or depend on bounds for these collections. By using a uniform version of one hypothesis we can get bounds on strains that enable us to use a weaker non-uniform complementary hypotheses for the subsequent development. We accordingly can start this process with different choices of uniform hypotheses.
Since many of these issues were treated in great detail in [12], we content ourselves in the main part of this paper with applying the mathematically simplest hypotheses to a problem with a specific set of boundary conditions. This policy enables us to focus on the novel aspects of our analysis. We defer to Sec. 19 a discussion of the manifold variants of our constitutive functions that also support the analysis.

Our crudest (but most transparent) constitutive restriction ensuring that frictional effects become infinitely large at a total compression and that (6.8) hold for $\dot{\delta} < 0$ is that

6.10. Hypothesis. *There are numbers $\varepsilon \in (0,1)$ and $A \geq 0$, and there is a continuously differentiable function ψ on $(0,\varepsilon)$ with $\psi(\zeta_3) \to \infty$ as $\delta \to 0$ such that*
(6.10)
$$\check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x) \leq -\psi'(\zeta_3)\dot{\zeta}_3 + A\psi(\zeta_3) \quad \forall\, \zeta_3 \in (0,\varepsilon), \quad \forall\, \zeta_1, \zeta_2, \zeta_4, \zeta_5, \zeta_6, \dot{\boldsymbol{\zeta}}, x.$$

With scarcely more generality, ψ and A could be allowed to depend on s. If $A = 0$, then for problems of free motion, the methods of Sec. 10 can immediately be used to show that (6.10) gives a positive lower bound for ζ_3 that is independent of t. The requirement that (6.10) hold for all values of $\dot{\zeta}_3$ unduly restricts the growth of the constitutive functions for large $\dot{\zeta}_3$. In Sec. 19, we shall formulate a refinement of Hypothesis 6.10 that does not suffer from this disadvantage.

The following hypothesis says that when the strains are suitably controlled, the "elasticity" $\partial\hat{\boldsymbol{\sigma}}/\partial\boldsymbol{\eta}$ is dominated by the "viscosity" $\partial\hat{\boldsymbol{\sigma}}/\partial\dot{\boldsymbol{\eta}}$. For this to occur, the viscosities must depend appropriately on the strains. This dependence is the underlying theme of our constitutive restrictions.

6.11. Hypothesis. *Let $\boldsymbol{\Lambda}$ be the positive-definite square root of the* (*positive-definite*) *symmetric part of $\hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}}$. For each $c > 0$ there is a number C such that*

$$
\begin{aligned}
&|\boldsymbol{\Lambda}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x)^{-1} \cdot \hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \cdot \dot{\boldsymbol{\eta}}|^2 \\
&\qquad \le C\,[1 + \boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x) \cdot \dot{\boldsymbol{\eta}} + \varphi(\boldsymbol{\eta}, x)] \quad \textit{when} \quad \zeta_3 \ge c.
\end{aligned}
\tag{6.11}
$$

Our basic constitutive hypotheses are Hypotheses 6.1, 6.2, 6.10, 6.11, namely, those that are numbered. It is important to note that these hypotheses are consistent. Since we characterize total compression by the single limit that $\zeta_3 \searrow 0$, a proof of consistency follows that of the *Note Added in Proof of* [12]. (For exhibiting a set of constitutive functions that meet these hypotheses it suffices to take these functions to be uncoupled in the sense that $\check{\sigma}_3$ depends only on $\zeta_3, \dot{\zeta}_3$, while the remaining $\check{\sigma}_i$ respectively depend only on $\zeta_i, \dot{\zeta}_i$, i.e., $\check{\sigma}_1$ depends only on $\zeta_1, \dot{\zeta}_1$, etc. Cf. (6.6).) Hypothesis 6.11 generalizes to our vectorial setting those hypotheses given in the *Note*, which, as noted there, are much weaker than the fundamental condition introduced by DAFERMOS [18].

7. Boundary and Initial Conditions. Weak Formulation

In the main part of this paper, we limit our attention to a single set of simple boundary conditions, commenting on other possibilities in Sec. 20. We assume here that the end $s = 0$ of the rod is fixed at the origin and that it is welded to a fixed plane spanned by vectors $\boldsymbol{d}_1^\circ(0)$ and $\boldsymbol{d}_2^\circ(0)$. Thus, the classical forms of these boundary conditions are

$$
\begin{gathered}
\boldsymbol{r}(0,t) = \boldsymbol{o}, \\
\boldsymbol{d}_k(0,t) \quad \text{are prescribed constant orthonormal vectors } \boldsymbol{d}_k^\circ(0)
\end{gathered}
\tag{7.1a}
$$

so that

$$
\boldsymbol{p}(0,t) = \boldsymbol{o}, \quad \boldsymbol{w}(0,t) = \boldsymbol{o}. \tag{7.1b}
$$

We assume that the end $s = 1$ is free and is subject to a given applied force $\bar{\boldsymbol{n}}$ and to a given applied couple $\bar{\boldsymbol{m}}$:

$$
\boldsymbol{n}(1,t) = \bar{\boldsymbol{n}}(t), \quad \boldsymbol{m}(1,t) = \bar{\boldsymbol{m}}(t), \tag{7.1c}
$$

with

$$
\bar{\boldsymbol{n}},\ \bar{\boldsymbol{m}} \in H^1_{\mathrm{loc}}[0,\infty). \tag{7.2}
$$

The classical form of the initial conditions are

$$
\boldsymbol{r}(s,0) = \boldsymbol{r}^\circ(s), \quad \boldsymbol{d}_k(s,0) = \boldsymbol{d}_k^\circ(s), \quad \boldsymbol{r}_t(s,0) = \boldsymbol{p}^\circ(s), \quad \boldsymbol{w}(s,0) = \boldsymbol{w}^\circ(s) \tag{7.3}
$$

with

$$\boldsymbol{p}^\circ(0) = \boldsymbol{o}, \quad \boldsymbol{w}^\circ(0) = \boldsymbol{o} \tag{7.4}$$

for compatibility. Let $\boldsymbol{\eta}^\circ := (\mathbf{v}^\circ, \mathbf{u}^\circ)$ be defined in terms of the initial data (7.3) by

$$\partial_s \boldsymbol{d}_k^\circ = u_j^\circ \boldsymbol{d}_j^\circ \times \boldsymbol{d}_k^\circ, \qquad v_k^\circ := \boldsymbol{r}_s^\circ \cdot \boldsymbol{d}_k^\circ. \tag{7.5}$$

Using (4.13) we find that the initial value of $\boldsymbol{\eta}_t \equiv (\mathbf{v}_t, \mathbf{u}_t)$ is given by

$$\boldsymbol{\eta}_t(\cdot, 0) = (\mathbf{p}_s^\circ + \mathbf{u}^\circ \times \mathbf{p}^\circ - \mathbf{w}^\circ \times \mathbf{v}^\circ,\ \mathbf{w}_s^\circ - \mathbf{w}^\circ \times \mathbf{u}^\circ). \tag{7.6}$$

7.7. Hypothesis. *The initial values* $\boldsymbol{r}^\circ$ *and* $\boldsymbol{d}_k^\circ$ *of* $\boldsymbol{r}$ *and* $\boldsymbol{d}_k$ *lie in* $H^2(0,1)$*, the initial values* $\boldsymbol{p}^\circ$ *and* $\boldsymbol{w}^\circ$ *of* $\boldsymbol{r}_t$ *and* $\boldsymbol{w}$ *lie in* $H^1(0,1)$*, and the initial values* $\boldsymbol{r}_{tt}(\cdot,0)$ *and* $\boldsymbol{w}_t(\cdot,0)$*, which are defined by the partial differential equations* (5.7d,e)*, lie in* $L^2(0,1)$*, i.e.,*

(7.8a)

$$\begin{aligned}\|\rho A \boldsymbol{r}_{tt}(\cdot,0)\| &:= \|\partial_s \hat{n}_k(\boldsymbol{\eta}^\circ, \boldsymbol{\eta}_t(\cdot,0), \cdot)\boldsymbol{d}_k^\circ\| \le C,\\ \|\rho \boldsymbol{J} \cdot \boldsymbol{w}_t(\cdot,0)\| &:= \|\hat{m}_k(\boldsymbol{\eta}^\circ, \boldsymbol{\eta}_t(\cdot,0), \cdot)\boldsymbol{d}_k^\circ + \boldsymbol{r}_s^\circ \times \hat{n}_k(\boldsymbol{\eta}^\circ, \boldsymbol{\eta}_t(\cdot,0), \cdot)\boldsymbol{d}_k^\circ\| \le C.\end{aligned}$$

The initial value $\zeta_3(\cdot,0) =: \zeta_3^\circ$*, induced by* (7.3)*, has a positive lower bound:*

$$\inf\{\zeta_3^\circ(s) : s \in [0,1]\} > 0 \tag{7.8b}$$

(*so that initially there is no total compression*).

The initial-boundary-value problem. Our initial-boundary-value problem for a viscoelastic rod consists of the kinematic relations (4.3)–(4.9), the equations of motion (5.1), (5.2), the constitutive equations (5.5) or (5.6), the boundary conditions (7.1), and the initial conditions (7.3).

In accord with the boundary conditions (7.1b) we define

(7.9)

$$\mathfrak{W} := \{\boldsymbol{x} \in H^1(0,1) : \boldsymbol{x}(0) = \boldsymbol{o}\} \quad \text{or} \quad \mathfrak{W} := \{\mathbf{x} \in H^1(0,1) : \mathbf{x}(0) = \mathbf{o}\},$$

the slight distinction being obvious from the context.

The weak formulation. A weak version of (5.7) subject to boundary conditions (7.1) may be formally obtained by multiplying the equations of (5.7) by test functions $\boldsymbol{x}$ and $\boldsymbol{y}$, depending only on s and vanishing at 0, and in the case of the momentum equations (5.7d), (5.7e) by integrating by parts. (The weak forms of the momentum equations correspond exactly to the Principle of Virtual Power [7, Chaps. 2, 8, 12, 16], which is not merely a formal mathematical artifice, but rather a general expression of fundamental laws of mechanics.) This process yields

(7.10a)
$$\int_0^1 \boldsymbol{v}_t \cdot \boldsymbol{x}\, ds = \int_0^1 \boldsymbol{p}_s \cdot \boldsymbol{x}\, ds = -\int_0^1 \boldsymbol{p} \cdot \boldsymbol{x}_s\, ds,$$
(7.10b)
$$\int_0^1 \boldsymbol{u}_t \cdot \boldsymbol{x}\, ds = \int_0^1 (\boldsymbol{w}_s + \boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{x}\, ds = -\int_0^1 [\boldsymbol{w} \cdot \boldsymbol{x}_s + (\boldsymbol{u} \times \boldsymbol{w}) \cdot \boldsymbol{x}]\, ds,$$
(7.10c)
$$\int_0^1 \rho A \boldsymbol{p}_t \cdot \boldsymbol{y}\, ds - \int_0^1 \boldsymbol{n} \cdot \boldsymbol{y}_s\, ds + \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{y}(1) \equiv -\int_0^1 [\boldsymbol{n} - \bar{\boldsymbol{n}}(t)] \cdot \boldsymbol{y}_s\, ds$$
(7.10d)
$$\begin{aligned}\int_0^1 (\rho \boldsymbol{J} \cdot \boldsymbol{w})_t \cdot \boldsymbol{y}\, ds &= \int_0^1 [-\boldsymbol{m} \cdot \boldsymbol{y}_s + (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{y}]\, ds + \bar{\boldsymbol{m}}(t) \cdot \boldsymbol{y}(1) \\ &\equiv -\int_0^1 [\boldsymbol{m} - \bar{\boldsymbol{m}}(t)] \cdot \boldsymbol{y}_s - (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{y}]\, ds,\end{aligned}$$

where $\boldsymbol{x}(1) = \boldsymbol{o}$, $\boldsymbol{y}(0) = \boldsymbol{o}$, $\boldsymbol{n}(s,t) := \hat{n}_k(\boldsymbol{\eta}(s,t), \boldsymbol{\eta}_t(s,t), s)\boldsymbol{d}_k(s,t)$, etc. The $\boldsymbol{d}_k$ are solutions of the initial-value problem for $\partial_t \boldsymbol{d}_k = \boldsymbol{w} \times \boldsymbol{d}_k$. In fact, we do not use the weak forms of the compatibility equations, (7.10a) and (7.10b); instead, we define $\boldsymbol{v}$ and $\boldsymbol{u}$ to be solutions of the compatibility equations (4.9), treated as ordinary differential equations in t. (The weak forms (7.10a) and (7.10b) would play a fundamental role in the treatment of the corresponding conservation laws for elastic rods. These forms are used to introduce artificial dissipative mechanisms to control shocks, which can be a tricky process for elastic rods [6].)

By using a version of the Fundamental Lemma of the Calculus of Variations, in particular, by taking $\boldsymbol{y}$ to be piecewise affine approximations to characteristic functions, we find that (7.10c,d) imply the integral forms of (5.7d,e):

$$\int_{s_1}^{s_2} \rho A \boldsymbol{p}_t\, ds = \boldsymbol{n}\Big|_{s_1}^{s_2}, \tag{7.11a}$$
$$\int_{s_1}^{s_2} (\rho \boldsymbol{J} \cdot \boldsymbol{w})_t\, ds = \boldsymbol{m}\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} \boldsymbol{v} \times \boldsymbol{n}\, ds, \tag{7.11b}$$

for almost every $s_1, s_2 \in [0,1]$, for $s_1 = 0$, for $s_2 = 1$, and for almost every $t \in [0,T]$. If $s_2 = 1$, we replace the boundary terms with their prescriptions from (7.1c). Conversely, the density of linear combinations of these $\boldsymbol{y} \in L^2(0,1)$ shows that (7.11) implies (7.10c,d), so that these two systems are equivalent.

We can replace the test functions $\boldsymbol{y}$ in (7.10c,d) with functions of both s and t in suitable function spaces, because arbitrary functions in these spaces can be approximated by finite linear combinations of products of functions of s with functions of t. Likewise, we can take the time derivative of (7.10c,d) and then replace the $\boldsymbol{y}$'s with functions of s and t, obtaining

$$\int_0^1 \rho A \boldsymbol{p}_{tt} \cdot \boldsymbol{y}\, ds = -\int_0^1 [\boldsymbol{n}_t - \bar{\boldsymbol{n}}_t] \cdot \boldsymbol{y}_s\, ds, \tag{7.12a}$$

$$\int_0^1 (\rho \boldsymbol{J} \cdot \boldsymbol{w})_{tt} \cdot \boldsymbol{y}\, ds = -\int_0^1 [\boldsymbol{m}_t - \bar{\boldsymbol{m}}_t(t)] \cdot \boldsymbol{y}_s - (\boldsymbol{v} \times \boldsymbol{n})_t \cdot \boldsymbol{y}]\, ds. \tag{7.12b}$$

III. A Priori Estimates

8. Plan of the Analysis

In the next section we begin the analysis of our initial-boundary-value problem leading to our fundamental existence theorem:

8.1. Theorem. *Let T be a fixed positive number. Let the initial data satisfy Hypothesis 7.7. Let the boundary data have the form* (7.1)*, satisfy* (7.2)*, and be compatible with the initial conditions. Then there is a unique solution* $(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{p}, \boldsymbol{w})$ *of* (7.10) *with*

$$\boldsymbol{v}, \boldsymbol{u} \in C^1(0, T, C^0[0,1]), \qquad \boldsymbol{p}, \boldsymbol{w} \in C^0(0, T, C^1[0,1]) \tag{8.2}$$

satisfying these initial conditions and boundary conditions pointwise and satisfying the equivalent systems (7.10) *and* (7.11).

The existence theory carried out in Part IV below shows that if the data are sufficiently regular, then so are the solutions as long as they exist. For the purpose of obtaining estimates, we may accordingly take the strain $\boldsymbol{\eta}$ and the strain-rate $\boldsymbol{\eta}_t$ to be continuous. We then obtain bounds for these quantities in Secs. 9–11. We avoid inconsistency in the existence theory by seeking solutions that satisfy these bounds. For simpler systems of differential equations, we could alternatively regard the estimates of Secs. 9–11 as purely heuristic substitutes for the estimates needed for the Galerkin approximations of Part IV. For our problem, there are important differences in the roles and in the derivations of the two kinds of estimates.

In Sec. 9, we obtain an energy estimate, based primarily on Hypothesis 6.2, which leads to a useful bound on the space-time integral of $|\boldsymbol{\eta}_t|^2$ and on the kinetic and stored energies. In Sec. 10, we use our energy estimate and Hypothesis 6.10 to show that ζ_3 is pointwise bounded below by a positive-valued function of t. In Sec. 11, we use Hypothesis 6.11 to get pointwise bounds on $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_t$, the arguments of the constitutive functions. The derivations of these bounds and their analogs for the Galerkin approximations form the technical heart of our paper. It is mainly here that we have to face the complications caused by the underlying mechanics.

Since we have these bounds on the arguments of the constitutive functions, we do not change the solutions of our governing equations if we modify our constitutive functions where their arguments do not obey these bounds. In Sec. 12 we effect such a modification that replaces $\boldsymbol{\sigma}$ with a modified function that is uniformly monotone in $\dot{\boldsymbol{\eta}}$ and that behaves regularly at a total compression. This replacement makes our problem more accessible to a version of the Faedo-Galerkin method designed to accommodate the technical challenges posed by the underlying mechanics. We use this method to carry out the existence theory in Part IV. In Sec. 18 we show that the solution depends continuously on the data and is therefore unique.

In Part V we discuss related problems, formulations, hypotheses, and methods. Sec. 19 treats alternative constitutive restrictions, some of which are both more physically natural and more complicated than those upon which we base our original treatment. In Sec. 20, we show how to treat other boundary conditions. In Sec. 21 we discuss a more traditional formulation in terms of componential equations, and we explain the difficulties that we would have encountered had we used more standard methods of analysis for it. In Sec. 22 we discuss related problems.

9. Energy Estimates

We introduce the kinetic energy $K(t)$, the stored energy $\Phi(t)$, and the work $\Omega(t)$ of the dissipative internal forces at time t:

(9.1a)

$$K(t) := \tfrac{1}{2}\langle \boldsymbol{p}(\cdot,t)\rho A\boldsymbol{p}(\cdot,t)\rangle + \tfrac{1}{2}\langle \boldsymbol{w}, \rho\boldsymbol{J}\cdot\boldsymbol{w}(s,t)\rangle,$$

(9.1b)

$$\Phi(t) := \int_0^1 \varphi(\mathsf{v}(s,t),\mathsf{u}(s,t),s)\,ds \equiv \int_0^1 \varphi(\boldsymbol{\eta}(s,t),s)\,ds,$$

(9.1c)

$$\Omega(t) := \int_0^t\int_0^1 \boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta}(s,\tau),\boldsymbol{\eta}_t(s,\tau),s)\cdot\boldsymbol{\eta}_t(s,\tau)\,ds\,d\tau \equiv \int_0^t \langle\boldsymbol{\sigma}^{\mathrm{D}},\boldsymbol{\eta}_t\rangle\,d\tau.$$

We substitute the constitutive equation (5.5b) into the weak equations of motion (7.12a,b), replace $\boldsymbol{y}$ in (7.12a) with $\boldsymbol{p}$, replace $\boldsymbol{y}$ in (7.12b) with $\boldsymbol{w}$, and add the resulting equations. Since

$$\begin{aligned}(9.2)\qquad \boldsymbol{n}\cdot\boldsymbol{p}_s &\equiv \boldsymbol{n}\cdot\boldsymbol{v}_t \equiv \boldsymbol{n}\cdot(v_k\boldsymbol{d}_k)_t = n_k\partial_t v_k + \boldsymbol{n}\cdot(\boldsymbol{w}\times\boldsymbol{v})\\ &\equiv \mathsf{n}\cdot\mathsf{v}_t + \boldsymbol{n}\cdot(\boldsymbol{w}\times\boldsymbol{v}) = \varphi_{\mathsf{v}}\cdot\mathsf{v}_t + \mathsf{n}^{\mathrm{D}}\cdot\mathsf{v}_t + \boldsymbol{n}\cdot(\boldsymbol{w}\times\boldsymbol{v}),\end{aligned}$$

etc., we obtain the energy equation
(9.3)

$$K(t)+\Phi(t)+\Omega(t) = K(0)+\Phi(0)+\int_0^t [\boldsymbol{n}(1,\tau)\cdot\boldsymbol{p}(1,\tau)+\boldsymbol{m}(1,\tau)\cdot\boldsymbol{w}(1,\tau)]\,d\tau.$$

Had we used equivalent componential forms in place of our vectorial equations, then the derivation of the energy equation (9.3) could be a formidable exercise. The basic difficulty, which recurs unavoidably in Sec. 11, is due to our introduction of the moving basis $\boldsymbol{d}_k$, which is essential for a simple description of constitutive equations invariant under rigid motions, but which complicates the equations of motion because it is responsible for the appearance of $\boldsymbol{w}$ in various time-derivatives.

We wish to bound the left-hand side (9.3), so that we must get sharp estimates for the integral on the right-hand side of (9.3), which gives the work done by the forces and couples at the end $s = 1$. For this purpose we use the monotonicity condition (6.3) (a consequence of Hypothesis 6.2), Hypothesis 7.7 giving the regularity of initial conditions (7.3), the requirement that the initial conditions satisfy $\Phi(0) < \infty$ (which is ensured by (7.8b)), and a weaker version of (7.2):

$$\bar{\boldsymbol{n}},\ \bar{\boldsymbol{m}} \in L^2_{\mathrm{loc}}[0,\infty). \tag{9.4}$$

Let us set

$$M := \left[\int_0^1 (\mathbf{n}^{\mathrm{D}}\cdot\mathbf{v}_t + \mathbf{m}^{\mathrm{D}}\cdot\mathbf{u}_t)\,ds\right]^{1/2} \equiv \left[\int_0^1 \boldsymbol{\sigma}^{\mathrm{D}}\cdot\boldsymbol{\eta}_t\,ds\right]^{1/2}, \tag{9.5a}$$

so that

$$\int_0^t M(\tau)^2\,d\tau = \Omega(t), \quad \int_0^t M(\tau)\,d\tau \le \sqrt{t\Omega(t)}. \tag{9.5b}$$

From the monotonicity condition (6.4), a consequence of Hypothesis 6.2, and from (9.1) we obtain

$$\int_0^1 |\boldsymbol{\eta}_t(s,t)|^2\,ds \equiv \int_0^1 \{|\mathbf{v}_t(s,t)|^2 + |\mathbf{u}_t(s,t)|^2\}\,ds \le CM(t)^2, \tag{9.6}$$

$$\int_0^1 (|\boldsymbol{p}(s,t)|^2 + |\boldsymbol{w}(s,t)|^2)\,ds \le CK(t), \tag{9.7}$$

$$\begin{aligned} \tfrac{1}{2}\int_0^1 |\boldsymbol{\eta}(s,t)|^2\,ds &\le \int_0^1 |\boldsymbol{\eta}(s,0)|^2\,ds + \int_0^1 \left[\left[\int_0^t |\boldsymbol{\eta}_t(s,\tau)|\,d\tau\right]^2\right]^2 ds \\ &\le C + \int_0^1 t\int_0^t |\boldsymbol{\eta}_t(s,\tau)|^2\,d\tau\,ds \le C + Ct\Omega(t). \end{aligned} \tag{9.8}$$

From these estimates, from identity $(4.8)_1$, and from the boundary conditions (7.1), we obtain

$$\begin{aligned} |\boldsymbol{p}(s,t)| &= \left|\int_0^s \boldsymbol{p}_s(\xi,t)\,d\xi\right| = \left|\int_0^s \boldsymbol{v}_t\,d\xi\right| = \left|\int_0^s [\boldsymbol{R}\cdot\mathbf{v}_t + \boldsymbol{w}\times\boldsymbol{v}]\,d\xi\right| \\ &\le \int_0^1 \left(|\mathbf{v}_t(\xi,t)| + \tfrac{1}{2}|\boldsymbol{w}(\xi,t)|^2 + \tfrac{1}{2}|\mathbf{v}(\xi,t)|^2\right)\,d\xi \\ &\le CM(t) + \Gamma(t)[K(t) + \Omega(t)], \end{aligned} \tag{9.9a}$$

$$|\boldsymbol{r}(s,t)| \le \Gamma(t)\sqrt{\Omega(t)} + \Gamma(t)\int_0^t [K(\tau) + \Omega(\tau)]\,d\tau. \tag{9.9b}$$

Likewise,

$$|\boldsymbol{w}(s,t)| \le \left| \int_0^s \partial_t u_k \boldsymbol{d}_k \, d\xi \right| \le \int_0^1 |\mathbf{u}_t(s,t)| \, ds \le CM(t). \tag{9.10}$$

In view of boundary conditions (7.1c) we estimate the boundary term on the right-hand side of (9.3) by

(9.11)

$$\begin{aligned}
&\left| \int_0^t [\bar{\boldsymbol{n}}(\tau) \cdot \boldsymbol{p}(1,\tau) + \bar{\boldsymbol{m}}(\tau) \cdot \boldsymbol{w}(1,\tau)] \, d\tau \right| \\
&\le \int_0^t \{ |\bar{\boldsymbol{n}}(\tau)| |\boldsymbol{p}(1,\tau)| + |\bar{\boldsymbol{m}}(\tau)| |\boldsymbol{w}(1,\tau)| \} \, d\tau \\
&\le C \int_0^t \{ |\bar{\boldsymbol{n}}(\tau)| + |\bar{\boldsymbol{m}}(\tau)| \} M(\tau) \, d\tau + \int_0^t \mathit{\Gamma}(t) |\bar{\boldsymbol{n}}(\tau)| [K(\tau) + \mathit{\Omega}(\tau)] \, d\tau \\
&\le C \int_0^t \{ |\bar{\boldsymbol{n}}(\tau)| + |\bar{\boldsymbol{m}}(\tau)| \}^2 \, d\tau + \varepsilon \mathit{\Omega}(t) + \int_0^t \mathit{\Gamma}(t) |\bar{\boldsymbol{n}}(\tau)| [K(\tau) + \mathit{\Omega}(\tau)] \, d\tau.
\end{aligned}$$

We now substitute (9.11) into (9.3) and invoke (9.4) and the positivity of φ (required by (6.1)) to obtain

(9.12)

$$K(t) + \mathit{\Omega}(t) \le K(t) + \mathit{\Phi}(t) + \mathit{\Omega}(t) \le \mathit{\Gamma}(t) + \mathit{\Gamma}(t) \int_0^t |\bar{\boldsymbol{n}}(\tau)| [K(\tau) + \mathit{\Omega}(\tau)] \, d\tau.$$

Since $\bar{\boldsymbol{n}}$ is locally integrable by assumption (9.4), the Gronwall inequality then implies that $K(t) + \mathit{\Omega}(t) \le \mathit{\Gamma}(t)$, so that $\mathit{\Phi}(t) \le \mathit{\Gamma}(t)$. Thus

9.13. Theorem. *Let the monotonicity condition* (6.2) *hold* (*so that* (6.4) *holds*) *and let the stored energy function satisfy* (6.1). *Let boundary conditions* (7.1) *hold subject to* (9.4) *and let the initial conditions satisfy Hypothesis 7.7. Then the energy estimate*

$$K(t) + \mathit{\Phi}(t) + \mathit{\Omega}(t) \le \mathit{\Gamma}(t) \tag{9.14}$$

holds. In particular,

$$|\boldsymbol{r}(\cdot,t)|, \ \|\boldsymbol{p}(\cdot,t)\|, \ \|\boldsymbol{w}(\cdot,t)\| \le \mathit{\Gamma}(t). \tag{9.15}$$

10. The Preclusion of Total Compression

We assume that Hypothesis 6.10 holds and that the hypotheses of Theorem 9.13 hold, so that the energy estimate (9.14) holds and so that ζ_3 is continuous. Condition (7.8b) allows us to choose the ε of Hypothesis 6.10 to satisfy $0 < \varepsilon < \inf_s \zeta_3^\circ(s) \equiv \inf_s \zeta_3(s,0)$, without loss of generality. To show that $\zeta_3(s,t)$ is positive for all (s,t) it suffices to show this only for all

(s,t) for which $\zeta_3(s,t) < \varepsilon$. Thus suppose that there is a ξ in $[0,1)$ and a $\tau_2 > 0$ such that $\zeta_3(\xi,\tau_2) < \varepsilon$. Since ζ_3 is taken to be continuous, there is a latest time τ_1 before τ_2 at which $\zeta_3(\xi,\tau_1) = \varepsilon$. From the equation of motion (5.1) and the constitutive hypothesis (6.10) we get

$$\boldsymbol{d}_3(\xi,t)\cdot\int_\xi^1 (\rho A)(s)\boldsymbol{p}(s,\tau)\,ds\Big|_{\tau_1}^{t} = \boldsymbol{d}_3(\xi,t)\cdot\int_{\tau_1}^t \boldsymbol{n}(s,\tau)\,d\tau\Big|_\xi^1 \tag{10.1}$$

$$\begin{aligned}
&\geq \boldsymbol{d}_3(\xi,t)\cdot\int_{\tau_1}^t \boldsymbol{n}(1,\tau)\,d\tau + \int_{\tau_1}^t \psi'(\zeta_3(\xi,\tau))\partial_t\zeta_3(\xi,\tau)\,d\tau \\
&\quad - A\int_{\tau_1}^t \psi(\zeta_3(\xi,\tau))\,d\tau \\
&\geq \psi(\zeta_3(\xi,t)) - \psi(\varepsilon) - A\int_{\tau_1}^t \psi(\zeta_3(\xi,\tau))\,d\tau - \Gamma(t)
\end{aligned} \tag{10.2}$$

for $\tau_1 \leq t \leq \tau_2$. Since (9.14) holds, we deduce from (9.9a) that

$$\left|\int_\xi^1 \rho A\boldsymbol{p}\,ds\right| \leq C\sqrt{\int_0^1 \rho A|\boldsymbol{p}|^2\,ds} \leq CK \leq \Gamma. \tag{10.3}$$

Thus we obtain from (10.1) and (10.3) that

$$\psi(\zeta_3(\xi,t)) \leq A\int_{\tau_1}^t \psi(\zeta_3(\xi,\tau))\,d\tau + \Gamma(t) \tag{10.4}$$

for $\tau_1 \leq t \leq \tau_2$. The Gronwall inequality then implies that

$$\psi(\zeta_3(\xi,\tau_2)) \leq \Gamma(\tau_2), \tag{10.5}$$

and the properties of ψ then yield the desideratum $\zeta_3(\xi,\tau_2) \geq \gamma(\tau_2)$. Since γ denotes a continuous positive-valued function on $(0,\infty)$, we have

10.6. Theorem. *Let the hypotheses of Theorem 9.13* (*namely,* (5.7a)*,* (6.1)*,* (7.1)*,* (9.4)*, Hypothesis 7.7, and the continuity of* $\boldsymbol{\eta}_t$) *hold, and further let Hypothesis 6.10 hold. Then*

$$\zeta_3(s,t) \geq \gamma(t) \quad \forall\,(s,t). \tag{10.6}$$

for any solution of the initial-boundary-value problem with the requisite smoothness.

11. Estimates of the Accelerations and the Strain Rates

As discussed in Sec. 8, we need pointwise bounds on both the strains $\boldsymbol{\eta}$ and the strain rates $\boldsymbol{\eta}_t$, which enter as arguments into the constitutive equations. In this section we use the bounds and (9.14) and (10.6) to derive such a priori estimates:

11.1. Theorem. *Let the hypotheses of Theorem 10.6 hold. Let the compatibility condition* (7.8a) *hold. Let T be any positive number. Then $\boldsymbol{\eta}$, $\boldsymbol{\eta}_t$, $\boldsymbol{p}_s$, $\boldsymbol{w}_s$ lie in a compact subset of $C^0([0,1]\times[0,T])$ that depends only on T, the constitutive functions, and bounds for the data.*

The pointwise bound on $\boldsymbol{\eta}_t$ is based on an analog of the energy estimate involving the accelerations rather than the velocities. Since this estimate is motivated by the needs of the analysis and is somewhat artificial from the viewpoint of mechanics, its derivation lacks the simplicity of that of the energy estimate. The many complications due to our use of the variable orthonormal basis $\{\boldsymbol{d}_k\}$ essentially occur because the functions defining a configuration of a rod take values in a manifold rather than in a vector space, and because our formulation makes explicit the requirement that material properties be invariant under rigid motions.

We shall obtain an energy-like estimate for the functional

$$
\begin{aligned}
&H[\boldsymbol{p}_t,\boldsymbol{w}_t,\boldsymbol{\eta},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{tt}](t) \\
(11.2)\qquad &:= \tfrac{1}{2}\langle \boldsymbol{p}_t, \rho A \boldsymbol{p}_t\rangle + \tfrac{1}{2}\langle \boldsymbol{w}_t, \rho \boldsymbol{J}\cdot \boldsymbol{w}_t\rangle + \int_0^t \langle \boldsymbol{\eta}_{tt}, \boldsymbol{\sigma}^{\mathrm{D}}_{\boldsymbol{\eta}}(\boldsymbol{\eta},\boldsymbol{\eta}_t,s)\cdot\boldsymbol{\eta}_{tt}\rangle\, d\tau.
\end{aligned}
$$

(H plays a role analogous to that played by $K+\Omega$ in the energy estimate (9.14).) We suppress the arguments $\boldsymbol{p}_t,\boldsymbol{w}_t,\boldsymbol{\eta},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{tt}$ of H. Note that (7.8a) implies that $H(0)\le C$. Our main effort in proving Theorem 11.1 lies in proving

11.3. Proposition. *Let the hypotheses of Theorem 10.6 hold. Let* (7.8a) *hold. Let T be any positive number. Then*

$$
(11.3)\qquad H(t)\le \Gamma(T) \quad \textit{for } 0\le t\le T
$$

with $\Gamma(T)$ depending only on T, the constitutive functions, and the bounds for the data.

Proof. We shall show that H satisfies an inequality of the form

$$
(11.4)\qquad H \le \Gamma\left[1+\int_0^t NH\,d\tau + \int_0^t N^2\sqrt{H}\,d\tau\right] \le \Gamma\left[1+\int_0^t N^2 H\,d\tau\right]
$$

where $N := 1+M$. Since $\int_0^t N^2 d\tau \le \Gamma(t)$ by (9.14), the Gronwall inequality implies the desired (11.3). We now derive (11.4).

We take $\boldsymbol{y} = \boldsymbol{p}_t$ in (7.12a) , take $\boldsymbol{y} = \boldsymbol{w}_t$ in (7.12b), add the resulting equations, and use the identity

(11.5)
$$\begin{aligned}&\boldsymbol{w}_t \cdot \partial_{tt}(\rho \boldsymbol{J} \cdot \boldsymbol{w}) \equiv \boldsymbol{w}_t \cdot \partial_{tt}(\rho J_{pq} w_q \boldsymbol{d}_p)\\ &\ = \tfrac{1}{2}\partial_t(\boldsymbol{w}_t \cdot \rho \boldsymbol{J} \cdot \boldsymbol{w}_t) + 2(\boldsymbol{w}_t \times \boldsymbol{w}) \cdot (\rho \boldsymbol{J} \cdot \boldsymbol{w}_t) + (\boldsymbol{w}_t \times \boldsymbol{w}) \cdot [\boldsymbol{w} \times (\rho \boldsymbol{J} \cdot \boldsymbol{w})]\end{aligned}$$

to obtain

$$\begin{aligned}\tfrac{1}{2}d_t\langle \boldsymbol{p}_t, \rho A \boldsymbol{p}_t\rangle + \tfrac{1}{2}d_t\langle \boldsymbol{w}_t, \rho \boldsymbol{J} \cdot \boldsymbol{w}_t\rangle - \bar{\boldsymbol{n}}_t \cdot \boldsymbol{p}_t(1,\cdot) - \bar{\boldsymbol{m}}_t \cdot \boldsymbol{w}_t(1,\cdot)&\\ = 2\langle \boldsymbol{w} \times \boldsymbol{w}_t, \rho \boldsymbol{J} \cdot \boldsymbol{w}_t + \boldsymbol{w} \times (\rho \boldsymbol{J} \cdot \boldsymbol{w})\rangle&\\ - \langle \boldsymbol{n}_t, \boldsymbol{p}_{st}\rangle - \langle (\boldsymbol{n} \times \boldsymbol{v})_t, \boldsymbol{w}_t\rangle - \langle \boldsymbol{m}_t, \boldsymbol{w}_{st}\rangle.&\end{aligned} \tag{11.6}$$

Using the identity (4.8) and the compatibility conditions (4.9) we obtain

(11.7)
$$\begin{aligned}&\boldsymbol{n}_t \cdot \boldsymbol{p}_{st} + (\boldsymbol{n} \times \boldsymbol{v})_t \cdot \boldsymbol{w}_t\\ &\quad = [\boldsymbol{R} \cdot \mathbf{n}_t + \boldsymbol{w} \times \boldsymbol{n}] \cdot [\boldsymbol{R} \cdot \mathbf{v}_{tt} + 2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t) + \boldsymbol{w}_t \times \boldsymbol{v} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{v})]\\ &\qquad + \boldsymbol{n}_t \cdot (\boldsymbol{v} \times \boldsymbol{w}_t) + \boldsymbol{n} \cdot \{[\boldsymbol{R} \cdot \mathbf{v}_t + \boldsymbol{w} \times \boldsymbol{v}] \times \boldsymbol{w}_t\}\\ &\quad = \mathbf{n}_t \cdot \mathbf{v}_{tt} + \boldsymbol{n}_t \cdot [2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t) + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{v})]\\ &\qquad - \boldsymbol{n} \cdot [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_{tt}) + \boldsymbol{w}_t \times (\boldsymbol{R} \cdot \mathbf{v}_t) + \boldsymbol{w}_t \times (\boldsymbol{w} \times \boldsymbol{v})]\\ &\quad = (\mathbf{n}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t + \mathbf{n}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt}) \cdot \mathbf{v}_{tt} + \partial_t\{\boldsymbol{n} \cdot [2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t) + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{v})]\}\\ &\qquad - \boldsymbol{n} \cdot \{3\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_{tt}) + 3\boldsymbol{w}_t \times (\boldsymbol{R} \cdot \mathbf{v}_t) + 3\boldsymbol{w} \times [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t)]\\ &\qquad\qquad + 2\boldsymbol{w}_t \times (\boldsymbol{w} \times \boldsymbol{v}) + \boldsymbol{w} \times (\boldsymbol{w}_t \times \boldsymbol{v}) + \boldsymbol{w} \times [\boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{v})]\},\end{aligned}$$

(11.8)
$$\begin{aligned}\boldsymbol{m}_t \cdot \boldsymbol{w}_{st} &= \boldsymbol{m}_t \cdot [\boldsymbol{u}_{tt} - \boldsymbol{w}_t \times \boldsymbol{u} - \boldsymbol{w} \times \boldsymbol{u}_t]\\ &= [\boldsymbol{R} \cdot \mathbf{m}_t + \boldsymbol{w} \times \boldsymbol{m}] \cdot [\boldsymbol{R} \cdot \mathbf{u}_{tt} + \boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)]\\ &= \mathbf{m}_t \cdot \mathbf{u}_{tt} + \boldsymbol{m}_t \cdot [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)] + (\boldsymbol{w} \times \boldsymbol{m}) \cdot (\boldsymbol{R} \cdot \mathbf{u}_{tt})\\ &= (\mathbf{m}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t + \mathbf{m}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt}) \cdot \mathbf{u}_{tt} + \partial_t\{\boldsymbol{m} \cdot [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)]\}\\ &\quad - \boldsymbol{m} \cdot \{2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_{tt}) + \boldsymbol{w}_t \times (\boldsymbol{R} \cdot \mathbf{u}_t) + \boldsymbol{w} \times [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)].\end{aligned}$$

We substitute the constitutive equations (5.5b) into (11.7) and (11.8), substitute these equations into (11.6) and use (11.2) to obtain

$$\begin{aligned}H_t = &-\langle \boldsymbol{\eta}_{tt}, \boldsymbol{\sigma}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t\rangle + \bar{\boldsymbol{n}}_t \cdot \boldsymbol{p}_t(1,\cdot) + \bar{\boldsymbol{m}}_t \cdot \boldsymbol{w}_t(1,\cdot)\\ &+ 2\langle \boldsymbol{w} \times \boldsymbol{w}_t, \rho \boldsymbol{J} \cdot \boldsymbol{w}_t + \boldsymbol{w} \times (\rho \boldsymbol{J} \cdot \boldsymbol{w})\rangle\\ &- \partial_t\langle \boldsymbol{n}, 2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t) + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{v})\rangle - \partial_t\langle \boldsymbol{m}, \boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)\rangle\\ &+ \langle \boldsymbol{n}, 3\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_{tt}) + 3\boldsymbol{w}_t \times (\boldsymbol{R} \cdot \mathbf{v}_t) + 3\boldsymbol{w} \times [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{v}_t)]\rangle\\ &+ \langle \boldsymbol{m}, 2\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_{tt}) + \boldsymbol{w}_t \times (\boldsymbol{R} \cdot \mathbf{u}_t) + \boldsymbol{w} \times [\boldsymbol{w} \times (\boldsymbol{R} \cdot \mathbf{u}_t)]\rangle.\end{aligned} \tag{11.9}$$

Hypothesis 6.11 was expressly designed to handle the term $\boldsymbol{\eta}_{tt} \cdot \hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t$, which appears in the right-hand side of (11.9):

(11.10)
$$|\boldsymbol{\eta}_{tt} \cdot \hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t| \equiv |(\boldsymbol{\Lambda} \cdot \boldsymbol{\eta}_{tt}) \cdot (\boldsymbol{\Lambda}^{-1} \cdot \hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}} \cdot \boldsymbol{\eta}_t)| \le \varepsilon \boldsymbol{\eta}_{tt} \cdot \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt} + C[1 + \boldsymbol{\sigma}^{\mathrm{D}} \cdot \boldsymbol{\eta}_t + \varphi].$$

Since $\rho\boldsymbol{J}$ is uniformly positive-definite, the Euclidean norm $|\boldsymbol{w}_t|$ is equivalent to $\sqrt{\boldsymbol{w}_t \cdot \rho\boldsymbol{J} \cdot \boldsymbol{w}_t}$. We integrate (11.9) with respect to t over $[0,t]$, use Hypothesis 7.7 to control initial values, use (11.10), and use the energy estimate (9.14) to control the integral of $\boldsymbol{\sigma}^{\mathrm{D}} \cdot \boldsymbol{\eta}_t + \varphi$ (which appears in (11.10)) to obtain

(11.11)

$$\begin{aligned}
H \le \Gamma &+ C \int_0^t \int_0^1 |\boldsymbol{w}|\boldsymbol{w}_t \cdot \rho\boldsymbol{J} \cdot \boldsymbol{w}_t \, ds\, d\tau \\
&+ C \int_0^t \int_0^1 |\boldsymbol{w}|^2 \sqrt{\boldsymbol{w}_t \cdot \rho\boldsymbol{J} \cdot \boldsymbol{w}_t}\sqrt{\boldsymbol{w} \cdot \rho\boldsymbol{J} \cdot \boldsymbol{w}} \, ds\, d\tau \\
&+ \Gamma \int_0^t \int_0^1 \{|\boldsymbol{w}|\,|\boldsymbol{\eta}_{tt}| + |\boldsymbol{w}|^2|\boldsymbol{\eta}_t| + |\boldsymbol{w}|^3|\boldsymbol{\eta}| + |\boldsymbol{w}_t|(|\boldsymbol{\eta}_t| + |\boldsymbol{w}|\,|\boldsymbol{\eta}|)\}|\boldsymbol{\sigma}| \, ds\, d\tau \\
&+ C \int_0^1 (|\boldsymbol{w}|\,|\boldsymbol{\eta}_t| + |\boldsymbol{w}|^2|\boldsymbol{\eta}|)|\boldsymbol{\sigma}| \, ds \\
&+ \int_0^t \{|\boldsymbol{p}_t(1,\tau)|\,|\bar{\boldsymbol{n}}_t(\tau)| + |\boldsymbol{w}_t(1,\tau)|\,|\bar{\boldsymbol{m}}_t(\tau)|\} \, d\tau.
\end{aligned}$$

We now estimate each term on the right-hand side of (11.11) to obtain from it the integral inequality (11.4) for H. It is evident that the presence of $\boldsymbol{w}$ and its derivatives is a primary source of difficulty in obtaining these estimates. It is reasonable to expect that these $\boldsymbol{w}$'s should cause no difficulty in our analysis because they appear in lower-order terms, albeit in powers or in products with other functions, but the demonstration of this fact here and in Section 15 requires quite a few tricky constructions. We recall that (9.10) and (9.14) imply that

$$|\boldsymbol{w}(s,t)| \le CM(t), \qquad \int_0^1 |\boldsymbol{w}(s,t)|^2 \, ds \le \Gamma(t). \tag{11.12}$$

To supplement these with another estimate on $\boldsymbol{w}$, we first observe that Hypothesis 7.7 on the initial data and the energy estimate (9.14) imply that

(11.13)

$$\begin{aligned}
\int_0^1 |\boldsymbol{\eta}_t|^2 \, ds &= \int_0^1 |\boldsymbol{\eta}_t(s,0)|^2 \, ds + 2\int_0^1 \int_0^t \boldsymbol{\eta}_t \cdot \boldsymbol{\eta}_{tt} \, d\tau\, ds \\
&\le C + 2\sqrt{\int_0^1 \int_0^t |\boldsymbol{\eta}_t|^2 \, d\tau\, ds \int_0^1 \int_0^t |\boldsymbol{\eta}_{tt}|^2 \, d\tau\, ds} \le \Gamma(1 + \sqrt{H}).
\end{aligned}$$

Then (4.9) and (11.13) yield

$$|\boldsymbol{w}(s,t)|^2 \le \int_0^1 |\mathbf{u}_t(\xi,t)|^2 d\xi \le \Gamma(t)[1 + \sqrt{H(t)}]. \tag{11.14}$$

We use (11.12) to bound the integrals in the first line of (11.11):

$$\int_0^t \int_0^1 |\boldsymbol{w}| \boldsymbol{w}_t \cdot \rho \boldsymbol{J} \cdot \boldsymbol{w}_t \, ds \, d\tau \le C \int_0^t MH \, d\tau, \tag{11.15a}$$

$$\int_0^t \int_0^1 |\boldsymbol{w}|^2 \sqrt{\boldsymbol{w}_t \cdot \rho \boldsymbol{J} \cdot \boldsymbol{w}_t} \sqrt{\boldsymbol{w} \cdot \rho \boldsymbol{J} \cdot \boldsymbol{w}} \, ds \, d\tau \tag{11.15b}$$

$$\le C \int_0^t M^2 \sqrt{K} \sqrt{H} \, d\tau \le \Gamma(t) \int_0^t M^2 \sqrt{H} \, d\tau.$$

To estimate the remaining terms, we need a bound on $\boldsymbol{\sigma}$: Let $s_2 \ge s_1$. In view of (7.1) with $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ in $L^\infty_{\text{loc}}[0,\infty)$, we obtain from equations of motion (5.1), (5.2), and estimates (9.8), (9.14), (11.14) that

(11.16a)
$$|\boldsymbol{n}(s_2,\cdot) - \boldsymbol{n}(s_1,\cdot)| \le \sqrt{\int_{s_1}^{s_2} \rho A \, ds} \sqrt{\langle \boldsymbol{p}_t, \rho A \boldsymbol{p}_t \rangle} \le C\sqrt{s_2 - s_1}\sqrt{H},$$

$$|\boldsymbol{n}(s,\cdot)| \le \Gamma(1+\sqrt{H}), \tag{11.16b}$$

(11.17a)
$$\begin{aligned}
|\boldsymbol{m}(s_2,\cdot) - \boldsymbol{m}(s_1,\cdot)| &\le \left| \int_{s_1}^{s_2} \boldsymbol{v} \times \boldsymbol{n} \, ds \right| + \left| \int_{s_1}^{s_2} [\rho \boldsymbol{J} \cdot \boldsymbol{w}_t + \boldsymbol{w} \times (\rho \boldsymbol{J} \cdot \boldsymbol{w})] \, ds \right| \\
&\le \Gamma(1+\sqrt{H}\,) \int_{s_1}^{s_2} |\mathbf{v}| \, ds + C \left| \int_{s_1}^{s_2} [|\boldsymbol{w}_t| + |\boldsymbol{w}|^2] \, ds \right| \\
&\le \Gamma \sqrt{s_2 - s_1} \big[(1+\sqrt{H}\,) + \sqrt{\langle \boldsymbol{w}_t, \rho \boldsymbol{J} \cdot \boldsymbol{w}_t \rangle} + (1+\sqrt{H}\,) \big],
\end{aligned}$$

$$|\boldsymbol{m}(s,\cdot)| \le \Gamma(1+\sqrt{H}). \tag{11.17b}$$

Let us examine the second line of (11.11). The monotonicity condition (6.3) implies that

$$|\boldsymbol{\eta}_{tt}| \le C\sqrt{\boldsymbol{\eta}_{tt} \cdot \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt}}. \tag{11.18}$$

Thus for any positive ε, (11.16b) and (11.18) yield

(11.19)
$$\begin{aligned}
\int_0^t \int_0^1 |\boldsymbol{w}|\,|\boldsymbol{\eta}_{tt}|\,|\boldsymbol{\sigma}| \, ds \, d\tau &\le \Gamma(t) \int_0^t \left[\int_0^1 \sqrt{\boldsymbol{w} \cdot \rho \boldsymbol{J} \cdot \boldsymbol{w}} |\boldsymbol{\eta}_{tt}| \, ds \right] (1+\sqrt{H}) \, d\tau \\
&\le \Gamma(t) \int_0^t \sqrt{K} \sqrt{\langle \boldsymbol{\eta}_{tt}, \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt} \rangle} \, ds \, (1+\sqrt{H}) \, d\tau \\
&\le \varepsilon \int_0^t \langle \boldsymbol{\eta}_{tt}, \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\eta}_{tt} \rangle \, d\tau + \Gamma(t) \int_0^t (1+H) \, d\tau \\
&\le \varepsilon H(t) + \Gamma(t) \int_0^t (1+H) \, d\tau.
\end{aligned}$$

By virtue of the energy estimates (9.6), (9.8), (9.14), the inequalities (11.11), and (11.15a) imply that

(11.20a)
$$\begin{aligned}\int_0^t\int_0^1 |\boldsymbol{w}|^2|\boldsymbol{\eta}_t|\,|\boldsymbol{\sigma}|\,ds\,d\tau &\le \Gamma(t)\int_0^t M\left[\int_0^1 |\boldsymbol{w}||\mathbf{v}_t|\,ds\right](1+\sqrt{H})\,d\tau\\ &\le \Gamma(t)\int_0^t M^2(1+\sqrt{H})\,d\tau,\end{aligned}$$

(11.20b)
$$\int_0^t\int_0^1 |\boldsymbol{w}|^3|\mathbf{v}|\,|\boldsymbol{\sigma}|\,ds\,d\tau \le \Gamma(t)\int_0^t M^2(1+\sqrt{H})\,d\tau,$$

(11.20c)
$$\int_0^t\int_0^1 |\boldsymbol{w}|^2|\boldsymbol{\eta}_t|\,|\boldsymbol{\sigma}|\,ds\,d\tau \le \Gamma(t)\int_0^t M^2(1+\sqrt{H})\,d\tau,$$

(11.20d)
$$\begin{aligned}\int_0^t\int_0^1 |\boldsymbol{w}_t|\,|\boldsymbol{\eta}_t|\,|\boldsymbol{\sigma}|\,ds\,d\tau &\le \Gamma(t)\int_0^t\left[\int_0^1 \sqrt{\boldsymbol{w}_t\cdot\rho\boldsymbol{J}\cdot\boldsymbol{w}_t}\,|\boldsymbol{\eta}_t|\,ds\right](1+\sqrt{H})\,d\tau\\ &\le \Gamma(t)\int_0^t M(1+H)\,d\tau,\end{aligned}$$

(11.20e)
$$\begin{aligned}\int_0^t\int_0^1 |\boldsymbol{w}|\,|\boldsymbol{w}_t|\,|\boldsymbol{\eta}|\,|\boldsymbol{\sigma}|\,ds\,d\tau &\le \Gamma(t)\int_0^t M\left[\int_0^1 \sqrt{\boldsymbol{w}_t\cdot\rho\boldsymbol{J}\cdot\boldsymbol{w}_t}|\mathbf{v}|\,ds\right](1+\sqrt{H})\,d\tau\\ &\le \Gamma(t)\int_0^t M(1+H)\,d\tau,\end{aligned}$$

We now study the penultimate integral of (11.11). Inequality (11.13) implies that

(11.21a)
$$\begin{aligned}&\int_0^1 |\boldsymbol{\eta}_t|\,|\boldsymbol{\sigma}|\,|\boldsymbol{w}|\,ds\\ &\le \Gamma(1+\sqrt{H})\int_0^1 |\boldsymbol{\eta}_t|\,|\boldsymbol{w}|\,ds \le \Gamma(1+\sqrt{H})\sqrt{\int_0^1 |\boldsymbol{\eta}_t|^2\,ds}\\ &\le \Gamma(1+\sqrt{H})\sqrt{C+\Gamma\sqrt{H}} \le \Gamma(1+H^{1/2})(1+H^{1/4}) \le \Gamma+\varepsilon H.\end{aligned}$$

Inequalities (9.8), (9.12), and (11.16) yield

(11.21b)
$$\int_0^1 |\boldsymbol{w}|^2|\boldsymbol{\eta}|\,|\boldsymbol{\sigma}|\,ds \le \Gamma(1+H^{1/4})(1+H^{1/2})\int_0^1 |\boldsymbol{w}|\,|\boldsymbol{\eta}|\,ds \le \Gamma+\varepsilon H.$$

In Sec. 19 we exhibit a constitutive assumption, complementary to Hypothesis 6.10, that would enable us to get a pointwise a priori bound on $\boldsymbol{\eta}$. In this case, we could get (11.21b) immediately from (9.14) and (11.16). In the absence of such an a priori bound on $\boldsymbol{\eta}$, we might have been led to use (9.10) and (9.8) to obtain

$$\int_0^1 |\boldsymbol{w}|^2|\boldsymbol{\eta}|\,|\boldsymbol{\sigma}|\,ds \le M\Gamma(1+\sqrt{H})\int_0^1 |\boldsymbol{w}||\boldsymbol{\eta}|\,ds \le M\Gamma(1+\sqrt{H}).$$

This inequality would produce a version of (11.4) that does not yield (11.3) because the bounds on M coming from (9.5) and (9.14) are insufficient to control the powers of M that would appear in this alternative version of (11.4).

We now estimate the boundary term of (11.11). From (4.8) and (7.1a) we obtain

(11.22)

$$|\boldsymbol{p}_t| \le \int_0^1 |\boldsymbol{p}_{st}|\,ds \le \int_0^1 |\mathbf{v}_{tt}|\,ds + 2\int_0^1 |\mathbf{v}_t|\,|\boldsymbol{w}|\,ds + \int_0^1 |\boldsymbol{w}_t|\,|\mathbf{v}|\,ds + \int_0^1 |\mathbf{v}|\,|\boldsymbol{w}|^2\,ds.$$

Thus (11.18), (11.13), (11.14), (9.6), (9.8) imply that

(11.23)

$$\begin{aligned}|\bar{\boldsymbol{n}}_t \cdot \boldsymbol{p}_t(1,\cdot)| &\le \Gamma\left[1 + \int_0^1 |\boldsymbol{\eta}_{tt}|\,ds + \sqrt{H} + \int_0^1 |\boldsymbol{w}_t|^2\,ds + M^2\right] \\ &\le \Gamma[1+M^2] + \varepsilon\langle \boldsymbol{w}_t, \rho \boldsymbol{J}\cdot \boldsymbol{w}_t\rangle + \varepsilon\langle \boldsymbol{\eta}_{tt}, \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}_{tt}\rangle + \varepsilon\sqrt{H}.\end{aligned}$$

Likewise, we obtain

$$|\bar{\boldsymbol{m}}_t \cdot \boldsymbol{w}_t(1,\cdot)| \le \Gamma[1+M^2] + \varepsilon\langle \boldsymbol{w}_t, \rho \boldsymbol{J}\cdot \boldsymbol{w}_t\rangle + \varepsilon\langle \boldsymbol{\eta}_{tt}, \hat{\boldsymbol{\sigma}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}_{tt}\rangle + \varepsilon\sqrt{H}. \tag{11.24}$$

We substitute all our estimates into (11.11) (not forgetting that (11.23) and (11.24) are to be integrated over $[0,t]$) and choose ε sufficiently small to get (11.4). □

Proof of Theorem 11.1. Proposition 11.3 and (11.16b) and (11.17b) now imply that $\boldsymbol{\sigma}$ is bounded:

$$|\boldsymbol{\sigma}(s,t)| \le \Gamma(t) \qquad \forall\, s,t. \tag{11.25}$$

Thus (6.4) implies that

$$\varphi_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}_t = \boldsymbol{\sigma}\cdot\boldsymbol{\eta}_t - \boldsymbol{\sigma}^{\mathrm{D}}\cdot\boldsymbol{\eta}_t \le \Gamma|\boldsymbol{\eta}_t| - c|\boldsymbol{\eta}_t|^2 \le \Gamma, \tag{11.26}$$

whence the boundedness of the initial data imply that $\varphi(\boldsymbol{\eta}, s) \le \Gamma$. Thus, Hypothesis 6.1 implies that the strain is pointwise bounded:

$$|\boldsymbol{\eta}(s,t)| \le \Gamma(t) \qquad \forall\, s,t. \tag{11.27}$$

Hypothesis 6.2 ensures that the finite-dimensional equation $\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s) = \boldsymbol{\sigma}$ can be uniquely solved for $\dot{\boldsymbol{\eta}}$, so that this equation is equivalent to an equation of the form

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\kappa}(\boldsymbol{\eta}, \boldsymbol{\sigma}, s). \tag{11.28}$$

Since we assumed that $\hat{\boldsymbol{\sigma}}$ is continuously differentiable in $\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}$ and continuous in s, it follows from the Local Implicit-Function Theorem that $\boldsymbol{\kappa}$ is continuously differentiable in $(\boldsymbol{\eta}, \boldsymbol{\sigma})$ and continuous in s. We conclude that the constitutive equation $\boldsymbol{\sigma}(s,t) = \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}(s,t), \boldsymbol{\eta}_t(s,t), s)$ is equivalent to

$$\boldsymbol{\eta}_t(s,t) = \boldsymbol{\kappa}(\boldsymbol{\eta}(s,t), \boldsymbol{\sigma}(s,t), s), \tag{11.29}$$

which, for given $\boldsymbol{\sigma}$, is an ordinary differential equation for $\boldsymbol{\eta}(s,\cdot)$ parametrized by s. Since both $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ are bounded, (11.29) implies that $\boldsymbol{\eta}_t$ is also bounded:

$$|\boldsymbol{\eta}_t(s,t)| \leq \Gamma(T) \qquad \forall\, (s,t) \in [0,1] \times [0,T]. \tag{11.30}$$

In view of (11.3) and (11.14) we deduce from (11.16a), (11.17a) that for each $t \in [0,T]$,

(11.31)

$\boldsymbol{\sigma}(\cdot,t)$ lies in a bounded subset of $C^{0,1/2}[0,1]$ independent of t for $t \leq T$.

Here $C^{0,1/2}[0,1]$ is the space of Hölder continuous functions with exponent $\frac{1}{2}$. The Arzelà-Ascoli Theorem implies that $C^{0,1/2}[0,1]$ is compactly embedded in $C^0[0,1]$.

From (11.31) it follows that $\tau \mapsto \boldsymbol{\kappa}(\boldsymbol{\eta}, \boldsymbol{\sigma}(s,\tau), s)$ is continuous. The standard theory of ordinary differential equations says that (11.29) has has a unique continuously differentiable solution for small t that satisfies the initial condition $\boldsymbol{\eta}(s,0) = \boldsymbol{\eta}^\circ(s)$. The boundedness of $\boldsymbol{\eta}$ implies that this solution can be continued to T.

By forming the integral equation for the difference $\boldsymbol{\eta}(s_1,t_1) - \boldsymbol{\eta}(s_2,t_2)$ and using (11.31), the Gronwall inequality, and the Arzelà-Ascoli Theorem, we find that $\boldsymbol{\eta}$ lies a compact subset of $C^0([0,1] \times [0,T])$. (This is one of the statements of Theorem 11.1.) Then (11.29) and the smoothness of $\boldsymbol{\kappa}$ imply that

(11.32)

$\boldsymbol{\eta}_t$ lies a bounded subset of $C^{0,1/2}[0,1]$ independent of t for each $t \in [0,T]$.

We now invoke a generalization of a lemma of Aubin [13]:

11.33. Lemma [29,34]**.** *Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be Banach spaces of functions with $\mathfrak{X}$ compactly embedded in $\mathfrak{Y}$ and with $\mathfrak{Y}$ embedded in $\mathfrak{Z}$. Let $\mathfrak{E}$ be a set of functions w for which w_t lies in a bounded subset of $L^p(0,T,\mathfrak{Z})$ with $p > 1$ and for which w lies in a bounded subset of $L^\infty(0,T,\mathfrak{X})$. Then $\mathfrak{E}$ lies in a compact subset of $C^0(0,T,\mathfrak{Y})$.*

To use this lemma, we take $\mathfrak{X} = C^{0,1/2}[0,1]$, $\mathfrak{Y} = C^0[0,1]$, $\mathfrak{Z} = L^2(0,1)$ $p = 2$, $w = \boldsymbol{\eta}_t$, and identify $\mathfrak{E}$ with those $\boldsymbol{\eta}_t$ satisfying $\int_0^t \|\boldsymbol{\eta}_{tt}\|^2\, d\tau \leq \Gamma(T)$ (which is a consequence of (11.3)). Thus $\boldsymbol{\eta}_{tt}$ lies in a bounded subset of $L^2([0,1] \times [0,T]) \equiv L^2(0,T,L^2(0,1)) \equiv L^2(0,T,\mathfrak{Z})$. Inequality (11.32) implies that $\boldsymbol{\eta}_t$ lies in a bounded subset of $L^\infty(0,T,C^{0,1/2}[0,1]) = L^\infty(0,T,\mathfrak{X})$. Hence $\boldsymbol{\eta}_t$ lies in a compact subset of $C^0(0,T,\mathfrak{Y}) = C^0(0,T,C^0[0,1])$. (This is another statement of Theorem 11.1.)

Having control of $\boldsymbol{\eta}_t$, we regard $(4.9)_2$ as a linear ordinary differential equation for $s \mapsto \boldsymbol{w}(s,t)$ parametrized by t. It is subject to the initial condition $\boldsymbol{w}(0,t) = \boldsymbol{o}$, which comes from the boundary condition (7.1b). By using the analog of (4.11), we find that

(11.34)
$$\boldsymbol{w}_s \text{ lies in a compact subset of } C^0(0,T,C^0[0,1]) \equiv C^0([0,1]\times[0,T]).$$

Likewise, we deduce the same result for $\boldsymbol{p}_s$ from (4.9) and (7.1b). □

12. The Modified Problem

Since we have shown that for any $T > 0$ there are $\bar{\Gamma}(T)$ and $\bar{\gamma}(T)$ such that

(12.1)
$$\bar{\gamma}(T) \le \zeta_3(\cdot,t) \le \bar{\Gamma}(T), \quad |\boldsymbol{\zeta}(\cdot,t)| \le \bar{\Gamma}(T), \quad |\boldsymbol{\eta}_t(\cdot,t)| \le \bar{\Gamma}(T) \quad \text{for } t \le T,$$

only the restriction of $\hat{\boldsymbol{\sigma}}(\cdot,\cdot,s)$ to the corresponding values of the arguments $(\boldsymbol{\eta},\dot{\boldsymbol{\eta}})$ actually intervenes in our initial-boundary-value problem for $t \le T$. This means that we can replace our constitutive functions, which were originally defined for all strains $\boldsymbol{\eta}$ satisfying (4.14) and for all strain rates $\dot{\boldsymbol{\eta}}$ and which exhibit unpleasant behavior at extreme values of these variables, by nicer constitutive functions, which are well behaved at the extreme values and which accordingly remove some of the difficulties in the existence theory. We now show how this can be done.

Let us set $\boldsymbol{\chi} := (\zeta_1,\zeta_2,\zeta_4,\zeta_5,\zeta_6) \equiv (v_1,v_2,u_1,u_2,u_3)$. For our fixed $\bar{\gamma}(T)$ and $\bar{\Gamma}(T)$ we introduce the cut-off functions

$$(12.2)\qquad \begin{aligned} [\zeta_3] &:= \begin{cases} \frac{1}{2}\bar{\gamma}(T) & \text{if} \quad \zeta_3 \le \frac{1}{2}\bar{\gamma}(T), \\ \zeta_3 & \text{if} \quad \frac{1}{2}\bar{\gamma}(T) \le \zeta_3 \le 2\bar{\Gamma}(T), \\ 2\bar{\Gamma}(T) & \text{if} \quad 2\bar{\Gamma}(T) \le \zeta_3, \end{cases} \\ [[\boldsymbol{\chi}]] &:= \begin{cases} \boldsymbol{\zeta}^- & \text{if} \quad |\boldsymbol{\chi}| \le 2\bar{\Gamma}(T), \\ 2\bar{\Gamma}(T)\boldsymbol{\chi}/|\boldsymbol{\chi}| & \text{if} \quad |\boldsymbol{\chi}| \ge 2\bar{\Gamma}(T). \end{cases} \end{aligned}$$

When $\boldsymbol{\zeta}$ is related to $\boldsymbol{\eta}$ by (6.5), then corresponding to (12.2) is the set of cut-off strains

$$(12.3)\quad \boldsymbol{\eta}^\sharp := \big([[v]]_1,\, [[v]]_2,\, [v_3 - \varUpsilon([[u]]_1,[[u]]_2,s)] + \varUpsilon([[u]]_1,[[u]]_2,s),\, [[\mathbf{u}]]\big)$$

where $[[v]]_1$ is the 1-component of $[[\boldsymbol{\zeta}]]$, and $[[\mathbf{u}]]$ is the triple of the last three components of $[[\boldsymbol{\zeta}]]$.

Guided by (6.6) we define the modified constitutive function $\boldsymbol{\sigma}^\sharp$:

(12.4)
$$\boldsymbol{\sigma}^\sharp(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},x) := \begin{cases} \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}^\sharp,\dot{\boldsymbol{\eta}},x) & \text{if} \quad |\dot{\boldsymbol{\eta}}| \le 2\bar{\Gamma}(T), \\ \hat{\boldsymbol{\sigma}}\left(\boldsymbol{\eta}^\sharp, \frac{2\bar{\Gamma}(T)\dot{\boldsymbol{\eta}}}{|\dot{\boldsymbol{\eta}}|}, x\right) + 2\mu\bar{\Gamma}(T)\big[1 - \frac{2\bar{\Gamma}(T)}{|\dot{\boldsymbol{\eta}}|}\big]\dot{\boldsymbol{\eta}} & \text{if} \quad |\dot{\boldsymbol{\eta}}| \ge 2\bar{\Gamma}(T) \end{cases}$$

where μ is a positive number. Thus the smoothness of our constitutive function $\hat{\boldsymbol{\sigma}}$ implies a uniform regularity for the modified constitutive function $\boldsymbol{\sigma}^\sharp$:

$$
\begin{gathered}
|\boldsymbol{\sigma}^\sharp(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x)| \leq \Gamma(T)(1 + |\dot{\boldsymbol{\eta}}|), \\
|\boldsymbol{\sigma}^\sharp_{\boldsymbol{\eta}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x)|, |\boldsymbol{\sigma}^\sharp_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x)|, |\boldsymbol{\sigma}^\sharp_x(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, x)| \leq \Gamma(T).
\end{gathered} \tag{12.5}
$$

Moreover, (6.3) implies that

$$
c|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi} \cdot \boldsymbol{\sigma}^\sharp_{\dot{\boldsymbol{\eta}}} \cdot \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \tag{12.6}
$$

whenever $|\dot{\boldsymbol{\eta}}| \leq 2\bar{\Gamma}(T)$. It was proved in [5] that the constitutive functions modified in this way have the virtue that (12.6) holds for all $\dot{\boldsymbol{\eta}}$ provided that μ is large enough. (This assertion is the basis of a surprisingly tricky proof that a continuously differentiable uniformly monotone mapping on a ball in $\mathbb{R}^n$ can be extended to a uniformly Lipschitz continuous, uniformly monotone mapping on all of $\mathbb{R}^n$ that is continuously differentiable everywhere except on the spherical boundary of the ball.)

In view of these remarks, we replace the actual problem with the modified problem. In doing so, we drop the sharp signs. This means that we are treating a problem of the same form as the original problem, but with the bounds (12.5), with the uniform monotonicity condition (12.6), and without the restriction (4.14).

IV. Existence and Uniqueness

In Secs. 13–17, we prove the existence part of Theorem 8.1 by a modification of the Faedo-Galerkin method [26, 39]. (We defer the proof of uniqueness to Sec. 18, where it is a consequence of the well-posedness proved there.) Our main effort is accordingly devoted to handling the characteristic difficulties, faced throughout this paper, due to the mathematical formulation of the underlying mechanics.

13. A Faedo-Galerkin Method

To avoid technical difficulties, we want approximations to the directors $\boldsymbol{d}_k$ to satisfy relations like (4.3). We accordingly replace a standard Faedo-Galerkin method based on the full set of weak equations (7.10) with one based on the weak momentum equations (7.10c,d) and on approximations of the geometric relations (5.7a,b,c). In Sec. 21 we discuss in detail the motivations for all of our modifications.

Galerkin approximations. We seek Galerkin approximations of standard type, not of the full set $(\boldsymbol{d}_k, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{p}, \boldsymbol{w})$ of unknowns of our initial-boundary-value problem, but only for $\boldsymbol{p}$ and $\boldsymbol{w}$, taking these in the very special form

$$\boldsymbol{p}^{\mathsf{N}}(s,t) := \sum_{\mathsf{a}=1}^{\mathsf{N}} \boldsymbol{p}_{\mathsf{a}}(t) y_{\mathsf{a}}(s), \qquad \boldsymbol{w}^{\mathsf{N}}(s,t) := \sum_{\mathsf{a}=1}^{\mathsf{N}} \boldsymbol{w}_{\mathsf{a}}(t) y_{\mathsf{a}}(s) \tag{13.1}$$

where

$$y_{\mathsf{a}}(s) := \sqrt{2}\sin \nu_{\mathsf{a}} s, \qquad \nu_{\mathsf{a}} := (2\mathsf{a}-1)\tfrac{\pi}{2}, \quad \text{whence} \quad \langle y_{\mathsf{a}}, y_{\mathsf{b}}\rangle = \delta_{\mathsf{ab}}, \tag{13.2}$$

with δ_{ab} the Kronecker delta. The functions $\boldsymbol{p}_{\mathsf{a}}, \boldsymbol{w}_{\mathsf{a}}$ are to be determined. (These unknown functions should also be indexed by N because the equations for them depend on N, but we suppress this index for visual clarity.) For notational convenience, we set

$$\begin{gathered} x_{\mathsf{a}}(s) := \sqrt{2}\cos \nu_{\mathsf{a}} s \quad \text{so that} \quad \langle x_{\mathsf{a}}, x_{\mathsf{b}}\rangle = \delta_{\mathsf{ab}}, \\ d_s x_{\mathsf{a}} = -\nu_{\mathsf{a}} y_{\mathsf{a}}, \quad d_s y_{\mathsf{a}} = \nu_{\mathsf{a}} x_{\mathsf{a}}, \qquad x_{\mathsf{a}}(s) = \nu_{\mathsf{a}} \int_s^1 y_{\mathsf{a}}(\xi)\, d\xi. \end{gathered} \tag{13.3}$$

System (13.1) then yields

$$\boldsymbol{w}_{\mathsf{a}} = \langle \boldsymbol{w}^{\mathsf{N}}, y_{\mathsf{a}}\rangle \quad \Leftrightarrow \quad \boldsymbol{w}^{\mathsf{N}} = \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{w}^{\mathsf{N}}, y_{\mathsf{a}}\rangle\, y_{\mathsf{a}}, \quad \text{etc.} \tag{13.4}$$

In view of (13.3) and (13.4), integration by parts yields

$$\boldsymbol{w}_s^{\mathsf{N}} = \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{w}^{\mathsf{N}}, y_{\mathsf{a}}\rangle\, d_s y_{\mathsf{a}} \equiv -\sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{w}^{\mathsf{N}}, d_s x_{\mathsf{a}}\rangle\, x_{\mathsf{a}} \equiv \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{w}_s^{\mathsf{N}}, x_{\mathsf{a}}\rangle\, x_{\mathsf{a}}. \tag{13.5}$$

We define orthogonal projectors $\boldsymbol{Y}^{\mathsf{N}}$ onto span $\{y_1, \dots, y_{\mathsf{N}}\}$ and $\boldsymbol{X}^{\mathsf{N}}$ onto span $\{x_1, \dots, x_{\mathsf{N}}\}$ by

$$\boldsymbol{Y}^{\mathsf{N}}\boldsymbol{f} := \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{f}, y_{\mathsf{a}}\rangle\, y_{\mathsf{a}}, \qquad \boldsymbol{X}^{\mathsf{N}}\boldsymbol{f} := \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{f}, x_{\mathsf{a}}\rangle\, x_{\mathsf{a}}, \tag{13.6}$$

so that $\boldsymbol{w}^{\mathsf{N}} \equiv \boldsymbol{Y}^{\mathsf{N}}\boldsymbol{w}^{\mathsf{N}}$ and $\boldsymbol{w}_s^{\mathsf{N}} \equiv \boldsymbol{X}^{\mathsf{N}}\boldsymbol{w}_s^{\mathsf{N}}$. Note that

$$\langle \boldsymbol{f}, \boldsymbol{X}^{\mathsf{N}}\boldsymbol{g}\rangle = \langle \boldsymbol{X}^{\mathsf{N}}\boldsymbol{f}, \boldsymbol{X}^{\mathsf{N}}\boldsymbol{g}\rangle = \langle \boldsymbol{X}^{\mathsf{N}}\boldsymbol{f}, \boldsymbol{g}\rangle, \qquad \|\boldsymbol{X}^{\mathsf{N}}\boldsymbol{f}\| \le \|\boldsymbol{f}\|; \tag{13.7}$$

the inequality is just Bessel's inequality.

Rather than representing approximations $\boldsymbol{d}_k^{\mathsf{N}}$, $\boldsymbol{v}^{\mathsf{N}}$, $\boldsymbol{u}^{\mathsf{N}}$ for $\boldsymbol{d}_k$, $\boldsymbol{v}$, $\boldsymbol{u}$ by sums like (13.1), we define them to be solutions of the following initial-value problems (based on (5.7a,b,c)) for ordinary differential equations with respect to t in which s is just a parameter:

$$\partial_t \boldsymbol{d}_k^{\mathsf{N}} = \boldsymbol{w}^{\mathsf{N}} \times \boldsymbol{d}_k^{\mathsf{N}}, \qquad \boldsymbol{d}_k^{\mathsf{N}}(s,0) = \boldsymbol{d}_k{}^{\circ}(s), \tag{13.8}$$

$$\boldsymbol{v}_t^{\mathsf{N}} = \boldsymbol{p}_s^{\mathsf{N}}, \qquad \boldsymbol{v}^{\mathsf{N}}(s,0) = \boldsymbol{v}^{\circ}(s), \tag{13.9}$$

$$\boldsymbol{u}_t^{\mathsf{N}} = \boldsymbol{w}_s^{\mathsf{N}} + \boldsymbol{w}^{\mathsf{N}} \times \boldsymbol{u}^{\mathsf{N}}, \qquad \boldsymbol{u}^{\mathsf{N}}(s,0) = \boldsymbol{u}^{\circ}(s). \tag{13.10}$$

(Were we to replace the boundary condition in (13.9) with $\boldsymbol{v}^{\mathsf{N}}(s,0) = \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{v}^\circ, x_{\mathsf{a}} \rangle x_{\mathsf{a}}(s)$, then $\boldsymbol{v}^{\mathsf{N}}(s,t)$ would have the form $\sum_{\mathsf{a}=1}^{\mathsf{N}} \boldsymbol{v}_{\mathsf{a}}(t) x_{\mathsf{a}}(s)$, which is a standard form of a Galerkin approximation. No such simplifications can be effected for (13.8) and (13.10).) By taking the dot product of (13.8) with $\boldsymbol{d}_p^{\mathsf{N}}$ we find that $\boldsymbol{d}_k^{\mathsf{N}} \cdot \boldsymbol{d}_p^{\mathsf{N}} = \boldsymbol{d}_k{}^\circ \cdot \boldsymbol{d}_p{}^\circ \equiv \delta_{kp}$, so that $\{\boldsymbol{d}_k^{\mathsf{N}}(s,t)\}$ is an orthonormal basis for each s, t.

We set

$$(13.11) \qquad v_k^{\mathsf{N}} := \boldsymbol{v}^{\mathsf{N}} \cdot \boldsymbol{d}_k^{\mathsf{N}}, \qquad u_k^{\mathsf{N}} := \boldsymbol{u}^{\mathsf{N}} \cdot \boldsymbol{d}_k^{\mathsf{N}},$$

$$(13.12) \qquad \mathbf{v}^{\mathsf{N}} := \boldsymbol{v}^{\mathsf{N}} \cdot \boldsymbol{d}_k^{\mathsf{N}} \mathbf{e}_k, \quad \mathbf{u}^{\mathsf{N}} := \boldsymbol{u}^{\mathsf{N}} \cdot \boldsymbol{d}_k^{\mathsf{N}} \mathbf{e}_k, \quad \boldsymbol{\eta}^{\mathsf{N}} := (\mathbf{v}^{\mathsf{N}}, \mathbf{u}^{\mathsf{N}}),$$
$$\text{so that} \quad \boldsymbol{u}^{\mathsf{N}} = \mathbf{u}^{\mathsf{N}} \cdot \mathbf{e}_k \boldsymbol{d}_k^{\mathsf{N}} =: \boldsymbol{R}^{\mathsf{N}} \cdot \mathbf{u}^{\mathsf{N}}, \quad \text{etc.},$$

(in consonance with (4.6) and (13.11)) where

$$(13.13) \quad \boldsymbol{R}^{\mathsf{N}} := \boldsymbol{d}_k^{\mathsf{N}} \mathbf{e}_k, \quad \boldsymbol{R}_t^{\mathsf{N}} = \boldsymbol{w}^{\mathsf{N}} \times \boldsymbol{R}^{\mathsf{N}} \quad \text{so that} \quad \boldsymbol{z}_t^{\mathsf{N}} = \boldsymbol{R}^{\mathsf{N}} \cdot \mathbf{z}_t + \boldsymbol{w}^{\mathsf{N}} \times \boldsymbol{z}^{\mathsf{N}}$$

for any function $\boldsymbol{z}^{\mathsf{N}}$ of the form $z_k^{\mathsf{N}} \boldsymbol{d}_k^{\mathsf{N}}$. Note that $\boldsymbol{R}^{\mathsf{N}}$ is an orthogonal transformation. (In (13.12) and (13.13), the summation convention applies to the Roman index k but not to the sanserif index N.)

We define

$$(13.14) \qquad \begin{gathered} n_k^{\mathsf{N}}(s,t) := \hat{n}_k(\mathbf{v}^{\mathsf{N}}(s,t), \mathbf{u}^{\mathsf{N}}(s,t), \mathbf{v}_t^{\mathsf{N}}(s,t), \mathbf{u}_t^{\mathsf{N}}(s,t), s), \\ \boldsymbol{n}^{\mathsf{N}} := n_k^{\mathsf{N}} \boldsymbol{d}_k^{\mathsf{N}}, \quad \mathbf{n}^{\mathsf{N}} := n_k^{\mathsf{N}} \mathbf{e}_k, \\ \mathbf{n}_{\mathrm{D}}^{\mathsf{N}}(s,t) := \hat{n}_k^{\mathrm{D}}(\mathbf{v}^{\mathsf{N}}(s,t), \mathbf{u}^{\mathsf{N}}(s,t), \mathbf{v}_t^{\mathsf{N}}(s,t), \mathbf{u}_t^{\mathsf{N}}(s,t), s) \mathbf{e}_k, \quad \text{etc.} \end{gathered}$$

In the theory of hyperbolic conservation laws, the weak forms (7.10a) and (7.10b) play a fundamental role. For the reasons mentioned above and in Sec. 21, we have replaced Galerkin equations based on them with (13.9) and (13.10). Presumably, such modifications would be unnecessary for Galerkin approximations supporting effective numerical methods, such as versions of the finite-element method.

Approximating ordinary differential equations. To get the approximating ordinary differential equations for the $\boldsymbol{p}_{\mathsf{a}}$ and $\boldsymbol{w}_{\mathsf{a}}$ from (7.10c) and (7.10d), we replace $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{p}$, $\boldsymbol{w}$, $\boldsymbol{d}_k$, $\boldsymbol{n}$, $\boldsymbol{m}$ with the same symbols bearing superposed indices N, replace $\rho \boldsymbol{J}$ with $\rho \boldsymbol{J}^{\mathsf{N}} := \rho J_{pq} \boldsymbol{d}_p^{\mathsf{N}} \boldsymbol{d}_q^{\mathsf{N}}$, replace $\boldsymbol{y}$ with $y_{\mathsf{a}} \boldsymbol{b}$ where $\boldsymbol{b}$ is an arbitrary constant vector, and then use the arbitrariness of $\boldsymbol{b}$ to obtain

(13.15)

$$\begin{aligned} \langle \boldsymbol{Y}^{\mathsf{N}} \rho A \boldsymbol{p}_t^{\mathsf{N}}, y_{\mathsf{a}} \rangle &\equiv \sum_{\mathsf{b}=1}^{\mathsf{N}} \int_0^1 y_{\mathsf{a}}(s) (\rho A)(s) y_{\mathsf{b}}(s)\, ds \ \frac{d\boldsymbol{p}_{\mathsf{b}}}{dt}(s,t) \\ &\equiv \frac{d}{dt} \int_0^1 y_{\mathsf{a}}(s) (\rho A)(s) \boldsymbol{p}^{\mathsf{N}}(s,t)\, ds \\ &= \bar{\boldsymbol{n}}(t) y_{\mathsf{a}}(1) - \int_0^1 \boldsymbol{n}^{\mathsf{N}} d_s y_{\mathsf{a}}\, ds \equiv - \int_0^1 [\boldsymbol{n}^{\mathsf{N}} - \bar{\boldsymbol{n}}] d_s y_{\mathsf{a}}\, ds \\ &\equiv -\nu_{\mathsf{a}} \langle \boldsymbol{X}^{\mathsf{N}} (\boldsymbol{n}^{\mathsf{N}} - \bar{\boldsymbol{n}}), x_{\mathsf{a}} \rangle, \end{aligned}$$

(13.16)

$$\begin{aligned}\langle \boldsymbol{Y}^{\mathsf{N}}(\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t, y_{\mathsf{a}}\rangle &\equiv \frac{d}{dt}\left[\int_0^1 \sum_{\mathsf{b}=1}^{\mathsf{N}} y_{\mathsf{a}}(s)\rho \boldsymbol{J}^{\mathsf{N}}(s,t) y_{\mathsf{b}}(s)\, ds \cdot \boldsymbol{w}_{\mathsf{b}}(t)\right]\\ &\equiv \frac{d}{dt}\int_0^1 y_{\mathsf{a}}(s)\rho \boldsymbol{J}^{\mathsf{N}}(s,t)\cdot \boldsymbol{w}^{\mathsf{N}}(s,t)\, ds\\ &= \bar{\boldsymbol{m}}(t) y_{\mathsf{a}}(1) - \int_0^1 \boldsymbol{m}^{\mathsf{N}} d_s y_{\mathsf{a}}\, ds + \int_0^1 (\boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}) y_{\mathsf{a}}\, ds\\ &\equiv -\int_0^1 \{[\boldsymbol{m}^{\mathsf{N}} - \bar{\boldsymbol{m}}] d_s y_{\mathsf{a}} - (\boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}) y_{\mathsf{a}}\}\, ds\\ &\equiv -\nu_{\mathsf{a}}\langle \boldsymbol{X}^{\mathsf{N}}(\boldsymbol{m}^{\mathsf{N}} - \bar{\boldsymbol{m}}), x_{\mathsf{a}}\rangle + \langle \boldsymbol{Y}^{\mathsf{N}}(\boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}), y_{\mathsf{a}}\rangle.\end{aligned}$$

(Recall that the boundary data $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ are introduced in (7.1c).) By virtue of (13.6), these equations hold not only for $\mathsf{a} = 1, \dots, \mathsf{N}$, but in fact for all a.

As initial conditions for $\boldsymbol{p}^{\mathsf{N}}$ and $\boldsymbol{w}^{\mathsf{N}}$ in (13.15) and (13.16) we take

$$\begin{aligned}\boldsymbol{p}^{\mathsf{N}}(s,0) &= \sum_{\mathsf{a}=1}^{\mathsf{N}} \boldsymbol{p}_{\mathsf{a}}(0) y_{\mathsf{a}}(s) = \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{p}^{\circ}, y_{\mathsf{a}}\rangle y_{\mathsf{a}}(s),\\ \boldsymbol{w}^{\mathsf{N}}(s,0) &= \sum_{\mathsf{a}=1}^{\mathsf{N}} \boldsymbol{w}_{\mathsf{a}}(0) y_{\mathsf{a}}(s) = \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{w}^{\circ}, y_{\mathsf{a}}\rangle y_{\mathsf{a}}(s).\end{aligned} \tag{13.17}$$

Let us multiply (13.15) and (13.16) by y_{a}, sum the resulting equations over a from 1 to N, use (13.6), and then integrate the resulting equations from s to 1 (or, alternatively, multiply (13.15) and (13.16) by x_{a}, sum the resulting equations over a from 1 to N, and use (13.3)) to obtain equivalent versions of these equations:

(13.18)

$$\begin{aligned}\int_s^1 \boldsymbol{Y}^{\mathsf{N}}\rho A \boldsymbol{p}_t^{\mathsf{N}}\, d\xi &\equiv \frac{d}{dt}\int_s^1 \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle y_{\mathsf{a}}, \rho A \boldsymbol{p}^{\mathsf{N}}\rangle y_{\mathsf{a}}\, d\xi\\ &= -\sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{n}^{\mathsf{N}} - \bar{\boldsymbol{n}}, x_{\mathsf{a}}\rangle x_{\mathsf{a}} \equiv -\boldsymbol{X}^{\mathsf{N}}(\boldsymbol{n}^{\mathsf{N}} - \bar{\boldsymbol{n}}),\end{aligned}$$

(13.19)

$$\begin{aligned}\int_s^1 \boldsymbol{Y}^{\mathsf{N}}\partial_t(\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})\, d\xi &\equiv \int_s^1 \frac{d}{dt}\left[\sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}}, y_{\mathsf{a}}\rangle y_{\mathsf{a}}\, d\xi\right]\\ &= -\sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{m}^{\mathsf{N}} - \bar{\boldsymbol{m}}, x_{\mathsf{a}}\rangle x_{\mathsf{a}} + \int_s^1 \sum_{\mathsf{a}=1}^{\mathsf{N}} \langle \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}, y_{\mathsf{a}}\rangle y_{\mathsf{a}}\, d\xi\\ &\equiv -\boldsymbol{X}^{\mathsf{N}}(\boldsymbol{m}^{\mathsf{N}} - \bar{\boldsymbol{m}}) + \int_s^1 \boldsymbol{Y}^{\mathsf{N}}(\boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}})\, d\xi.\end{aligned}$$

14. Global Existence of Solutions of the Approximating Ordinary Differential Equations

The convergence of the Faedo-Galerkin method for our problem hinges on obtaining sharp estimates for (13.1), some of which are analogous to those obtained in Sections 9–11. In all these estimates it is important to note that the constant C and the functions Γ and γ are independent of N.

System (13.8)–(13.17) is a well-defined initial-value problem for a finite-dimensional system of ordinary differential equations for the variables $\boldsymbol{p}_{\mathsf{a}}, \boldsymbol{w}_{\mathsf{a}}, \boldsymbol{d}_k^{\mathsf{N}}, \boldsymbol{v}^{\mathsf{N}}, \boldsymbol{u}^{\mathsf{N}}$. Since there is a positive number c such that $\langle \boldsymbol{p}, \rho A \boldsymbol{p}\rangle \geq c|\boldsymbol{p}|^2$ and $\langle \boldsymbol{w}, \rho \boldsymbol{J}^{\mathsf{N}} \cdot \boldsymbol{w}\rangle \geq c|\boldsymbol{w}|^2$, equations (13.16) and (13.15) can be put into standard form in which the $d_t \boldsymbol{p}_{\mathsf{a}}$ and $d_t \boldsymbol{w}_{\mathsf{a}}$ are expressed as functions of $\boldsymbol{p}_{\mathsf{a}}, \boldsymbol{w}_{\mathsf{a}}, \boldsymbol{d}_k^{\mathsf{N}}, \mathbf{v}^{\mathsf{N}}, \mathbf{u}^{\mathsf{N}}$. Since $\hat{\boldsymbol{\sigma}}(\cdot, x)$ is assumed to be continuously differentiable, and since $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ are continuous by (7.2), the standard theory of ordinary differential equations implies that this initial-value problem has a unique classical solution defined on a neighborhood of $t = 0$. The continuation theory for ordinary differential equations says that this solution is defined for all time provided that no component of it can blow up in finite time. We now show this. We have already shown that the $\boldsymbol{d}_k^{\mathsf{N}}$ are bounded because they are orthonormal.

To control $\boldsymbol{p}_{\mathsf{a}}, \boldsymbol{w}_{\mathsf{a}}$ we obtain an energy estimate corresponding to (9.14) by taking the dot product of(13.15) with $\boldsymbol{p}_{\mathsf{a}}$, taking the dot product of (13.16) with $\boldsymbol{w}_{\mathsf{a}}$, and summing the resulting two equations over a from 1 to N, adding the resulting sums, and invoking the compatibility equations (13.9) and (13.10):

(14.1)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 [\rho A \boldsymbol{p}^{\mathsf{N}} \cdot \boldsymbol{p}^{\mathsf{N}} + \boldsymbol{w}^{\mathsf{N}} \cdot \boldsymbol{J}^{\mathsf{N}} \cdot \boldsymbol{w}^{\mathsf{N}}]\, ds - \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{p}^{\mathsf{N}}(1,t) - \bar{\boldsymbol{m}}(t) \cdot \boldsymbol{w}^{\mathsf{N}}(1,t) \\
&\quad = \int_0^1 [-\boldsymbol{n}^{\mathsf{N}} \cdot \boldsymbol{p}_s^{\mathsf{N}} - \boldsymbol{m}^{\mathsf{N}} \cdot \boldsymbol{w}_s^{\mathsf{N}} + (\boldsymbol{v}^{\mathsf{N}} \times \boldsymbol{n}^{\mathsf{N}}) \cdot \boldsymbol{w}^{\mathsf{N}}]\, ds \\
&\quad = \int_0^1 \left\{ -\boldsymbol{n}^{\mathsf{N}} \cdot \boldsymbol{v}_t^{\mathsf{N}} - \boldsymbol{m}^{\mathsf{N}} \cdot [\boldsymbol{u}_t^{\mathsf{N}} - \boldsymbol{w}^{\mathsf{N}} \times \boldsymbol{u}^{\mathsf{N}}] + [\boldsymbol{v}^{\mathsf{N}} \times \boldsymbol{n}^{\mathsf{N}}] \cdot \boldsymbol{w}^{\mathsf{N}} \right\} ds \\
&\quad = \int_0^1 [-\boldsymbol{n}^{\mathsf{N}} \cdot \partial_t v_k^{\mathsf{N}} \boldsymbol{d}_k^{\mathsf{N}} - \boldsymbol{m}^{\mathsf{N}} \cdot \partial_t u_k^{\mathsf{N}} \boldsymbol{d}_k^{\mathsf{N}}]\, ds \\
&\quad = -\int_0^1 [\mathbf{n}^{\mathsf{N}} \cdot \mathbf{v}_t^{\mathsf{N}} + \mathbf{m}^{\mathsf{N}} \cdot \mathbf{u}_t^{\mathsf{N}}]\, ds.
\end{aligned}
$$

In analogy with (9.1) and (9.5) we define

(14.2a)

$$
K^{\mathsf{N}} := \frac{1}{2}\int_0^1 [\rho A \mathbf{p}^{\mathsf{N}} \cdot \mathbf{p}^{\mathsf{N}} + \mathbf{w}^{\mathsf{N}} \cdot \rho \mathbf{J} \cdot \mathbf{w}^{\mathsf{N}}]\, ds, \qquad \Phi^{\mathsf{N}} := \int_0^1 \varphi(\mathbf{v}^{\mathsf{N}}, \mathbf{u}^{\mathsf{N}}, s)\, ds,
$$

$$M^{\mathsf{N}} := \left[\int_0^1 \boldsymbol{\sigma}_{\mathrm{D}}^{\mathsf{N}} \cdot \boldsymbol{\eta}_t^{\mathsf{N}}\, ds\right]^{1/2} \equiv \left[\int_0^1 \boldsymbol{\sigma}^{\mathrm{D}}(\boldsymbol{\eta}^{\mathsf{N}}, \boldsymbol{\eta}_t^{\mathsf{N}}, s) \cdot \boldsymbol{\eta}_t^{\mathsf{N}}\, ds\right]^{1/2}, \tag{14.2b}$$

$$\Omega^{\mathsf{N}}(t) := \int_0^t M^{\mathsf{N}}(\tau)^2\, d\tau. \tag{14.2c}$$

We use constitutive equation (5.6) as in Section 9 to convert (14.1) into the energy equation

$$\tfrac{d}{dt}[K^{\mathsf{N}}(t) + \Phi^{\mathsf{N}}(t) + \Omega^{\mathsf{N}}(t)] = \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{p}^{\mathsf{N}}(1,t) + \bar{\boldsymbol{m}}(t) \cdot \boldsymbol{w}^{\mathsf{N}}(1,t), \tag{14.3}$$

whence the restriction (7.2) on the boundary data $\bar{\boldsymbol{n}}, \bar{\boldsymbol{m}}$ implies that

(14.4)
$$\begin{aligned} K^{\mathsf{N}}(t) + \Phi^{\mathsf{N}}(t) + \Omega^{\mathsf{N}}(t) &\le C + \int_0^t |\bar{\boldsymbol{n}}(\tau) \cdot \boldsymbol{p}^{\mathsf{N}}(1,\tau) + \bar{\boldsymbol{m}}(\tau) \cdot \boldsymbol{w}^{\mathsf{N}}(1,\tau)|\, d\tau \\ &\le C + \Gamma(t) \int_0^t (|\boldsymbol{p}^{\mathsf{N}}(1,\tau)| + |\boldsymbol{w}^{\mathsf{N}}(1,\tau)|)\, d\tau. \end{aligned}$$

To estimate the right-hand side of (14.4), we first use (6.4) to obtain

$$\int_0^1 |\boldsymbol{\eta}_t^{\mathsf{N}}(s,t)|^2\, ds \equiv \int_0^1 \{|\mathbf{v}_t^{\mathsf{N}}(s,t)|^2 + |\mathbf{u}_t^{\mathsf{N}}(s,t)|^2\}\, ds \le C M^{\mathsf{N}}(t)^2, \tag{14.5}$$

$$\int_0^1 \{|\boldsymbol{p}^{\mathsf{N}}(s,t)|^2 + \boldsymbol{w}^{\mathsf{N}}(s,t)|^2\}\, ds \le C K^{\mathsf{N}}(t), \tag{14.6}$$

(14.7)
$$\int_0^1 |\boldsymbol{\eta}^{\mathsf{N}}(s,t)|^2\, ds \le C + 2\int_0^1 t \int_0^t \{|\boldsymbol{\eta}_t^{\mathsf{N}}(s,\tau)|^2\, d\tau\, ds \le C + Ct\Omega^{\mathsf{N}}(t),$$

in analogy with (9.6)–(9.8).

In view of the compatibility equation (13.9) and these estimates, we obtain

(14.8)
$$\begin{aligned} |\boldsymbol{p}^{\mathsf{N}}(s,t)| &\le \int_0^1 |\boldsymbol{p}_s^{\mathsf{N}}(\xi,t)|\, d\xi = \int_0^1 |\boldsymbol{v}_t^{\mathsf{N}}|\, d\xi \le \int_0^1 \left(|\mathbf{v}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}||\mathbf{v}^{\mathsf{N}}|\right)\, d\xi \\ &\le \int_0^1 \left(|\mathbf{v}_t^{\mathsf{N}}| + \tfrac{1}{2}|\boldsymbol{w}^{\mathsf{N}}|^2 + \tfrac{1}{2}|\mathbf{v}^{\mathsf{N}}|^2\right)\, d\xi \\ &\le C M^{\mathsf{N}}(t) + \Gamma(t)[K^{\mathsf{N}}(t) + \Omega^{\mathsf{N}}(t)]. \end{aligned}$$

Likewise,

$$|\boldsymbol{w}^{\mathsf{N}}(s,t)| \le \int_0^1 \{|\mathbf{u}_t^{\mathsf{N}}(\xi,t)|\}\, d\xi \le C M^{\mathsf{N}}(t). \tag{14.9}$$

The substitution of (14.8) and (14.9) into the right-hand side of (14.4) and the use of the Gronwall inequality yields

$$K^{\mathsf{N}} + \Phi^{\mathsf{N}} + \Omega^{\mathsf{N}} \le \Gamma. \tag{14.10}$$

Let us again emphasize that here and throughout this section Γ is independent of N.

It then follows from the continuation theory for ordinary differential equations that (13.8)–(13.15) has a solution defined for all t that satisfies

$$\max_{t\in[0,T]}\big(\|\boldsymbol{p}^{\mathsf{N}}\|+\|\boldsymbol{w}^{\mathsf{N}}\|\big)\le\Gamma(T), \tag{14.11a}$$

$$\int_0^T\int_0^1|\boldsymbol{\eta}_t^{\mathsf{N}}|^2\,ds\,dt\equiv\int_0^T\int_0^1\big(|\mathbf{v}_t^{\mathsf{N}}|^2+|\mathbf{u}_t^{\mathsf{N}}|^2\big)\,ds\,dt\le\Gamma(T), \tag{14.11b}$$

whence

$$\max_{t\in[0,T]}\|\boldsymbol{\eta}^{\mathsf{N}}\|\equiv\max_{t\in[0,T]}\big(\|\mathbf{v}^{\mathsf{N}}\|+\|\mathbf{u}^{\mathsf{N}}\|\big)\le\Gamma(T). \tag{14.11c}$$

15. Estimates of Higher Derivatives

The derivation of the energy estimate (14.10) for the Galerkin approximation closely follows the pattern of that for the corresponding estimate (9.14) for the full partial differential equations. There are some subtle difficulties, however, in likewise extending the results of Sec. 11 to the Galerkin approximation:

Since $\hat{\boldsymbol{\sigma}}(\cdot,x)$ is continuously differentiable and since $\bar{\boldsymbol{m}},\bar{\boldsymbol{n}}\in H^1_{\mathrm{loc}}$ by (7.2), we can differentiate system (13.15), (13.16) with respect to t, finding that the second derivatives $d_{tt}\boldsymbol{p}_{\mathsf{a}}$ and $d_{tt}\boldsymbol{w}_{\mathsf{a}}$ thereby appearing on the left-hand sides belong to L^2_{loc}. We multiply the resulting equations respectively by $d_t\boldsymbol{p}_{\mathsf{a}}$ and $d_t\boldsymbol{w}_{\mathsf{a}}$, sum them over a, and add them to obtain the analog of (11.9) with all the symbols bearing the superscript N:

$$\begin{aligned}&\tfrac{1}{2}d_t\langle\rho A\boldsymbol{p}_t^{\mathsf{N}},\boldsymbol{p}_t^{\mathsf{N}}\rangle+\langle\partial_{tt}(\rho\boldsymbol{J}^{\mathsf{N}}\cdot\boldsymbol{w}_t^{\mathsf{N}}),\boldsymbol{w}_t^{\mathsf{N}}\rangle-\bar{\boldsymbol{n}}_t\cdot\boldsymbol{p}_t^{\mathsf{N}}(1,t)-\bar{\boldsymbol{m}}_t\cdot\boldsymbol{w}_t^{\mathsf{N}}(1,t)\\&\qquad=-\langle\boldsymbol{n}_t^{\mathsf{N}},\boldsymbol{p}_{st}^{\mathsf{N}}\rangle-\langle\boldsymbol{m}_t^{\mathsf{N}},\boldsymbol{w}_{st}^{\mathsf{N}}\rangle+\langle(\boldsymbol{v}^{\mathsf{N}}\times\boldsymbol{n}^{\mathsf{N}})_t,\boldsymbol{w}_t^{\mathsf{N}}\rangle.\end{aligned} \tag{15.1}$$

In imitation of (11.2) we define

$$H^{\mathsf{N}}[\boldsymbol{p}_t^{\mathsf{N}},\boldsymbol{w}_t^{\mathsf{N}},\boldsymbol{\eta}^{\mathsf{N}},\boldsymbol{\eta}_t^{\mathsf{N}},\boldsymbol{\eta}_{tt}^{\mathsf{N}}](t):=\tfrac{1}{2}\langle\boldsymbol{p}_t^{\mathsf{N}},\rho A\boldsymbol{p}_t^{\mathsf{N}}\rangle+\tfrac{1}{2}\langle\boldsymbol{w}_t^{\mathsf{N}},\rho\boldsymbol{J}^{\mathsf{N}}\cdot\boldsymbol{w}_t^{\mathsf{N}}\rangle+\int_0^t\|\boldsymbol{\eta}_{tt}^{\mathsf{N}}\|^2\,d\tau. \tag{15.2}$$

Since the compatibility equations (13.9) and (13.10) (coming from our definition of $\boldsymbol{v}^{\mathsf{N}}$ and $\boldsymbol{u}^{\mathsf{N}}$ as their solutions) are exactly satisfied (and consequently the $\boldsymbol{d}_k^{\mathsf{N}}$ are unambiguously defined), we can use the constitutive assumptions (6.2) and (6.11) to obtain the exact analog of (11.11):

$$\begin{aligned}(15.3)\qquad H^{\mathsf{N}} \le{}& \Gamma + C\int_0^t\!\!\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{w}_t^{\mathsf{N}}|^2\,ds\,d\tau + C\int_0^t\!\!\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|^3|\boldsymbol{w}_t^{\mathsf{N}}|\,ds\,d\tau \\ &+ \Gamma\int_0^t\!\!\int_0^1 \big(\,|\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}_{tt}^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|^2|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|^3|\boldsymbol{\eta}^{\mathsf{N}}| \\ &\qquad + |\boldsymbol{w}_t^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{w}_t^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}|\,\big)\,|\boldsymbol{\sigma}^{\mathsf{N}}|\,ds\,d\tau \\ &+ C\int_0^1 (|\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|^2|\boldsymbol{\eta}^{\mathsf{N}}|)|\boldsymbol{\sigma}^{\mathsf{N}}|\,ds \\ &+ \int_0^t [|\boldsymbol{p}_t^{\mathsf{N}}(1,\tau)|\,|\bar{\boldsymbol{n}}_t(\tau)| + |\boldsymbol{w}_t^{\mathsf{N}}(1,\tau)|\,|\bar{\boldsymbol{m}}_t(\tau)|]\,d\tau.\end{aligned}$$

As in Sec. 11 we must estimate the right-hand side of (15.3) to obtain the analog of Proposition 11.3:

15.4. Proposition. *Let the hypotheses of Theorem 10.6 hold. Let* (7.8a) *hold. Then*

$$(15.4)\qquad H^{\mathsf{N}}(t) \le \Gamma\left[1 + \int_0^t (1+M^{\mathsf{N}})^2 H^{\mathsf{N}}\,d\tau\right] \quad \textit{whence} \quad H^{\mathsf{N}}(t) \le \Gamma(T)$$

for all $T > 0$ and for all $t \in [0,T]$.

Proof. As in Sec. 11, we obtain the exact analogs of (11.12)–(11.14):

$$(15.5)\qquad \begin{gathered} |\boldsymbol{w}^{\mathsf{N}}| \le CM^{\mathsf{N}}, \quad \|\boldsymbol{w}^{\mathsf{N}}\| \le \Gamma, \quad |\boldsymbol{w}^{\mathsf{N}}|^2 \le \Gamma[1+\sqrt{H^{\mathsf{N}}}], \\ \|\boldsymbol{\eta}_t^{\mathsf{N}}\| \le CM^{\mathsf{N}}, \quad \|\boldsymbol{\eta}_t^{\mathsf{N}}\|^2 \le \Gamma\sqrt{H^{\mathsf{N}}}. \end{gathered}$$

We supplement these with (14.11).

Exactly as in (11.15), (11.23), (11.24), we estimate all the terms not containing $\boldsymbol{\sigma}^{\mathsf{N}}$:

$$\begin{aligned}(15.6)\qquad &\int_0^t\!\!\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{w}_t^{\mathsf{N}}|^2\,ds\,d\tau + C\int_0^t\!\!\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|^3|\boldsymbol{w}_t^{\mathsf{N}}|\,ds\,d\tau \\ &\qquad \le \Gamma\int_0^t [M^{\mathsf{N}}H^{\mathsf{N}} + (M^{\mathsf{N}})^2\sqrt{H^{\mathsf{N}}}]\,d\tau,\end{aligned}$$

$$(15.7)\qquad \int_0^t \{|\boldsymbol{p}_t^{\mathsf{N}}(1,\tau)|\,|\bar{\boldsymbol{n}}_t(\tau)| + |\boldsymbol{w}_t^{\mathsf{N}}(1,\tau)|\,|\bar{\boldsymbol{m}}_t(\tau)|\}\,d\tau \le \Gamma + C\int_0^t H^{\mathsf{N}}\,d\tau.$$

Our development so far parallels that of the proof of Proposition 11.3 up to (11.16). To get useful estimates of the remaining integrals on the right-hand side of (15.3), we cannot continue following that proof because we lack pointwise estimates on $\boldsymbol{n}^{\mathsf{N}}$ and $\boldsymbol{m}^{\mathsf{N}}$ like those of (11.16) and (11.17). In particular, (11.16) and (11.17) are based on integrals of acceleration terms over intervals $[s_1, s_2]$ on the s-axis, i.e., on integrals over $[0,1]$ of the product of the acceleration terms with the characteristic function for $[s_1, s_2]$. We cannot get such integrals in the weak formulation of equations supporting our Galerkin approximation because these characteristic functions do not lie in the span of the test functions (13.2) and (13.3). Rather than attempting to approximate the characteristic functions by combinations of these test functions, we instead use estimates for $\boldsymbol{X}^{\mathsf{N}}\boldsymbol{n}^{\mathsf{N}}$ and $\boldsymbol{X}^{\mathsf{N}}\boldsymbol{m}^{\mathsf{N}}$ available from (13.18) and (13.19), the bounds (12.5) and (12.6) for the modified problem, the bounds (14.11) and (15.5) just obtained, and the constitutive assumption (6.11). Since the derivation of the requisite estimates of the remaining integrals on the right-hand side of (15.3) depends sensitively on the choice of estimates for $\boldsymbol{w}^{\mathsf{N}}$ and $\boldsymbol{\eta}^{\mathsf{N}}$ from (15.5), and on the choice and ordering of standard mathematical inequalities, we supply the tricky details:

(15.8)
$$
\begin{aligned}
&\int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}_{tt}|\,|\boldsymbol{\sigma}^{\mathsf{N}}|\,ds\,d\tau\\
&\le \Gamma(T)\int_0^t M^{\mathsf{N}}\int_0^1 |\boldsymbol{\eta}^{\mathsf{N}}_{tt}|\,(1+|\boldsymbol{\eta}^{\mathsf{N}}_t|)\,ds\,d\tau \le \Gamma(T)\int_0^t M^{\mathsf{N}}\|\boldsymbol{\eta}^{\mathsf{N}}_{tt}\|\,(1+\|\boldsymbol{\eta}^{\mathsf{N}}_t\|)\,d\tau\\
&\le \Gamma(T)\int_0^t [M^{\mathsf{N}}]^2\,d\tau+\varepsilon\int_0^t \|\boldsymbol{\eta}^{\mathsf{N}}_{tt}\|^2\,d\tau+\Gamma(T)\sqrt{\int_0^t \|\boldsymbol{\eta}^{\mathsf{N}}_{tt}\|^2\,d\tau}\sqrt{\int_0^t (M^{\mathsf{N}})^2\sqrt{H^{\mathsf{N}}}\,d\tau}\\
&\le \Gamma(T)+\varepsilon H^{\mathsf{N}}+\Gamma(T)\sqrt{H^{\mathsf{N}}}\sqrt{\int_0^t (M^{\mathsf{N}})^2[1+H^{\mathsf{N}}]\,d\tau}\\
&\le \Gamma(T)+\varepsilon H^{\mathsf{N}}+\Gamma(T)\int_0^t (M^{\mathsf{N}})^2 H^{\mathsf{N}}\,d\tau.
\end{aligned}
$$

(15.9)
$$
\begin{aligned}
&\int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf{N}}|^2|\boldsymbol{\eta}^{\mathsf{N}}_t|\,|\boldsymbol{\sigma}^{\mathsf{N}}|\,ds\,d\tau \le \Gamma(T)\int_0^t (M^{\mathsf{N}})^2\|\boldsymbol{\eta}^{\mathsf{N}}_t\|(1+\|\boldsymbol{\eta}^{\mathsf{N}}_t\|)\,d\tau\\
&\le \Gamma(T)\int_0^t (M^{\mathsf{N}})^2[(H^{\mathsf{N}})^{1/4}+(H^{\mathsf{N}})^{1/2}]\,d\tau \le \Gamma(T)+\Gamma(T)\int_0^t (M^{\mathsf{N}})^2 H^{\mathsf{N}}\,d\tau,
\end{aligned}
$$

(15.10)
$$
\begin{aligned}
&\int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf{N}}_t|\,|\boldsymbol{\eta}^{\mathsf{N}}_t|\,|\boldsymbol{\sigma}^{\mathsf{N}}|\,ds\,d\tau \le \int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf{N}}_t|\,|\boldsymbol{\eta}^{\mathsf{N}}_t|\,(1+|\boldsymbol{\eta}^{\mathsf{N}}_t|)\,ds\,d\tau\\
&\le \int_0^1\left[C+\Gamma(T)\int_0^t |\boldsymbol{\eta}^{\mathsf{N}}_{tt}|)\,d\tau\right]\left[\int_0^t |\boldsymbol{w}^{\mathsf{N}}_t|\,|\boldsymbol{\eta}^{\mathsf{N}}_t|\;d\tau\right]ds\\
&\le \Gamma(T)\sqrt{\left[1+\int_0^1\int_0^t |\boldsymbol{\eta}^{\mathsf{N}}_{tt}|\,d\tau\,ds\right]^2}\sqrt{\left[\int_0^1\int_0^t |\boldsymbol{w}^{\mathsf{N}}_t|\,|\boldsymbol{\eta}^{\mathsf{N}}_t|\,d\tau\,ds\right]^2}\\
&\le \Gamma(T)\sqrt{\left[1+\int_0^1\int_0^t |\boldsymbol{\eta}^{\mathsf{N}}_{tt}|^2\,d\tau\,ds\right]}\left[\int_0^1\int_0^t |\boldsymbol{w}^{\mathsf{N}}_t|^2\,d\tau\,ds\right]^{1/4}\left[\int_0^1\int_0^t |\boldsymbol{\eta}^{\mathsf{N}}_t|^2\,d\tau\,ds\right]^{1/4}\\
&\le \Gamma(T)\left[1+\sqrt{H^{\mathsf{N}}(t)}\right]\left[\int_0^t H^{\mathsf{N}}\,d\tau\right]^{1/4}\left[\int_0^t H^{\mathsf{N}}\,d\tau\right]^{1/8}\\
&= \Gamma(T)\left[1+\sqrt{H^{\mathsf{N}}(t)}\right]\left[\int_0^t H^{\mathsf{N}}\,d\tau\right]^{3/8} \le \varepsilon H^{\mathsf{N}}+\Gamma(T)\left[1+\int_0^t H^{\mathsf{N}}\,d\tau\right],
\end{aligned}
$$

(15.11)
$$\int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf N}|^3|\boldsymbol{\eta}^{\mathsf N}|\,|\boldsymbol{\sigma}^{\mathsf N}|\,ds\,d\tau \le \Gamma(T)\int_0^t (M^{\mathsf N})^2[1+(H^{\mathsf N})^{1/4}]\,\|\boldsymbol{\eta}^{\mathsf N}\|\,(1+\|\boldsymbol{\eta}_t^{\mathsf N}\|)\,d\tau$$
$$\le \Gamma(T)+\Gamma(T)\int_0^t (M^{\mathsf N})^2[(H^{\mathsf N})^{1/4}+(H^{\mathsf N})^{1/2}]\,d\tau \le \Gamma(T)+\Gamma(T)\int_0^t (M^{\mathsf N})^2 H^{\mathsf N}\,d\tau,$$

(15.12)
$$\begin{aligned}&\int_0^t\int_0^1 |\boldsymbol{w}^{\mathsf N}|\,|\boldsymbol{w}_t^{\mathsf N}|\,|\boldsymbol{\eta}^{\mathsf N}|\,|\boldsymbol{\sigma}^{\mathsf N}|\,ds\,d\tau\\ &\le \int_0^1\left[C+\Gamma(T)\int_0^t |\boldsymbol{\eta}_{tt}^{\mathsf N}|)\,d\tau\right]\left[\int_0^t |\boldsymbol{w}^{\mathsf N}|\,|\boldsymbol{w}_t^{\mathsf N}|\,|\boldsymbol{\eta}^{\mathsf N}|\,\,d\tau\right]ds\\ &\le \Gamma(T)\left[1+\sqrt{H^{\mathsf N}(t)}\right]\sqrt{\int_0^t M^{\mathsf N}\int_0^1 \|\boldsymbol{w}_t^{\mathsf N}\|\,\|\boldsymbol{\eta}^{\mathsf N}\|\,ds\,d\tau}\\ &\le \Gamma(T)\left[1+\sqrt{H^{\mathsf N}(t)}\right]\left[\int_0^t (M^{\mathsf N})^2\,d\tau\right]^{1/4}\left[\int_0^t H^{\mathsf N}\,d\tau\right]^{1/4}\\ &\le \varepsilon H^{\mathsf N}+\Gamma(T)\left[1+\int_0^t H^{\mathsf N}\,d\tau\right],\end{aligned}$$

(15.13)
$$\begin{aligned}&\int_0^1 |\boldsymbol{w}^{\mathsf N}|\,|\boldsymbol{\eta}_t^{\mathsf N}|\,|\boldsymbol{\sigma}^{\mathsf N}|\,ds \le \Gamma(T)[1+H^{\mathsf N}(t)^{1/4}]\int_0^1 |\boldsymbol{\eta}_t^{\mathsf N}|\,[1+|\boldsymbol{\eta}_t^{\mathsf N}|]\,ds\\ &\le \Gamma(T)[1+H^{\mathsf N}(t)^{3/4}] \le \Gamma(T)+\varepsilon H^{\mathsf N}(t),\end{aligned}$$

(15.14)
$$\begin{aligned}&\int_0^1 |\boldsymbol{w}^{\mathsf N}|^2|\boldsymbol{\eta}^{\mathsf N}|\,|\boldsymbol{\sigma}^{\mathsf N}|\,ds \le \Gamma(T)[1+H^{\mathsf N}(t)^{1/2}]\,\|\boldsymbol{\eta}^{\mathsf N}\|(1+\|\boldsymbol{\eta}_t^{\mathsf N}\|)\\ &\le \Gamma(T)[1+H^{\mathsf N}(t)^{1/2}][1+H^{\mathsf N}(t)^{1/4}] \le \Gamma(T)+\varepsilon H^{\mathsf N}(t).\end{aligned}$$

These estimates ensure (15.4). □

An immediate consequence of (15.4) is the strengthening of (15.5):

(15.15)
$$|\boldsymbol{p}^{\mathsf N}| \le \Gamma(T),\quad |\boldsymbol{w}^{\mathsf N}| \le \Gamma(T),\quad \|\boldsymbol{p}_t^{\mathsf N}\| \le \Gamma(T),\quad \|\boldsymbol{w}_t^{\mathsf N}\| \le \Gamma(T),$$
$$\|\boldsymbol{\eta}_t^{\mathsf N}\| \le \Gamma(T),\quad \|\boldsymbol{p}_s^{\mathsf N}\| \le \Gamma(T),\quad \|\boldsymbol{w}_s^{\mathsf N}\| \le \Gamma(T),\quad \int_0^t \|\boldsymbol{\eta}_{tt}^{\mathsf N}\|^2\,d\tau \le \Gamma(T).$$

To justify the convergence of our Galerkin approximations to the solution, we must obtain an estimate for $\|\boldsymbol{\eta}_{st}^{\mathsf N}\|$. Toward this end, we use (13.7) to replace $\int_0^1[\boldsymbol{n}^{\mathsf N}-\bar{\boldsymbol{n}}]d_s y_{\mathsf a}\,ds$ in (13.15) with $\int_0^1 \boldsymbol{X}^{\mathsf N}[\boldsymbol{n}^{\mathsf N}-\bar{\boldsymbol{n}}]d_s y_{\mathsf a}\,ds$, take the dot product of the resulting version of this equation with $\nu_{\mathsf a}^2\boldsymbol{p}_{\mathsf a}$, use the identity $d_{ss}y_{\mathsf a}=-\nu_{\mathsf a}^2 y_{\mathsf a}$, sum the product over $\mathsf a$ from 1 to $\mathsf N$, and integrate the sum by parts to obtain

$$\langle \boldsymbol{p}_{ss}^{\mathsf N},\rho A\boldsymbol{p}_t^{\mathsf N}\rangle = -\boldsymbol{X}^{\mathsf N}(\boldsymbol{n}^{\mathsf N}-\bar{\boldsymbol{n}})\big|_{s=1}\cdot\boldsymbol{p}_{ss}^{\mathsf N}(1,\cdot)+\langle \boldsymbol{p}_{ss}^{\mathsf N},\boldsymbol{n}_s^{\mathsf N}\rangle. \tag{15.16}$$

Likewise, we obtain

(15.17)

$$\langle \boldsymbol{w}^{\mathsf{N}}_{ss}, (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle = -\boldsymbol{X}^{\mathsf{N}}(\boldsymbol{m}^{\mathsf{N}}-\bar{\boldsymbol{m}})\big|_{s=1}\cdot \boldsymbol{w}^{\mathsf{N}}_{ss}(1,\cdot)+\langle \boldsymbol{w}^{\mathsf{N}}_{ss}, \boldsymbol{m}^{\mathsf{N}}_s\rangle+\langle \boldsymbol{w}^{\mathsf{N}}_{ss}, \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}\rangle.$$

We note with satisfaction that (13.18) and (13.19) imply that the boundary terms in (15.16) and (15.17) vanish. Now,

(15.18a)

$$\boldsymbol{n}^{\mathsf{N}}_s = \boldsymbol{R}^{\mathsf{N}}\cdot[\hat{\mathbf{n}}_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_s + \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \hat{\mathbf{n}}_x] + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}}, \quad \text{etc.},$$

(15.18b)

$$\begin{aligned}\boldsymbol{p}^{\mathsf{N}}_{ss} = \boldsymbol{v}^{\mathsf{N}}_{st} &= \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st} + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_t\\ &+ \boldsymbol{w}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_s - \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t + \boldsymbol{w}^{\mathsf{N}}\times(\boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{v}^{\mathsf{N}}),\end{aligned}$$

(15.18c)

$$\boldsymbol{w}^{\mathsf{N}}_{ss} = \boldsymbol{u}^{\mathsf{N}}_{st} - (\boldsymbol{u}^{\mathsf{N}}_t - \boldsymbol{w}^{\mathsf{N}}\times \boldsymbol{u}^{\mathsf{N}})\times \boldsymbol{u}^{\mathsf{N}} - \boldsymbol{w}^{\mathsf{N}}\times \boldsymbol{u}^{\mathsf{N}}_s = \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st} + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t$$

where the derivatives of $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ are evaluated at $(\boldsymbol{\eta}^{\mathsf{N}}, \boldsymbol{\eta}^{\mathsf{N}}_t, s)$ and where $\boldsymbol{R}^{\mathsf{N}} := \boldsymbol{d}^{\mathsf{N}}_k \mathbf{e}_k$. The identities (15.18b) and (15.18c) follow from the compatibility equations (13.9) and (13.10). We add (15.16) and (15.17) (without the boundary terms) and substitute (15.18) into the sum to obtain

(15.19)

$$\begin{aligned}&\langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st} + \boldsymbol{p}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st},\ \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \boldsymbol{n}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} - \rho A \boldsymbol{p}^{\mathsf{N}}_t\rangle\\ &+\langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st} + \boldsymbol{w}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st},\\ &\quad \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \boldsymbol{m}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle = 0,\end{aligned}$$

whence

(15.20)

$$\begin{aligned}&\langle \boldsymbol{\eta}^{\mathsf{N}}_{st}, \boldsymbol{\sigma}^{\mathsf{N}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle \equiv \langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle + \langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle\\ &= -\langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st}, \boldsymbol{n}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} - \rho A \boldsymbol{p}^{\mathsf{N}}_t\rangle\\ &\quad - \langle \boldsymbol{p}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle\\ &\quad - \langle \boldsymbol{p}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st}, \boldsymbol{n}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} - \rho A \boldsymbol{p}^{\mathsf{N}}_t\rangle\\ &\quad - \langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st}, \boldsymbol{m}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle\\ &\quad - \langle \boldsymbol{w}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle\\ &\quad - \langle \boldsymbol{w}^{\mathsf{N}}_{ss} - \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st}, \boldsymbol{m}^{\mathsf{N}}_s - \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st} + \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle\\ &= -\langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot[\hat{\mathbf{n}}_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_s + \hat{\mathbf{n}}_x] + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - \rho A \boldsymbol{p}^{\mathsf{N}}_t\rangle\\ &\quad - \langle \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_t + \boldsymbol{w}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_s - \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t + \boldsymbol{w}^{\mathsf{N}}\times(\boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{v}^{\mathsf{N}}), \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{n}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle\\ &\quad - \langle \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_t + \boldsymbol{w}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{v}^{\mathsf{N}}_s - \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t + \boldsymbol{w}^{\mathsf{N}}\times(\boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{v}^{\mathsf{N}}),\\ &\qquad \boldsymbol{R}^{\mathsf{N}}\cdot[\hat{\mathbf{n}}_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_s + \hat{\mathbf{n}}_x] + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - \rho A \boldsymbol{p}^{\mathsf{N}}_t\rangle\\ &\quad - \langle \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_{st}, \boldsymbol{R}^{\mathsf{N}}\cdot[\hat{\mathbf{m}}_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_s + \hat{\mathbf{m}}_x] + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{m}^{\mathsf{N}} + \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle\\ &\quad - \langle \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t, \boldsymbol{R}^{\mathsf{N}}\cdot \hat{\mathbf{m}}_{\dot{\boldsymbol{\eta}}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_{st}\rangle\\ &\quad - \langle \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{R}^{\mathsf{N}}\cdot \mathbf{u}^{\mathsf{N}}_t, \boldsymbol{R}^{\mathsf{N}}\cdot[\hat{\mathbf{n}}_{\boldsymbol{\eta}}\cdot \boldsymbol{\eta}^{\mathsf{N}}_s + \hat{\mathbf{n}}_x] + \boldsymbol{u}^{\mathsf{N}}\times \boldsymbol{m}^{\mathsf{N}} + \boldsymbol{v}^{\mathsf{N}}\times \boldsymbol{n}^{\mathsf{N}} - (\rho \boldsymbol{J}^{\mathsf{N}}\cdot \boldsymbol{w}^{\mathsf{N}})_t\rangle.\end{aligned}$$

Applying the inequalities (12.5), (12.6), (15.15) to (15.20) yields
(15.21)

$$
\begin{aligned}
\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2 \le \Gamma \int_0^1 & |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\{|\boldsymbol{\eta}_s^{\mathsf{N}}| + 1 + |\boldsymbol{\eta}^{\mathsf{N}}|\,(1 + |\boldsymbol{\eta}_t^{\mathsf{N}}|) + |\boldsymbol{p}_t^{\mathsf{N}}|\}\,ds \\
&+ \Gamma \int_0^1 \{|\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}_s^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}|^2\}|\boldsymbol{\eta}_{st}^{\mathsf{N}}|\,ds \\
&+ \Gamma \int_0^1 \{|\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}_s^{\mathsf{N}}| + |\boldsymbol{w}^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}|^2\} \\
&\qquad \{|\boldsymbol{\eta}_s^{\mathsf{N}}| + 1 + |\boldsymbol{\eta}^{\mathsf{N}}|(1 + |\boldsymbol{\eta}_t^{\mathsf{N}}|) + |\boldsymbol{p}_t^{\mathsf{N}}|\}\,ds \\
&+ \Gamma \int_0^1 |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\{|\boldsymbol{\eta}_s^{\mathsf{N}}| + 1 + |\boldsymbol{\eta}^{\mathsf{N}}|(1 + |\boldsymbol{\eta}_t^{\mathsf{N}}|) + |\boldsymbol{w}^{\mathsf{N}}|^2 + |\boldsymbol{w}_t^{\mathsf{N}}|\}\,ds \\
&+ \Gamma \int_0^1 \{|\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}|\}|\boldsymbol{\eta}_{st}^{\mathsf{N}}|\,ds \\
&+ \Gamma \int_0^1 |\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}|\{|\boldsymbol{\eta}_s^{\mathsf{N}}| + 1 + |\boldsymbol{\eta}^{\mathsf{N}}|(1 + |\boldsymbol{\eta}_t^{\mathsf{N}}|) + |\boldsymbol{w}^{\mathsf{N}}|^2 + |\boldsymbol{w}_t^{\mathsf{N}}|\}\,ds \\
\le \Gamma \int_0^1 & \{|\boldsymbol{\eta}_{st}^{\mathsf{N}}| + |\boldsymbol{\eta}^{\mathsf{N}}|^2 + |\boldsymbol{\eta}_s^{\mathsf{N}}| + |\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}|\} \\
&\qquad \{1 + |\boldsymbol{\eta}^{\mathsf{N}}|^2 + |\boldsymbol{\eta}_s^{\mathsf{N}}| + |\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}| + |\boldsymbol{p}_t^{\mathsf{N}}| + |\boldsymbol{w}_t^{\mathsf{N}}|\}\,ds.
\end{aligned}
$$

To estimate the terms in the last line of (15.21) we use the pointwise estimates
(15.22)

$$
\begin{gathered}
|\boldsymbol{\eta}_t^{\mathsf{N}}(s,t)| \le C + \|\boldsymbol{\eta}_{st}^{\mathsf{N}}(\cdot,t)\|, \quad |\boldsymbol{\eta}^{\mathsf{N}}(s,t)| \le \Gamma(t) + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}(\cdot,\tau)\|\,d\tau, \\
|\boldsymbol{\eta}_s^{\mathsf{N}}(s,t)| \le C + \int_0^t |\boldsymbol{\eta}_{st}^{\mathsf{N}}(s,\tau)|\,d\tau, \quad \|\boldsymbol{\eta}_s^{\mathsf{N}}\|^2 \le C + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}(\cdot,\tau)\|^2\,d\tau,
\end{gathered}
$$

valid for all $s \in [o,1]$ and for all $t \ge 0$, together with (12.5), (12.6), (15.15):

(15.23a)

$$
\begin{aligned}
&\int_0^1 |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}|^2\,ds \le \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\,\|\boldsymbol{\eta}^{\mathsf{N}}\|\left[\Gamma(T) + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\,d\tau\right] \\
&\le \varepsilon\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2 + \Gamma(T)\left[1 + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2\,d\tau\right],
\end{aligned}
$$

(15.23b)

$$
\begin{aligned}
&\int_0^1 |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\,|\boldsymbol{\eta}_s^{\mathsf{N}}|\,ds \le \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\sqrt{C + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2\,d\tau} \\
&\le \varepsilon\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2 + \Gamma(T)\left[1 + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2\,d\tau\right],
\end{aligned}
$$

(15.23c)

$$\int_0^1 |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\,|\boldsymbol{\eta}^{\mathsf{N}}|\,|\boldsymbol{\eta}_t^{\mathsf{N}}|\,ds \le \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\|\boldsymbol{\eta}_t^{\mathsf{N}}\|\left[\Gamma(T) + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\,d\tau\right]$$

$$\le \varepsilon\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2 + \Gamma(T)\left[1 + \int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2\,d\tau\right],$$

(15.23d)

$$\int_0^1 |\boldsymbol{\eta}_{st}^{\mathsf{N}}|\{|\boldsymbol{p}_t^{\mathsf{N}}| + |\boldsymbol{w}_t^{\mathsf{N}}|\}\,ds \le \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|\{\|\boldsymbol{p}_t^{\mathsf{N}}\| + \|\boldsymbol{w}_t^{\mathsf{N}}\|\} \le \varepsilon\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2 + \Gamma(T)$$

for all $T \ge 0$ and for all $t \in [0,T]$. The remaining terms in the last line of (15.21) are easily shown to have the same bounds. Thus $\|\boldsymbol{\eta}_{st}^{\mathsf{N}}\| \le \Gamma(T) + \Gamma(T)\int_0^t \|\boldsymbol{\eta}_{st}^{\mathsf{N}}\|^2\,d\tau$, so that the Gronwall inequality implies that

$$\|\boldsymbol{\eta}_{st}^{\mathsf{N}}(\cdot,t)\| \le \Gamma(T) \quad \forall\, t \in [0,T]. \tag{15.24}$$

16. Convergence

We adopt the convention that a convergent subsequence of a given sequence is denoted the same way as the parent sequence. Let $\mathfrak{X} = H^1(0,1)$, $\mathfrak{Y} = C[0,1]$, $\mathfrak{Z} = L^2(0,1)$, so that $\mathfrak{X}$ is compactly embedded in $\mathfrak{Y}$ and $\mathfrak{Y}$ is embedded in $\mathfrak{Z}$. Since (15.24) gives a bound on $\boldsymbol{\eta}_{st}^{\mathsf{N}}(\cdot,t)$ in $\mathfrak{Z}$ and, equivalently, a bound on $\boldsymbol{\eta}_t^{\mathsf{N}}(\cdot,t)$ in $\mathfrak{X}$ that is uniform in t for $t \in [0,T]$, it follows that $\boldsymbol{\eta}_t^{\mathsf{N}}$ is bounded in $L^2(0,T,\mathfrak{Z})$. Thus Lemma 11.33 implies that

$$\boldsymbol{\eta}_t^{\mathsf{N}} \text{ lies in a compact subset of } C(0,T,\mathfrak{Y}) \equiv C([0,1]\times[0,T]), \tag{16.1}$$

so that

$$\begin{gathered}\boldsymbol{\eta}_t^{\mathsf{N}} \text{ has a subsequence converging uniformly}\\ \text{to a continuous limit } \boldsymbol{\eta}_t^{\infty} \text{ on } [0,1]\times[0,T].\end{gathered} \tag{16.2}$$

Since Hypothesis 7.7 implies that the initial values of $\boldsymbol{\eta}$ are continuous, it follows that

$$\begin{gathered}\boldsymbol{\eta}^{\mathsf{N}} \text{ itself lies in a compact subset of } C([0,1]\times[0,T]),\\ \text{and converges uniformly to a continuous limit } \boldsymbol{\eta}^{\infty},\end{gathered} \tag{16.3}$$

whose t-derivative is $\boldsymbol{\eta}_t^{\infty}$. Since the values $(\boldsymbol{\eta}^{\mathsf{N}}(s,t), \boldsymbol{\eta}_t^{\mathsf{N}}(s,t)$ lie in a compact subset of $\mathbb{R}^{12}$, the continuity of $\hat{\boldsymbol{\sigma}}$ implies that

$$\begin{gathered}\boldsymbol{\sigma}^{\mathsf{N}} \equiv \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}^{\mathsf{N}}, \boldsymbol{\eta}_t^{\mathsf{N}}, \cdot) \text{ converges uniformly}\\ \text{to a continuous limit } \boldsymbol{\sigma}^{\infty} \equiv \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}^{\infty}, \boldsymbol{\eta}_t^{\infty}, \cdot).\end{gathered} \tag{16.4}$$

We now regard the compatibility equation (13.10) as a linear ordinary differential equation for $s \mapsto \boldsymbol{w}^{\mathsf{N}}(s,t)$ subject to the boundary condition that $\boldsymbol{w}^{\mathsf{N}}(0,t)=\boldsymbol{o}$, which comes from (13.2). Then (13.9) (cf. (4.9)–(4.11)) implies that

$$|\boldsymbol{w}^{\mathsf{N}}(s,t)-\boldsymbol{w}^{\mathsf{N}}(\xi,t)|\le\left|\int_{\xi}^{s}|\mathbf{u}_t^{\mathsf{N}}|\,d\xi\right|, \tag{16.5}$$

so that (16.1) implies that $\boldsymbol{w}^{\mathsf{N}}$ is uniformly pointwise bounded and equicontinuous. It accordingly has a uniformly convergent subsequence, and (13.9) implies that $\boldsymbol{w}_s^{\mathsf{N}}$ has a uniformly convergent subsequence:

(16.6)

$\boldsymbol{w}^{\mathsf{N}}$ and $\boldsymbol{w}_s^{\mathsf{N}}$ converge uniformly to continuous limits $\boldsymbol{w}^{\infty}$ and $\boldsymbol{w}_s^{\infty}$,

with $\boldsymbol{w}_s^{\infty}$ the s-derivative of $\boldsymbol{w}^{\infty}$, and with $\boldsymbol{w}^{\infty}$ satisfying

$$\boldsymbol{w}_s^{\infty}=\boldsymbol{u}_t^{\infty}-\boldsymbol{w}^{\infty}\times\boldsymbol{u}^{\infty}. \tag{16.7}$$

We likewise deduce completely analogous results about the convergence of $\boldsymbol{p}_s^{\mathsf{N}}$ to $\boldsymbol{p}_s^{\infty}$:

(16.8) $\boldsymbol{p}^{\mathsf{N}}$ and $\boldsymbol{p}_s^{\mathsf{N}}$ converge uniformly to continuous limits $\boldsymbol{p}^{\infty}$ and $\boldsymbol{p}_s^{\infty}$,

with $\boldsymbol{p}_s^{\infty}$ the s-derivative of $\boldsymbol{p}^{\infty}$, and with $\boldsymbol{p}^{\infty}$ satisfying

$$\boldsymbol{p}_s^{\infty}=\boldsymbol{v}_t^{\infty}. \tag{16.9}$$

Since the $\boldsymbol{d}_k^{\mathsf{N}}$ satisfy the integral version of the initial-value problem (13.8), and since $\boldsymbol{w}^{\mathsf{N}}$ is uniformly convergent, the $\boldsymbol{d}_k^{\mathsf{N}}$ are uniformly bounded and equicontinuous, so that they have a uniformly convergent subsequence. The differential equation (13.8) then implies that the $\partial_t\boldsymbol{d}_k^{\mathsf{N}}$ also converge uniformly to the t derivative of $\boldsymbol{d}_k^{\infty}$:

(16.10)

$\boldsymbol{d}_k^{\mathsf{N}}$ and $\partial_t\boldsymbol{d}_k^{\mathsf{N}}$ converge uniformly to continuous limits $\boldsymbol{d}_k^{\infty}$ and $\partial_t\boldsymbol{d}_k^{\infty}$,

with $\boldsymbol{d}_k^{\infty}$ satisfying (4.3):

$$\partial_t\boldsymbol{d}_k^{\infty}=\boldsymbol{w}^{\infty}\times\boldsymbol{d}_k^{\infty}. \tag{16.11}$$

It follows from the uniform convergence given by (16.3), (16.4), (16.6), (16.8), (16.10) that all the cross products appearing in the Galerkin equations (13.8)–(13.10), (13.15), (13.16) have uniformly convergent subsequences:

$$\begin{gathered}\boldsymbol{w}^{\mathsf{N}}\times\boldsymbol{d}_k^{\mathsf{N}}\to\boldsymbol{w}^{\infty}\times\boldsymbol{d}_k^{\infty},\quad \boldsymbol{w}^{\mathsf{N}}\times\boldsymbol{u}^{\mathsf{N}}\to\boldsymbol{w}^{\infty}\times\boldsymbol{u}^{\infty},\\ \boldsymbol{v}^{\mathsf{N}}\times\boldsymbol{n}^{\mathsf{N}}\to\boldsymbol{v}^{\infty}\times\boldsymbol{n}^{\infty}\quad\text{uniformly.}\end{gathered} \tag{16.12}$$

From (15.2)–(15.4) we obtain that $\langle\boldsymbol{p}_t^{\mathsf{N}},\rho A\boldsymbol{p}_t^{\mathsf{N}}\rangle,\langle\boldsymbol{w}_t^{\mathsf{N}},\rho\boldsymbol{J}\cdot\boldsymbol{w}_t^{\mathsf{N}}\rangle\le\Gamma(T)$. Thus $\boldsymbol{w}_t^{\mathsf{N}}$ and $\boldsymbol{p}_t^{\mathsf{N}}$ are bounded sequences in the reflexive space $L^2((0,1)\times(0,T))$ and accordingly have weakly convergent subsequences:

$$\boldsymbol{w}_t^{\mathsf{N}}\rightharpoonup(\boldsymbol{w}_t)^{\infty},\quad \boldsymbol{p}_t^{\mathsf{N}}\rightharpoonup(\boldsymbol{p}_t)^{\infty}\quad\text{in}\quad L^2((0,1)\times(0,T)). \tag{16.13}$$

Since $\boldsymbol{w}^{\mathsf{N}}$ converges uniformly to $\boldsymbol{w}^{\infty}$, it follows that $(\boldsymbol{w}_t)^{\infty}$ is the distributional t-derivative of $\boldsymbol{w}^{\infty}$, i.e., $(\mathbf{w}_t)^{\infty}=(\mathbf{w}^{\infty})_t$ in the sense of distributions. We accordingly drop the parentheses in (16.13). The uniform convergence of $\boldsymbol{d}_k^{\mathsf{N}}$ and $\boldsymbol{w}^{\mathsf{N}}$ then implies the weak convergence

$$(\boldsymbol{J}^{\mathsf{N}}\cdot\boldsymbol{w}^{\mathsf{N}})_t\rightharpoonup(\boldsymbol{J}^{\infty}\cdot\boldsymbol{w}^{\infty})_t\quad\text{in}\quad L^2((0,1)\times(0,T)). \tag{16.14}$$

17. Weak Solutions

We have just shown that $\boldsymbol{d}_k^\infty$, $\boldsymbol{p}^\infty$, $\boldsymbol{w}^\infty$, $\boldsymbol{v}^\infty$, $\boldsymbol{u}^\infty$ satisfy (5.7a–c) in the classical sense. We now show that these functions also satisfy the momentum equations in a suitable way. The definitions (13.6) of the orthogonal projectors $\boldsymbol{Y}^{\mathsf{N}}$ and $\boldsymbol{X}^{\mathsf{N}}$ show that the Galerkin approximations (13.15) and (13.16) of the momentum equations hold not only for $\mathsf{a} = 1, \dots, \mathsf{N}$, but for any a. Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \dots$ be arbitrary functions of t in $C^1[0,T]$. We take the dot products of (13.15) and (13.16) each with $\boldsymbol{y}_{\mathsf{a}}$ and sum the resulting equations from 1 to L, to obtain

$$\text{(17.1a)}\quad \int_0^T \langle \rho A \boldsymbol{p}_t^{\mathsf{N}}, \boldsymbol{y}\rangle\, dt = -\int_0^T \langle \boldsymbol{n}^{\mathsf{N}} - \bar{\boldsymbol{n}}, \boldsymbol{y}_s\rangle\, dt,$$

(17.1b)

$$\int_0^T \langle (\rho \boldsymbol{J}^{\mathsf{N}} \cdot \mathbf{w}^{\mathsf{N}})_t, \boldsymbol{y}\rangle\, dt = -\int_0^T \langle \boldsymbol{m}^{\mathsf{N}} - \bar{\boldsymbol{m}}, \boldsymbol{y}_s\rangle\, dt + \int_0^T \langle \boldsymbol{v}^{\mathsf{N}} \times \boldsymbol{n}^{\mathsf{N}}, \boldsymbol{y}\rangle\, dt$$

for all $\boldsymbol{y}$ of the form $\sum_{\mathsf{a}=1}^{\mathsf{L}} \boldsymbol{y}_{\mathsf{a}} y_{\mathsf{a}}$. But such finite sums $\boldsymbol{y}$ are dense in $L^2(0,T,\mathfrak{W})$. Thus the approximations $\boldsymbol{d}_k^{\mathsf{N}}, \boldsymbol{p}^{\mathsf{N}}, \boldsymbol{w}^{\mathsf{N}}, \boldsymbol{v}^{\mathsf{N}}, \boldsymbol{u}^{\mathsf{N}}$ satisfy the weak momentum equations (17.1) for all time-dependent test functions $\boldsymbol{y} \in L^2(0,T,\mathfrak{W})$. (This system has a form more general than (7.10c,d). We could make this system even more general by putting time derivatives on the test functions, but our convergence results make doing this unnecessary.)

We use (16.2)–(16.13) to take the subsequential limits of these equations as $\mathsf{N} \to \infty$:

(17.2a)

$$\int_0^T \langle \rho A \boldsymbol{p}_t^\infty, \boldsymbol{y}\rangle\, dt = -\int_0^T \langle \boldsymbol{n}^\infty - \bar{\boldsymbol{n}}, \boldsymbol{y}_s\rangle\, dt,$$

(17.2b)

$$\int_0^T \langle (\rho \boldsymbol{J}^\infty \cdot \mathbf{w}^\infty)_t, \boldsymbol{y}\rangle\, dt = -\int_0^T \langle \boldsymbol{m}^\infty - \bar{\boldsymbol{m}}, \boldsymbol{y}_s\rangle\, dt + \int_0^T \langle \boldsymbol{v}^\infty \times \boldsymbol{n}^\infty, \boldsymbol{y}\rangle\, dt.$$

By using the arbitrariness of $\boldsymbol{y}$ in (17.2) (i.e., by using a version of the Fundamental Lemma of the Calculus of Variations), we readily find that $\boldsymbol{d}_k^\infty, \boldsymbol{p}^\infty, \boldsymbol{w}^\infty, \boldsymbol{v}^\infty, \boldsymbol{u}^\infty$ satisfy (7.10c,d). The method for showing that these functions satisfy the boundary and initial conditions is standard, is exactly the same as in [12, p. 177], and is accordingly omitted. We have thus proved the existence part of Theorem 8.1.

By imposing more restrictions on the data, we can ensure that the solution enjoys more regularity. The methods are standard (see, e.g., [24, 26, 39]) and are accordingly omitted. In Proposition 19.20, however, we shall show how to obtain further estimates from a slight variant of our constitutive hypotheses.

18. Continuous Dependence on the Data. Uniqueness

In this section we show that solutions depend continuously on the initial and boundary data, from which it follows that solutions are unique. Our approach is almost standard; the chief novelty is our treatment of the difficulties that come from our use of a moving basis.

We continue to treat the boundary conditions (7.1). We examine the difference of solutions of two problems for the same material in which the nonzero initial and boundary data are distinguished by superscripts 1 and 2. We denote the corresponding solutions by the same superscripts: $\boldsymbol{p}^1, \dots$ and $\boldsymbol{p}^2, \dots$, and set

(18.1)
$$\delta\boldsymbol{p} := \boldsymbol{p}^1 - \boldsymbol{p}^2, \qquad \delta\boldsymbol{n} := \boldsymbol{n}^1 - \boldsymbol{n}^2, \quad \boldsymbol{n}^1 := \boldsymbol{R}^1 \cdot \hat{\mathbf{n}}(\mathbf{v}^1, \mathbf{u}^1, \mathbf{v}^1_t, \mathbf{u}^1_t, s),$$
$$\delta\boldsymbol{p}^\circ := \boldsymbol{p}^1_\circ - \boldsymbol{p}^2_\circ, \quad \delta\bar{\boldsymbol{n}} := \bar{\boldsymbol{n}}^1 - \bar{\boldsymbol{n}}^2,$$

etc. (For notational clarity, the initial data bear subscripts $\circ$, instead of such superscripts.)

In the weak linear momentum equation (7.10c), we replace $\boldsymbol{p}, \boldsymbol{n}, \bar{\boldsymbol{n}}$ with $\delta\boldsymbol{p}, \delta\boldsymbol{n}, \delta\bar{\boldsymbol{n}}$ and replace $\boldsymbol{y}$ by $\delta\boldsymbol{p}$ to obtain

$$\text{(18.2)} \quad \frac{1}{2}\frac{d}{dt}\int_0^1 \rho A|\delta\boldsymbol{p}|^2\,ds + \int_0^1 \delta\boldsymbol{n}\cdot\delta\boldsymbol{p}_s\,ds = \delta\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{p}(1,\cdot) \equiv \delta\bar{\boldsymbol{n}}\cdot\int_0^1 \delta\boldsymbol{p}_s\,ds.$$

Likewise, from (7.10c) we get

(18.3)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \delta\boldsymbol{w}\cdot\rho\boldsymbol{J}^1\cdot\delta\boldsymbol{w}\,ds &+ \int_0^1 \delta\boldsymbol{m}\cdot\delta\boldsymbol{w}_s\,ds \\ &= \int_0^1 [\tfrac{1}{2}\delta\boldsymbol{w}\cdot(\rho\boldsymbol{J}^1)_t\cdot\delta\boldsymbol{w} - \delta\boldsymbol{w}\cdot\delta\rho\boldsymbol{J}_t\cdot\boldsymbol{w}^2 - \delta\boldsymbol{w}\cdot\delta\rho\boldsymbol{J}\cdot\boldsymbol{w}^2_t]\,ds \\ &\quad + \int_0^1 [\delta\boldsymbol{v}\times\boldsymbol{n}^1 + \boldsymbol{v}^2\times\delta\boldsymbol{n}]\cdot\delta\boldsymbol{w}\,ds + \delta\bar{\boldsymbol{m}}\cdot\int_0^1 \delta\boldsymbol{w}_s\,ds.\end{aligned}$$

Since the $\boldsymbol{d}_k$ are unit vectors, we immediately obtain the crude estimate

$$\text{(18.4)} \qquad |\delta\boldsymbol{d}_k(s,t)| \le 2,$$

but we need a sharper estimate: Let us set $\Delta := \max_k |\delta\boldsymbol{d}_k|$, $\Delta^\circ := \max_k |\delta\boldsymbol{d}^\circ_k|$. Note that $|\delta\boldsymbol{R}| \le \Delta$ (where $|\delta\boldsymbol{R}|$ is the operator norm). From (6.3) we get

$$\text{(18.5a)} \qquad \delta\boldsymbol{d}_k(\cdot,t) = \delta\boldsymbol{d}^\circ_k + \int_0^t [\delta\boldsymbol{w}\times\boldsymbol{d}^1_k + \boldsymbol{w}^2\times\delta\boldsymbol{d}_k]\,d\tau,$$

so that

$$\text{(18.5b)} \qquad \Delta(\cdot,t) \le \Delta^\circ + \int_0^t |\delta\boldsymbol{w}|\,d\tau + \Gamma(t)\int_0^t \Delta\,d\tau.$$

The Gronwall inequality then yields

$$\Delta(\cdot,t) \le \Gamma(t)\left[\Delta^\circ + \int_0^t |\delta \boldsymbol{w}|\,d\tau\right]. \tag{18.5c}$$

For λ,μ ranging over 1,2, set

$$\mathsf{n}^{\lambda\mu} := \hat{\mathsf{n}}(\boldsymbol{\eta}^\lambda, \boldsymbol{\eta}_t^\mu, \cdot), \tag{18.6a}$$

$$\mathsf{n}_{\boldsymbol{\eta}}^{\mu} := \int_0^1 \hat{\mathsf{n}}_{\boldsymbol{\eta}}(\alpha\boldsymbol{\eta}^1 + (1-\alpha)\boldsymbol{\eta}^2, \boldsymbol{\eta}_t^\mu, \cdot)\,d\alpha, \tag{18.6b}$$

$$\mathsf{n}_{\dot{\boldsymbol{\eta}}}^{\lambda} := \int_0^1 \hat{\mathsf{n}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta}^\lambda, \alpha\boldsymbol{\eta}_t^1 + (1-\alpha)\boldsymbol{\eta}_t^2, \cdot)\,d\alpha, \tag{18.6c}$$

etc. For regular solutions we thus obtain from the compatibility conditions(4.9), the Mean-Value Theorem, and the Monotonicity Condition (6.2) that

(18.6d)

$$\begin{aligned}
&\delta\boldsymbol{n}\cdot\delta\boldsymbol{p}_s + \delta\boldsymbol{m}\cdot\delta\boldsymbol{w}_s \\
&= [\boldsymbol{R}^1\cdot\mathsf{n}^{11} - \boldsymbol{R}^2\cdot\mathsf{n}^{22}]\cdot\delta\boldsymbol{p}_s + [\boldsymbol{R}^1\cdot\mathsf{m}^{11} - \boldsymbol{R}^2\cdot\mathsf{m}^{22}]\cdot\delta\boldsymbol{w}_s \\
&= [\boldsymbol{R}^1\cdot\mathsf{n}^{11} - \boldsymbol{R}^1\cdot\mathsf{n}^{12} + \boldsymbol{R}^1\cdot(\mathsf{n}^{12} - \mathsf{n}^{22}) + \delta\boldsymbol{R}\cdot\mathsf{n}^{22}] \\
&\quad \cdot[\boldsymbol{R}^1\cdot\delta\mathsf{v}_t + \delta\boldsymbol{R}\cdot\mathsf{v}_t^2 + \delta\boldsymbol{w}\times\boldsymbol{v}^1 + \boldsymbol{w}^2\times\delta\boldsymbol{v}] \\
&\quad + [\boldsymbol{R}^1\cdot\mathsf{m}^{11} - \boldsymbol{R}^1\cdot\mathsf{m}^{12} + \boldsymbol{R}^1\cdot(\mathsf{m}^{12} - \mathsf{m}^{22}) + \delta\boldsymbol{R}\cdot\mathsf{m}^{22}] \\
&\quad \cdot[\boldsymbol{R}^1\cdot\delta\mathsf{u}_t + \delta\boldsymbol{R}\cdot\mathsf{u}_t^2] \\
&\ge c|\delta\boldsymbol{\eta}_t|^2 \\
&\quad + [\boldsymbol{R}^1\cdot\mathsf{n}_{\dot{\boldsymbol{\eta}}}^1\cdot\delta\boldsymbol{\eta}_t + \boldsymbol{R}^1\cdot\mathsf{n}_{\boldsymbol{\eta}}^1\cdot\delta\boldsymbol{\eta} + \delta\boldsymbol{R}\cdot\mathsf{n}^{22}]\cdot[\delta\boldsymbol{R}\cdot\mathsf{v}_t^2 + \delta\boldsymbol{w}\times\boldsymbol{v}^1 + \boldsymbol{w}^2\times\delta\boldsymbol{v}] \\
&\quad + [\boldsymbol{R}^1\cdot\mathsf{n}_{\boldsymbol{\eta}}^1\cdot\delta\boldsymbol{\eta} + \delta\boldsymbol{R}\cdot\mathsf{n}^{22}]\cdot\boldsymbol{R}^1\cdot\delta\mathsf{v}_t \\
&\quad + [\boldsymbol{R}^1\cdot\mathsf{m}_{\dot{\boldsymbol{\eta}}}^1\cdot\delta\boldsymbol{\eta}_t + \boldsymbol{R}^1\cdot\mathsf{m}_{\boldsymbol{\eta}}^1\cdot\delta\boldsymbol{\eta} + \delta\boldsymbol{R}\cdot\mathsf{m}^{22}]\cdot\delta\boldsymbol{R}\cdot\mathsf{u}_t^2 \\
&\quad + [\boldsymbol{R}^1\cdot\mathsf{m}_{\boldsymbol{\eta}}^1\cdot\delta\boldsymbol{\eta} + \delta\boldsymbol{R}\cdot\mathsf{m}^{22}]\cdot\boldsymbol{R}^1\cdot\delta\mathsf{u}_t \\
&\ge (c-\varepsilon)|\delta\boldsymbol{\eta}_t|^2 - \Gamma[\Delta^2 + |\delta\boldsymbol{\eta}|^2 + |\delta\boldsymbol{w}|^2].
\end{aligned}$$

We add (18.2) and (18.3), integrate the sum from 0 to t, use (18.6), and use inequalities like

$$\begin{aligned}
\int_0^t \delta\bar{\boldsymbol{n}}\cdot\int_0^1 \delta\boldsymbol{v}_t\,ds\,d\tau &\le \left[\int_0^t |\delta\bar{\boldsymbol{n}}|^2\,d\tau\right]^{1/2}\left[\int_0^t\!\!\int_0^1 |\delta\boldsymbol{v}_t|^2\,ds\,d\tau\right]^{1/2} \\
&\le C\int_0^t |\delta\bar{\boldsymbol{n}}|^2\,d\tau + \varepsilon\int_0^t\int_0^1 |\delta\boldsymbol{v}_t|^2\,ds\,d\tau
\end{aligned} \tag{18.7}$$

to obtain

$$\begin{aligned}(18.8)\qquad &\int_0^1 (|\delta \boldsymbol{w}|^2 + |\delta \boldsymbol{p}|^2)\,ds + c\int_0^t\int_0^1 (|\delta \boldsymbol{w}_s|^2 + |\delta \boldsymbol{v}_t|^2)\,ds\,d\tau \\ &\le \int_0^1 (|\delta \boldsymbol{w}^\circ|^2 + |\delta \boldsymbol{p}^\circ|^2)\,ds + C\int_0^t (|\delta \bar{\boldsymbol{n}}|^2 + |\delta \bar{\boldsymbol{m}}|^2)\,d\tau \\ &\quad + C\int_0^t\int_0^1 (|\delta \boldsymbol{w}|^2 + |\delta \boldsymbol{u}|^2 + |\delta \boldsymbol{v}|^2 + \varDelta^2)\,ds\,d\tau.\end{aligned}$$

We want to manipulate this inequality into a form to which we can apply the Gronwall inequality. From (4.9) we obtain

$$(18.9)\qquad |\delta \boldsymbol{u}(\cdot,t)| \le |\delta \boldsymbol{u}^\circ| + \int_0^t (|\delta \boldsymbol{w}_s| + \varGamma|\delta \boldsymbol{w}| + \varGamma|\delta \boldsymbol{u}|)\,d\tau,$$

whence the Gronwall inequality implies that

$$(18.10)\qquad |\delta \boldsymbol{u}(\cdot,t)| \le \varGamma(t)|\delta \boldsymbol{u}^\circ| + \varGamma(t)\int_0^t (|\delta \boldsymbol{w}_s| + |\delta \boldsymbol{w}|)\,d\tau.$$

We substitute (18.5c) and (18.10) into (18.8), and use the (Poincaré-type) inequality $|\boldsymbol{v}(\cdot,t)| \le |\boldsymbol{v}^\circ| + \sqrt{t\int_0^t |\boldsymbol{v}_t(\cdot,\tau)|^2\,d\tau}$ to obtain

$$\begin{aligned}(18.11)\qquad &\int_0^1 |\delta \boldsymbol{w}|^2\,ds + c\int_0^t\int_0^1 (|\delta \boldsymbol{w}_s|^2 + |\delta \boldsymbol{v}_t|^2)\,ds\,d\tau \\ &\le \varGamma\int_0^1 (|\delta \boldsymbol{w}^\circ|^2 + |\delta \boldsymbol{p}^\circ|^2 + |\delta \boldsymbol{u}^\circ|^2 + |\delta \boldsymbol{v}_t^\circ|^2 + (\varDelta^\circ)^2)\,ds \\ &\quad + C\int_0^t (|\delta \bar{\boldsymbol{n}}|^2 + |\delta \bar{\boldsymbol{m}}|^2)\,d\tau \\ &\quad + \varGamma\int_0^t\int_0^1 \left[|\delta \boldsymbol{w}|^2 + \int_0^\tau |\delta \boldsymbol{w}_s|^2\,d\zeta + \int_0^\tau |\delta \boldsymbol{v}_t|^2\,d\zeta\right] ds\,d\tau.\end{aligned}$$

The Gronwall inequality implies that this inequality holds with the last line omitted. It then follows from (18.8) that

$$\begin{aligned}(18.12)\qquad &\int_0^1 (|\delta \boldsymbol{w}|^2 + |\delta \boldsymbol{p}|^2)\,ds + c\int_0^t\int_0^1 (|\delta \boldsymbol{w}_s|^2 + |\delta \boldsymbol{v}_t|^2)\,ds\,d\tau \\ &\le \varGamma\int_0^1 (|\delta \boldsymbol{w}^\circ|^2 + |\delta \boldsymbol{r}_t^\circ|^2 + |\delta \boldsymbol{u}^\circ|^2 + |\delta \boldsymbol{v}_t^\circ|^2 + (\varDelta^\circ)^2)\,ds + C\int_0^t (|\delta \bar{\boldsymbol{n}}|^2 + |\delta \bar{\boldsymbol{m}}|^2)\,d\tau.\end{aligned}$$

This inequality gives continuous dependence on the initial and boundary conditions with respect to to the norms that intervene here. Obviously the vanishing of the left-hand side of (18.12) when the right-hand side vanishes ensures uniqueness.

V. Commentary

19. Alternative Constitutive Hypotheses and their Consequences

In this section we give variants of our constitutive assumptions, several of which are less physically restrictive (and also more complicated) than those used above, and we describe their consequences.

Preclusion of total compression. The requirement of Hypothesis 6.10 that (6.10) hold for all positive values of $\dot{\zeta}_3$ unduly restricts the growth of the constitutive functions for large $\dot{\zeta}_3$. We now formulate a refinement of this hypothesis that does not suffer from this disadvantage. We set

(19.1) $$\boldsymbol{\chi} := (\zeta_1, \zeta_2, \zeta_4, \zeta_5, \zeta_6) \equiv (\eta_1, \eta_2, \eta_4, \eta_5, \eta_6).$$

19.2. Hypothesis. (i) *There is a positive number*

$$\varepsilon < \inf\{\zeta_3 : \check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x) = 0,\ \dot{\zeta}_3 = 1,\ \boldsymbol{\zeta} \in \mathbb{R}^6,\ \dot{\boldsymbol{\chi}} \in \mathbb{R}^5,\ x \in [0,1]\},$$

there is a number

$$n_* > \sup\{\check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, x) : \zeta_3 = \varepsilon,\ \dot{\zeta}_3 = 1,\ \boldsymbol{\chi} \in \mathbb{R}^5,\ \dot{\boldsymbol{\chi}} \in \mathbb{R}^5,\ x \in [0,1]\},$$

there is a positive number A_+, and there is a continuously differentiable, strictly decreasing, convex, positive-valued function ψ_+ on $(0,\varepsilon)$ with

(19.2a) $$\psi_+(\zeta_3) \to \infty \quad \textit{as} \quad \zeta_3 \to 0,$$

such that

(19.2b) $$\check{n}_3^{\mathrm{D}}(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, s) \le -\psi_+'(\zeta_3)\dot{\zeta}_3 - A_+$$

for

(19.2c) $$\zeta_3 \le \varepsilon, \quad 1 \le \dot{\zeta}_3, \quad \check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, s) \le n_*.$$

(ii) *Inequality* (6.10) *holds when $\zeta_3 \le \varepsilon$ and $\dot{\zeta}_3 \le 1$.*
(iii) *There is a number $n_{**} < n_*$ with the property that $\dot{\zeta}_3 \ge 1$ if $\check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, s) = n_{**}$ and $\zeta_3 \le \varepsilon$, there is a number $A_- \ge A_+$, and there is a continuously differentiable, strictly decreasing, convex, positive-valued function ψ_- on $(0,\varepsilon)$ with*

(19.2d) $$\psi_-(\zeta_3) \to \infty \quad \textit{as} \quad \zeta_3 \to 0$$

such that

(19.2e) $$\check{n}_3^{\mathrm{D}}(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, s) \ge -\psi_-'(\zeta_3)\dot{\zeta}_3 - A_-$$

for

(19.2f) $$|\boldsymbol{\eta}| \le C, \quad \zeta_3 \le \varepsilon, \quad 1 \le \dot{\zeta}_3, \quad n_{**} \le \check{n}_3(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, s) \le n_*.$$

(iv) *The numbers* $A_{\pm}$, n_*, n_{**} *satisfy*

$$0 < A_+ + n_{**} < A_- + n_*. \tag{19.2g}$$

This hypothesis is a complicated analog of Hypothesis 3.9 of [12]. See Figure 3.8 of that paper for the underlying geometry used in this hypothesis. Since the preclusion of total compression requires the control merely of a scalar variable, the exploitation of Hypothesis 19.2 to get the requisite bound follows directly from the quite technical treatment (using a phase-plane analysis) in Sec. 7 of [12]. Hypothesis 19.2 posits a uniformity with respect to the variables $\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}$. Presumably this uniformity could be weakened in the context of the balancing of constitutive hypotheses discussed below.

Additional constitutive restrictions to handle boundary conditions on the configuration. If $\boldsymbol{p}(0,t)$ and $\boldsymbol{p}(1,t)$ are prescribed, but not equal, or if $\boldsymbol{w}(0,t)$ and $\boldsymbol{w}(1,t)$ are prescribed, but not equal, then the energy estimate (9.14) need not hold, because the terms on the right-hand side of (9.3) containing $\boldsymbol{n}$ or $\boldsymbol{m}$ can no longer be conveniently estimated: These terms involve $\partial\varphi/\partial\boldsymbol{\eta}$, which behaves worse than φ itself near total compression, and so cannot be dominated by a term involving Φ. A method for handling this difficulty for scalar problems, employing strengthened constitutive hypotheses and an intricate analysis involving the solution of a simultaneous system of integral inequalities, is given by [12]. Again, since the preclusion of total compression just requires the control of a scalar variable, the methods of Sec. 6 of [12] may very well carry over to our problem, but we do not pursue this question.

A priori bound on the strain. In Sec. 11 we obtained an a priori bound on the strain as a byproduct of Hypothesis 6.11. We now show that the next hypothesis enables to get such a bound more directly. The availability of this bound enables us to weaken several of the other constitutive hypotheses.

19.3. Hypothesis. *For each* $c > 0$, *there is a continuously differentiable function* $\kappa \mapsto \beta(\kappa)$ *with* $\beta(\kappa) \to \infty$ *as* $\kappa \to \infty$ *and there are numbers* $D \geq 0$ *and* $\varepsilon > 0$ *such that*

$$\frac{\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s)\cdot\boldsymbol{\eta}}{|\boldsymbol{\eta}|} \geq \beta'(|\boldsymbol{\eta}|)\frac{\boldsymbol{\eta}\cdot\dot{\boldsymbol{\eta}}}{|\boldsymbol{\eta}|} - D\beta(|\boldsymbol{\eta}|) + \varepsilon|\boldsymbol{\eta}|\,|\dot{\boldsymbol{\eta}}| \tag{19.3}$$

for all $\dot{\boldsymbol{\eta}}$ *and for all* $\boldsymbol{\eta}$ *and* s *such that* $\hat{\zeta}_3(\boldsymbol{\eta}, s) \geq c$.

Hypothesis 19.3 is suggested by the form of constitutive equations for transversely isotropic rods [7]. If $\hat{m}_3$ changes sign while $\hat{m}_\alpha$, $\hat{n}_\alpha$, $\hat{n}_3$ remain unchanged when u_3 and $\dot{u}_3$ change sign, then these equations have the form

$$\begin{aligned}
\hat{m}_\alpha(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s) &= m(\mathcal{I}, s)u_\alpha + m^\times(\mathcal{I}, s)v_\alpha + \mu(\mathcal{I}, s)\dot{u}_\alpha + \mu^\times(\mathcal{I}, s)\dot{v}_\alpha,\\
\hat{n}_\alpha(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s) &= n^\times(\mathcal{I}, s)u_\alpha + n(\mathcal{I}, s)v_\alpha + \nu^\times(\mathcal{I}, s)\dot{u}_\alpha + \nu(\mathcal{I}, s)\dot{v}_\alpha,\\
\hat{m}_3(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s) &= \xi(\mathcal{I}, s)u_3 + \eta(\mathcal{I}, s)\dot{u}_3,\\
\hat{n}_3(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, s) &= \zeta(\mathcal{I}, s),
\end{aligned} \tag{19.4}$$

where $m, m^\times, \mu, \mu^\times, n, n^\times, \nu, \nu^\times, \xi, \eta, \zeta$ are scalar-valued constitutive functions, and $\mathcal{I}$ is the collection of invariants

(19.5)
$$u_\alpha u_\alpha, u_\alpha v_\alpha, v_\alpha v_\alpha, u_\alpha \dot{u}_\alpha, u_\alpha \dot{v}_\alpha, v_\alpha \dot{u}_\alpha, v_\alpha \dot{v}_\alpha, \dot{u}_\alpha \dot{u}_\alpha, \dot{u}_\alpha \dot{v}_\alpha, \dot{v}_\alpha \dot{v}_\alpha, {u_3}^2, u_3 \dot{u}_3, {\dot{u}_3}^2, v_3, \dot{v}_3.$$

If we substitute (19.4) into the left-hand side of (19.3), we can discern some of the physical content of these hypotheses. The term $\beta'(|\boldsymbol{\eta}|)$ gives a strain-dependent lower bound on the viscosity. In contrast to the corresponding term in (6.10), the presence of this bound is not very restrictive: Indeed, $\beta'(|\boldsymbol{\eta}|)$ can approach 0 as $|\boldsymbol{\eta}| \to \infty$ so that the viscosity could decrease in large extensions. E.g., we could have $\beta(|\boldsymbol{\eta}|) = \log |\boldsymbol{\eta}|$. On the other hand, the presence of the positive term $\varepsilon|\boldsymbol{\eta}|\,|\dot{\boldsymbol{\eta}}|$ on the right-hand side of (19.3) suggests that the "elastic moduli" m, $m^\times$, etc., of (19.4) become larger with the strain rate.

Now we show how to use this hypothesis to get the desired bound on $\boldsymbol{\eta}$. We continue to assume that the hypotheses of Theorem 10.6 hold. We now use Hypothesis 19.3 get a pointwise bound on $|\boldsymbol{\eta}|$. For any $\xi \in [0,1]$ we obtain from the equations of motion (5.1) and (5.2) (with $\boldsymbol{f} = \boldsymbol{o} = \boldsymbol{l}$) the following identities whose sum is the left-hand side of (19.3):

(19.6)
$$\begin{aligned} \frac{\boldsymbol{n}(\xi,t)\cdot\boldsymbol{v}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|} = \frac{\bar{\boldsymbol{n}}(t)\cdot\boldsymbol{v}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|} &- \partial_t\left[\frac{\boldsymbol{v}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1 (\rho A)(s)\boldsymbol{p}(s,t)\,ds\right] \\ &+ \partial_t\left[\frac{\boldsymbol{v}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\right]\cdot\int_\xi^1 (\rho A)(s)\boldsymbol{p}(s,t)\,ds, \end{aligned}$$

(19.7)
$$\begin{aligned} \frac{\boldsymbol{m}(\xi,t)\cdot\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|} = \frac{\bar{\boldsymbol{m}}(t)\cdot\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|} &+ \frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1 \boldsymbol{v}(s,t)\times\boldsymbol{n}(s,t)\,ds \\ &- \partial_t\left\{\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1 \partial_t[(\rho\boldsymbol{J})(s,t)\cdot\boldsymbol{w}(s,t)]\,ds\right\} \\ &+ \partial_t\left[\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\right]\cdot\int_\xi^1 (\rho\boldsymbol{J})(s,t)\cdot\boldsymbol{w}(s,t)\,ds. \end{aligned}$$

Since $|\boldsymbol{\eta}(\xi,t)| \geq \gamma(t)$ by (10.6), the identities

$$\begin{aligned} \partial_t\left(\frac{\boldsymbol{v}}{|\boldsymbol{\eta}|}\right) &= \frac{\boldsymbol{R}\cdot\mathbf{v}_t}{|\boldsymbol{\eta}|} + \frac{\boldsymbol{w}\times\boldsymbol{v}}{|\boldsymbol{\eta}|} - \frac{\boldsymbol{v}(\boldsymbol{\eta}\cdot\boldsymbol{\eta}_t)}{|\boldsymbol{\eta}|^3}, \\ \partial_t\left(\frac{\boldsymbol{u}}{|\boldsymbol{\eta}|}\right) &= \frac{\boldsymbol{R}\cdot\mathbf{u}_t}{|\boldsymbol{\eta}|} + \frac{\boldsymbol{w}\times\boldsymbol{u}}{|\boldsymbol{\eta}|} - \frac{\boldsymbol{u}(\boldsymbol{\eta}\cdot\boldsymbol{\eta}_t)}{|\boldsymbol{\eta}|^3} \end{aligned} \tag{19.8}$$

immediately imply that

$$\left|\partial_t\left(\frac{\boldsymbol{v}}{|\boldsymbol{\eta}|}\right)\right|,\ \left|\partial_t\left(\frac{\boldsymbol{u}}{|\boldsymbol{\eta}|}\right)\right| \leq |\boldsymbol{w}| + \Gamma(t)|\boldsymbol{\eta}_t|. \tag{19.9}$$

Let us examine the second term on the right-hand side of (19.7). Again using (5.1) we obtain

(19.10)
$$\begin{aligned}
&\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1 \boldsymbol{r}_s(s,t)\times\boldsymbol{n}(s,t)\,ds\\
&\quad=\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1 \boldsymbol{r}_s(s,t)\times\left[\bar{\boldsymbol{n}}(t)-\int_s^1(\rho A)(\xi)\boldsymbol{r}_{tt}(\xi,t)\,d\xi\right]ds\\
&\quad=\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot[\boldsymbol{r}(1,t)-\boldsymbol{r}(\xi,t)]\times\bar{\boldsymbol{n}}(t)\\
&\qquad+\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1[\boldsymbol{r}(s,t)-\boldsymbol{r}(\xi,t)]\times(\rho A)(s)\boldsymbol{r}_{tt}(s,t)\,ds.
\end{aligned}$$

We write the last term of (19.10) as

$$\begin{aligned}
&\partial_t\left\{\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1[\boldsymbol{r}(s,t)-\boldsymbol{r}(\xi,t)]\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\right\}\\
&-\partial_t\left[\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\right]\cdot\int_\xi^1[\boldsymbol{r}(s,t)-\boldsymbol{r}(\xi,t)]\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\\
&-\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1[\boldsymbol{r}_t(s,t)-\boldsymbol{r}_t(\xi,t)]\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds.
\end{aligned}\tag{19.11}$$

When we insert the constitutive equations (5.5) into the sum of the left-hand sides of (19.6) and (19.7), we obtain the left-hand side of the constitutive restriction (19.3). We substitute (19.6) and (19.7) into (19.3) and use (10.3), (19.9)–(19.11) to find that there is a function Γ_1 such that

(19.12)
$$\begin{aligned}
&\beta'(|\boldsymbol{\eta}(\xi,t)|)\frac{\boldsymbol{\eta}(\xi,t)\cdot\boldsymbol{\eta}_t(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}-D\beta(|\boldsymbol{\eta}(\xi,t)|)+\varepsilon|\boldsymbol{\eta}(\xi,t)|\,|\boldsymbol{\eta}_t(\xi,t)|\\
&\quad\le\frac{\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta},\boldsymbol{\eta}_t,s)\cdot\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\\
&\quad\le\Gamma(t)+\tfrac12\Gamma_1(t)(|\boldsymbol{w}|+|\mathbf{u}_t|+|\mathbf{v}_t|)\\
&\qquad-\partial_t\left\{\frac{\boldsymbol{r}_s(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\right.\\
&\qquad\qquad+\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1\partial_t\big[(\rho\boldsymbol{J})(s,t)\cdot\boldsymbol{w}(s,t)\big]\,ds\\
&\qquad\qquad\left.+\frac{\boldsymbol{u}(\xi,t)}{|\boldsymbol{\eta}(\xi,t)|}\cdot\int_\xi^1[\boldsymbol{r}(s,t)-\boldsymbol{r}(\xi,t)]\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\right\}.
\end{aligned}$$

Since (9.10) implies that $|\boldsymbol{w}|\le CM(t)$, it follows from (9.5b) and (9.14) that $\int_0^t\Gamma_1(\tau)|\boldsymbol{w}(s,\tau)|\,d\tau\le\Gamma(t)$. Since the term in braces in (19.12) is

bounded, we integrate (19.10) with respect to t from τ_1 to t and use the last observation to get
(19.13)

$$\beta(|\boldsymbol{\eta}(\xi,t)|) - \beta(|\boldsymbol{\eta}(\xi,\tau_1)|) \le D\int_{\tau_1}^{t} \beta(|\boldsymbol{\eta}(\xi,\tau)|)\,d\tau + \int_{\tau_1}^{t} \Gamma_1(\tau)|\boldsymbol{\eta}_t(\xi,t)|\,d\tau.$$

Let $\Gamma_2(t)$ be a continuous function that is bounded below by $\Gamma_1(t)/\varepsilon + \max_s |\boldsymbol{\eta}(s,0)|$. To show that $|\boldsymbol{\eta}(s,t)|$ is bounded for all (s,t) it suffices to show that it is bounded only for all (s,t) for which $|\boldsymbol{\eta}(s,t)| > \Gamma_2(t)$. Thus, suppose that there is some ξ in $(0,1]$ and some $\tau_2 > 0$ such that $|\boldsymbol{\eta}(\xi,\tau_2)| > \Gamma_2(\tau_2)$. Since we are presuming that $|\boldsymbol{\eta}|$ is continuous, there is a last time τ_1 at which $|\boldsymbol{\eta}(\xi,\tau_1)| = \Gamma_2(\tau_2)$. For $t \in [\tau_1,\tau_2]$ the last term of (19.13) is not positive, so it can be dropped. The Gronwall inequality can then be applied to the resulting form of (19.13) to show that $\sup_\xi |\boldsymbol{\eta}(\xi,t)| \le \Gamma(t)$ for $t \in [\tau_1,\tau_2]$. Thus we obtain

19.14. Theorem. *Let Hypothesis 19.3 and the hypotheses of Theorem 10.6 hold. Let the initial values of $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_t$ lie in $L^2(0,1)$. Then*

$$|\boldsymbol{\eta}(s,t)| \le \Gamma(t) \quad \forall\,(s,t). \tag{19.14}$$

Balancing of constitutive assumptions. The condition that the β in Hypothesis 19.3 depends upon a lower bound c for ζ_3 means in practice that this hypothesis, just like Hypothesis 6.11, should not be invoked until after a lower bound (10.6) is obtained for ζ_3 by the use of Hypothesis 6.10. Were we to strengthen Hypothesis 19.3 by requiring that it hold with ε, β, D independent of c, then we could construct our system of estimates by first using this strengthened 19.3 to bound $\boldsymbol{\eta}$. In this case, we could use a weaker version of Hypothesis 6.10 to ensure that a total compression cannot occur:

19.15. Hypothesis. *For each $C > 0$ there are numbers $\varepsilon \in (0,1)$ and $A \ge 0$, and there is a continuously differentiable function ψ on $(0,\varepsilon)$ with $\psi(\zeta_3) \to \infty$ as $\zeta_3 \to 0$ such that (6.10) holds if $|\boldsymbol{\eta}| \le C$ and $0 < \zeta_3 \le \varepsilon$.*

Indeed, with the strengthened 19.3 we could weaken other hypotheses. E.g., we could weaken Hypothesis 6.11 by requiring that it hold under the further restriction that $|\boldsymbol{\eta}| \le C$.

Control of $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}$. An additional estimate. A variant of Hypothesis 6.11 having the same effect of controlling (5.8) is

19.16. Hypothesis. *For each c there is a number C such that*

(19.16)

$$\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\frac{\partial\hat{\boldsymbol{\sigma}}}{\partial\boldsymbol{\eta}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\right| \le C\left[1 + \frac{\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\boldsymbol{\xi}}}{|\boldsymbol{\xi}|}\;\frac{\sqrt{\sigma^{\mathrm{D}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\dot{\boldsymbol{\eta}}}}{|\dot{\boldsymbol{\eta}}|}\;\right]$$

for all $\boldsymbol{\xi} \ne \mathbf{0}$ when $\zeta_3 \ge c$.

This hypothesis, just as Hypothesis 6.11, could be weakened by requiring it to hold only under the further restriction that $|\boldsymbol{\eta}|$ be bounded, when such a bound follows from an hypothesis like Hypothesis 19.3.

An intimately related condition is that

19.17. Hypothesis. *For each c there is a number C such that*

$$|\boldsymbol{\xi}\cdot\hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)| \le C[1+\boldsymbol{\xi}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\boldsymbol{\xi}] \quad \forall\boldsymbol{\xi} \quad \text{with} \quad |\boldsymbol{\xi}|=1 \tag{19.17}$$

when $\zeta_3 \ge c$.

Let us supplement these conditions with the requirements that the condition number for $\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}$ be bounded and that the dependence of $\hat{\boldsymbol{\sigma}}$ on s not be pathological:

19.18. Hypothesis. *Let*

$$q^{\pm}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s) := \begin{Bmatrix}\max\\ \min\end{Bmatrix}\{\boldsymbol{\xi}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\boldsymbol{\xi} : |\boldsymbol{\xi}|=1\}. \tag{19.18a}$$

For each $c, C > 0$ there is a number G such that

$$q^+ \le Gq^-. \tag{19.18b}$$

if $\zeta_3 \ge c$ and $|\boldsymbol{\eta}| < C$.

19.19. Hypothesis. *For each $c > 0$, there is a number C such that*

$$|\hat{\boldsymbol{\sigma}}_x| \le C(1+|\hat{\boldsymbol{\sigma}}|). \tag{19.19}$$

when $\zeta_3 \ge c$.

These conditions support

19.20. Proposition. *Let Hypotheses 19.18 and 19.19 and the hypotheses of Proposition 11.3 hold. Then*

$$\|\boldsymbol{\eta}_{st}\| \le \Gamma. \tag{19.20}$$

Proof. From (5.1), (5.2), (5.5) we obtain

$$\begin{aligned}\boldsymbol{\eta}_{st}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\boldsymbol{\eta}_{st} = {} & \rho A\boldsymbol{p}_t\cdot\boldsymbol{R}\cdot\mathbf{v}_{st} + \partial_t(\rho\boldsymbol{J}\cdot\boldsymbol{w})\cdot\boldsymbol{R}\cdot\mathbf{u}_{st}\\ & - \boldsymbol{\eta}_{st}\cdot\hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}_s - \boldsymbol{\eta}_{st}\cdot\hat{\boldsymbol{\sigma}}_x\\ & - \mathbf{u}_{st}\cdot(\mathbf{u}\times\hat{\mathbf{m}}+\mathbf{v}\times\hat{\mathbf{n}}) - \mathbf{v}_{st}\cdot(\mathbf{u}\times\hat{\mathbf{n}}).\end{aligned} \tag{19.21}$$

We deduce from (19.18a), (6.3), and (19.21) that

$$\begin{aligned}|\boldsymbol{\eta}_{st}|^2 &\le \frac{1}{q^-}\boldsymbol{\eta}_{st}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}},s)\cdot\boldsymbol{\eta}_{st}\\ &\le \frac{1}{q^-}|\boldsymbol{\eta}_{st}\cdot\hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}_s| + \frac{1}{c}|\rho A\boldsymbol{p}_t\cdot\boldsymbol{R}\cdot\mathbf{v}_{st} + \partial_t(\rho\boldsymbol{J}\cdot\boldsymbol{w})\cdot\boldsymbol{R}\cdot\mathbf{u}_{st}|\\ &\quad + \frac{1}{c}\left|\boldsymbol{\eta}_{st}\cdot\hat{\boldsymbol{\sigma}}_x + \mathbf{u}_{st}\cdot(\mathbf{u}\times\hat{\mathbf{m}}+\mathbf{v}\times\hat{\mathbf{n}}) + \mathbf{v}_{st}\cdot(\mathbf{u}\times\hat{\mathbf{n}})\right|.\end{aligned} \tag{19.22}$$

Inequalities (6.11), (19.18b) imply that

(19.23)
$$\frac{1}{q^-}|\boldsymbol{\eta}_{st}\cdot\hat{\boldsymbol{\sigma}}_{\boldsymbol{\eta}}\cdot\boldsymbol{\eta}_s| \le \frac{G}{q^-}\left\{|\boldsymbol{\eta}_{st}| + \frac{\boldsymbol{\eta}_{st}\cdot\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}\cdot\boldsymbol{\eta}_{st}}{|\boldsymbol{\eta}_{st}|}\right\}|\boldsymbol{\eta}_s| \le G\left(1+\frac{q^+}{q^-}\right)|\boldsymbol{\eta}_{st}|\,|\boldsymbol{\eta}_s|$$
$$\le \varepsilon|\boldsymbol{\eta}_{st}|^2 + C|\boldsymbol{\eta}_s|^2 \le \varepsilon|\boldsymbol{\eta}_{st}|^2 + C + C\int_0^t |\boldsymbol{\eta}_{st}|^2\,d\tau.$$

Since $\boldsymbol{m}(1,\cdot)$ and $\boldsymbol{n}(1,\cdot)$ are prescribed, estimates (10.6), (19.14), (11.11), (11.15a), (19.23) and Hypothesis 19.19 enable us to deduce from (19.22) that

(19.24)
$$|\boldsymbol{\eta}_{st}|^2 \le C + C\rho A|\boldsymbol{p}_t|^2 + C\boldsymbol{w}_t\cdot\rho\boldsymbol{J}\cdot\boldsymbol{w}_t$$
$$+ \varGamma\langle\boldsymbol{p}_t, \rho A\boldsymbol{p}_t\rangle + \varGamma\langle\boldsymbol{w}_t, \rho\boldsymbol{J}\cdot\boldsymbol{w}_t\rangle + C\int_0^t |\boldsymbol{\eta}_{st}|^2\,d\tau.$$

Integrating (19.24) with respect to s over $(0,1)$, using (11.4), and using the Gronwall inequality, we obtain (19.20).

20. The Treatment of Other Boundary Conditions

If we replace the specific boundary conditions (7.1) with various complementary combinations of components of $\boldsymbol{r}_t$ and $\boldsymbol{n}$ and of $\boldsymbol{w}$ and $\boldsymbol{n}$ at each end of the rod (see [7,8,10]), we would require

20.1. Hypothesis. *The prescribed components of* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *at* $s=0$ *and* $s=1$ *lie in the Sobolev space* $H^2_{\mathrm{loc}}[0,\infty)$. *The prescribed components of* $\boldsymbol{n}$ *and* $\boldsymbol{m}$ *at* $s=0$ *and* $s=1$ *lie in* $H^1_{\mathrm{loc}}[0,\infty)$.

(Weaker conditions suffice for the energy estimates like those we now obtain.)

We now obtain energy estimates like those of Sec. 9 for some slightly trickier cases. First suppose that $\boldsymbol{r}(0,\cdot)$ is prescribed in $W^{1,\infty}_{\mathrm{loc}}[0,\infty)$ and $\boldsymbol{n}(1,\cdot) \equiv \bar{\boldsymbol{n}}$ is prescribed in $L^\infty_{\mathrm{loc}}[0,\infty)$, say. Then (9.9) again holds. To estimate the integral on the right-hand side of (9.3) for $s=0$ we use (5.1) to obtain

(20.2)
$$\int_0^t \boldsymbol{n}(0,\tau)\cdot\boldsymbol{p}(0,\tau)\,d\tau$$
$$= \int_0^t \boldsymbol{p}(0,\tau)\cdot\Big[\boldsymbol{n}(1,\tau) - \int_0^1 (\rho A)(s)\boldsymbol{p}_t(s,\tau)\,ds\Big]\,d\tau$$
$$= \int_0^t \boldsymbol{p}(0,\tau)\cdot\boldsymbol{n}(1,\tau)\,d\tau - \int_0^1 \boldsymbol{p}(0,\tau)\cdot(\rho A)(s)\boldsymbol{p}(s,\tau)\,ds\bigg|_{\tau=0}^{\tau=t}$$
$$+ \int_0^t \boldsymbol{p}_t(0,\tau)\cdot\int_0^1 (\rho A)(s)\boldsymbol{p}(s,\tau)\,ds\,d\tau.$$

By the techniques we have been using, we easily find that the absolute value of (20.2) is dominated by $\varGamma(t) + \varepsilon CK(t) + \int_0^t K(\tau)\,d\tau$. Thus the integrals of

(9.3) are dominated by the tractable bound $\Gamma(t)+\varepsilon CK(t)+\Gamma(t)\int_0^t[K(\tau)+\Omega(\tau)]\,d\tau$.

Now suppose that both ends are free and subjected to prescribed forces:

$$\boldsymbol{n}(0,t) \quad\text{and}\quad \boldsymbol{n}(1,t) \quad \text{are prescribed in} \quad L^\infty_{\text{loc}}[0,\infty). \tag{20.3}$$

The mass center of the rod at time t is

$$\boldsymbol{c}(t) = \frac{\int_0^1 (\rho A)(s)\boldsymbol{r}(s,t)\,ds}{\int_0^1 (\rho A)(s)\,ds}. \tag{20.4}$$

We estimate it and its derivatives by integrating (5.1) with respect to s over $[0,1]$ to get

$$\left[\int_0^1 (\rho A)(s)\,ds\right]\boldsymbol{c}_{tt}(t) = \boldsymbol{n}(1,t) - \boldsymbol{n}(0,t), \tag{20.5}$$

whence

$$|\boldsymbol{c}_t(t)|,\ |\boldsymbol{c}(t)| \le \Gamma(t). \tag{20.6}$$

Let $\boldsymbol{e}$ be an arbitrary constant unit vector. Then for fixed t there is an $s^*(\boldsymbol{e},t)\in(0,1)$ such that $\boldsymbol{e}\cdot[\boldsymbol{r}_t(s^*,t)-\boldsymbol{c}_t(t)]=0$, for if not, then $\boldsymbol{e}\cdot\boldsymbol{r}_t(s,t) > \boldsymbol{e}\cdot\boldsymbol{c}_t(t)$, say, for all s, and the integration of this inequality with respect to s over $(0,1)$ would contradict (20.4). We use this property of s^* to get

$$\boldsymbol{e}\cdot[\boldsymbol{r}_t(s,t)-\boldsymbol{c}_t(t)] = \boldsymbol{e}\cdot\int_{s^*}^s \boldsymbol{r}_{st}(\xi,t)\,d\xi = \boldsymbol{e}\cdot\int_{s^*}^s [\boldsymbol{R}\cdot\mathbf{v}_t + \boldsymbol{w}\times\boldsymbol{v}]\,d\xi, \tag{20.7}$$

whence we obtain (9.9a) with $\boldsymbol{r}_t(0,t)$ replaced with $\boldsymbol{c}_t$. From (20.3) and (9.9a) we obtain the requisite replacement for (9.11):

$$\int_0^t \boldsymbol{n}(s,\tau)\cdot\boldsymbol{r}_t(s,\tau)\,d\tau\Big|_{s=0}^{s=1} \le \Gamma(t) + \Gamma(t)\sqrt{\Omega(t)} + \Gamma(t)\int_0^t [K(\tau)+\Omega(\tau)]\,d\tau. \tag{20.8}$$

We now show how to get such estimates for the work done by the end couples when

$$\boldsymbol{m}(0,t) \quad\text{and}\quad \boldsymbol{m}(1,t) \quad \text{are prescribed in } L^\infty_{\text{loc}}[0,\infty). \tag{20.9}$$

In analogy with (20.4) we set

$$\boldsymbol{a}(t) := \int_0^1 (\rho\boldsymbol{J})(s,t)\cdot\boldsymbol{w}(s,t)\,ds, \tag{20.10}$$

integrate (5.2) with respect to s and t by parts, and use (5.1) to obtain

$$|\boldsymbol{a}(t)| \le \Gamma(t) + \int_0^t \boldsymbol{r}(s,\tau)\times\boldsymbol{n}(s,\tau)\,d\tau\Big|_{s=0}^{s=1} + \left|\int_0^1 \boldsymbol{r}(s,t)\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\right|. \tag{20.11}$$

By (9.9b) we have

(20.12)

$$\int_0^t \boldsymbol{r}(s,\tau)\times\boldsymbol{n}(s,\tau)\,d\tau\bigg|_{s=0}^{s=1} \le \Gamma(t)+\Gamma(t)\sqrt{\Omega(t)}+\Gamma(t)\int_0^t[K(\tau)+\Omega(\tau)]\,d\tau.$$

We estimate the rightmost term in (20.11) by

$$\begin{aligned}(20.13)\quad \left|\int_0^1 \boldsymbol{r}(s,t)\times(\rho A)(s)\boldsymbol{r}_t(s,t)\,ds\right| &\le \frac{1}{2}C\int_0^1|\boldsymbol{r}(s,t)|^2\,ds+\frac{\varepsilon}{2}\int_0^1(\rho A)(s)|\boldsymbol{r}_t(s,t)|^2\,ds\\ &\le C\int_0^1|\boldsymbol{r}(s,0)|^2\,ds+C\int_0^1\!\!\int_0^t|\boldsymbol{r}_t(s,\tau)|^2\,d\tau\,ds+\varepsilon K(t)\\ &\le C+C\int_0^t K(\tau)\,d\tau+\varepsilon K(t).\end{aligned}$$

Thus

$$(20.14)\quad |\boldsymbol{a}(t)|\le\Gamma(t)+\varepsilon K(t)+\Gamma(t)\sqrt{\Omega(t)}+\Gamma(t)\int_0^t[K(\tau)+\Omega(\tau)]\,d\tau.$$

Integrating (4.9) from ξ to s, and operating on the resulting equation with $\int_0^1 d\xi(\rho\boldsymbol{J})(\xi,t)\cdot$, we obtain

$$(20.15\text{a})\quad \left[\int_0^1(\rho\boldsymbol{J})(\xi,t)\,d\xi\right]\cdot\boldsymbol{w}(s,t)-\boldsymbol{a}(t)=\int_0^1\left[(\rho\boldsymbol{J})(\xi,t)\cdot\int_\xi^s\boldsymbol{R}\cdot\mathbf{u}_t\,d\xi\right]d\xi.$$

Since $\rho\boldsymbol{J}$ is positive-definite, so is $\int_0^1(\rho\boldsymbol{J})(\xi,t)\,d\xi$. Thus

$$(20.15\text{b})\qquad |\boldsymbol{w}(s,t)|\le C|\boldsymbol{a}(t)|+C\int_0^1|\mathbf{u}_t(s,t)|\,ds\le C|\boldsymbol{a}(t)|+CM(t).$$

Thus by combining (20.14) and (20.15) we get

(20.16)

$$\int_0^t\boldsymbol{m}(s,\tau)\cdot\boldsymbol{w}(s,\tau)\,d\tau\bigg|_{s=0}^{s=1}\le\Gamma(t)+\varepsilon K(t)+\varepsilon\Omega(t)+\Gamma(t)\int_0^t[K(\tau)+\Omega(\tau)]\,d\tau.$$

We now substitute (20.8) and (20.16) into (9.3) and invoke hypothesis (6.1) to obtain (9.12) and thus the energy estimate (9.14).

We likewise handle the conditions

$$(20.17)\qquad \boldsymbol{w}(0,t)\quad\text{and}\quad\boldsymbol{m}(1,t)\quad\text{are prescribed.}$$

and the conditions

$$(20.18)\qquad \boldsymbol{p}(0,t)=\boldsymbol{p}(1,t)\quad\text{is prescribed,}$$

$$(20.19)\qquad \boldsymbol{w}(0,t)=\boldsymbol{w}(1,t)\quad\text{is prescribed.}$$

In summary, we obtain the analogs of Theorem 9.13 for each of these kinds of boundary conditions, under slightly different regularity hypotheses.

In Sec. 19, we discussed the tricky case in which $\boldsymbol{p}(0,t)$ and $\boldsymbol{p}(1,t)$ are prescribed, but not equal, or $\boldsymbol{w}(0,t)$ and $\boldsymbol{w}(1,t)$ are prescribed, but not equal.

We now show how to get the estimates of Sec. 11 for other boundary conditions. We modify the treatment beginning with (11.22) to estimate the boundary term of (11.11).

First suppose that $\boldsymbol{n}(0,\cdot)$ and $\boldsymbol{n}(1,\cdot)$ are prescribed. We integrate the inequality

$$|\boldsymbol{p}_t(s,t)| - |\boldsymbol{p}_t(\xi,t)| \le \left|\int_\xi^s |\boldsymbol{p}_{st}(\xi,t)|\,d\xi\right| \le \int_0^1 |\boldsymbol{p}_{st}(\xi,t)|\,d\xi \tag{20.20}$$

with respect to ξ over $[0,1]$ and use (5.1e), (11.18), (9.6) to get

$$\begin{aligned} |\boldsymbol{p}_t(\cdot,\cdot)| &\le \int_0^1 |\boldsymbol{p}_t|\,d\xi + \int_0^1 |\mathbf{v}_{tt}|\,d\xi \\ &\quad + 2\int_0^1 |\mathbf{v}_t|\,|\boldsymbol{w}|\,d\xi + \int_0^1 |\boldsymbol{w}_t|\,|\mathbf{v}|\,d\xi + \int_0^1 |\mathbf{v}|\,|\boldsymbol{w}|^2\,d\xi. \end{aligned} \tag{20.21}$$

Thus (11.18), (9.6), (9.8) imply that

$$|\bar{\boldsymbol{n}}_t\cdot\boldsymbol{p}_t(1,\cdot)| \le \varGamma[1+M] + \langle\boldsymbol{p}_t,\rho A\boldsymbol{p}_t\rangle + \langle\boldsymbol{w}_t,\rho\boldsymbol{J}\cdot\boldsymbol{w}_t\rangle + \varepsilon\int_0^t \langle\boldsymbol{\eta}_{tt},\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}\cdot\boldsymbol{\eta}_{tt}\rangle\,d\tau, \text{ etc.} \tag{20.22}$$

Let us generalize initial conditions (7.1b) slightly by assuming that $\boldsymbol{r}(0,\cdot)$ is a prescribed nonzero function. We then use the equation of motion (5.1) to get

$$\begin{aligned} &\boldsymbol{p}_t(1,\cdot)\cdot\boldsymbol{n}_t(1,\cdot) - \boldsymbol{p}_t(0,\cdot)\cdot\boldsymbol{n}_t(0,\cdot) \\ &= [\boldsymbol{p}_t(1,\cdot) - \boldsymbol{p}_t(0,\cdot)]\cdot\bar{\boldsymbol{n}}_t + \partial_t\left[\boldsymbol{p}_t(0,\cdot)\cdot\int_0^1 \rho A\boldsymbol{p}_t\,ds\right] - \boldsymbol{p}_{tt}(0,\cdot)\cdot\int_0^1 \rho A\boldsymbol{p}_t\,ds. \end{aligned} \tag{20.23}$$

We assume that $\boldsymbol{r}(0,\cdot)\in H^3_{\mathrm{loc}}(0,\infty)$. In estimating the time integral of this boundary term, we employ $|\boldsymbol{p}_t\cdot\int_0^1\rho A\boldsymbol{p}_t\,ds| \le \varGamma(t) + \varepsilon\langle\boldsymbol{p}_t,\rho A\boldsymbol{p}_t\rangle$.

We likewise generalize (7.1b) slightly by assuming that $\boldsymbol{w}(0,\cdot)$ is a prescribed nonzero function. As in (20.20), (20.22), we find that

$$|\boldsymbol{w}_t(s,t)| \le \int_0^1 |\boldsymbol{w}_t(\xi,t)|\,d\xi + \int_0^1 |\mathbf{u}_{tt}(\xi,t)|\,d\xi + \varGamma(t)\int_0^1 |\boldsymbol{w}(\xi,t)|\,|\mathbf{u}_t(\xi,t)|\,d\xi, \tag{20.24}$$

so that (20.15) and the analog of (11.18) yield

$$|\bar{\boldsymbol{m}}_t(t)\cdot\boldsymbol{w}_t(1,t)| \le \varGamma(t)[1+M(t)^2] + \langle\boldsymbol{w}_t,\rho\boldsymbol{J}\cdot\boldsymbol{w}_t\rangle + \varepsilon\int_0^t \langle\boldsymbol{\eta}_{tt},\boldsymbol{\sigma}^{\mathrm{D}}_{\dot{\boldsymbol{\eta}}}\cdot\boldsymbol{\eta}_{tt}\rangle\,d\tau. \tag{20.25}$$

To handle the boundary term at 0 we use (5.2) and (11.20) to get

$$
\begin{aligned}
&\boldsymbol{m}_t(0,t)\cdot\boldsymbol{w}_t(0,t)\\
&=\bar{\boldsymbol{m}}_t\cdot\boldsymbol{w}_t(0,t)+\boldsymbol{w}_t(0,t)\cdot\int_0^1 \boldsymbol{v}_t\times\boldsymbol{n}\,ds\\
&\quad+\partial_t\left[\int_0^1[\boldsymbol{w}_t(0,t)\times\boldsymbol{v}]\cdot\boldsymbol{n}\,ds\right]-\int_0^1\partial_t[\boldsymbol{w}_t(0,t)\times\boldsymbol{v}]\cdot\boldsymbol{n}\,ds\\
&\quad-\partial_t\left[\boldsymbol{w}_t(0,t)\cdot\int_0^1(\rho\boldsymbol{J}\cdot\boldsymbol{w})_t\,ds\right]-\boldsymbol{w}_{tt}(0,t)\cdot\int_0^1(\rho\boldsymbol{J}\cdot\boldsymbol{w})_t\,ds\\
&\le\partial_t\left[\int_0^1[\boldsymbol{w}_t(0,t)\times\boldsymbol{r}_s]\cdot\boldsymbol{n}\,ds\right]-\partial_t\left[\boldsymbol{w}_t(0,t)\cdot\int_0^1(\rho\boldsymbol{J}\cdot\boldsymbol{w})_t\,ds\right]\\
&\quad+\Gamma(t)[1+M(t)^2]+\langle\boldsymbol{p}_t,\rho A\boldsymbol{p}_t\rangle+\langle\boldsymbol{w}_t,\rho\boldsymbol{J}\cdot\boldsymbol{w}_t\rangle.
\end{aligned}
\tag{20.26}
$$

It is clear that our methods handle a much wider variety of boundary conditions, including, e.g., those in which $\boldsymbol{r}(0,t)$ is confined to a moving surface or a moving curve and in which the complementary components of $\boldsymbol{n}(0,t)$ tangent to these manifolds are prescribed, and those in which a rigid body is attached to one end of the rod (cf. [10]). The rest of our analysis goes through provided that the boundary conditions support the energy estimate (9.14) and that special provisions are made to handle the difficulties caused by boundary conditions on the configuration mentioned in the fourth paragraph of Sec. 19. For each distinctive set of boundary conditions, we would replace the y_{a} introduced in in Sec. 13 with a richer class of orthonormal trigonometric functions (possibly vectorial, like those defined by (21.4) and (21.5) below, but with different homogeneous boundary conditions).

21. Componential Versions of the Equations

If we take the componential version of (5.7) with respect to the basis $\{\boldsymbol{d}_k\}$, we can uncouple the equation (5.7a) for the $\boldsymbol{d}_k$ from the remaining equations: Let e_{klm} denote the alternating symbol. Then (5.7a) has the componential form

$$\partial_t\boldsymbol{d}_k=e_{kij}w_j\boldsymbol{d}_i, \tag{21.1}$$

and the remaining equations of system (5.7) are equivalent to

$$\partial_t v_i=\partial_s p_i+e_{ijk}(u_jp_k-w_jv_k), \tag{21.2a}$$

$$\partial_t u_i=\partial_s w_i-e_{ijk}w_ju_k, \tag{21.2b}$$

$$\partial_t(\rho Ap_i)=\partial_s\hat{n}_i+e_{ijk}(u_j\hat{n}_k-w_j\rho Ap_k), \tag{21.2c}$$

$$\partial_t(\rho J_{ij}w_j)=\partial_s\hat{m}_i+e_{ijk}(u_j\hat{m}_k+v_j\hat{n}_k-w_j\rho J_{kq}w_q), \tag{21.2d}$$

where the arguments of $\hat{n}_k$ and $\hat{m}_k$ are $\mathbf{v}, \mathbf{u}, \mathbf{v}_t, \mathbf{u}_t, s$. We can write this system in the compact form

$$\mathbf{v}_t = \mathbf{p}_s + \mathbf{u} \times \mathbf{p} - \mathbf{w} \times \mathbf{v}, \tag{21.3a}$$

$$\mathbf{u}_t = \mathbf{w}_s - \mathbf{w} \times \mathbf{u}, \tag{21.3b}$$

$$\partial_t(\rho A \mathbf{p}) \equiv \rho A \mathbf{p}_t = \partial_s \hat{\mathbf{n}} + \mathbf{u} \times \hat{\mathbf{n}} - \mathbf{w} \times (\rho A \mathbf{p}), \tag{21.3c}$$

$$\partial_t(\rho \mathbf{J} \cdot \mathbf{w}) \equiv \rho \mathbf{J} \cdot \mathbf{w}_t = \partial_s \hat{\mathbf{m}} + \mathbf{u} \times \hat{\mathbf{m}} + \mathbf{v} \times \hat{\mathbf{n}} - \mathbf{w} \times (\rho \mathbf{J} \cdot \mathbf{w}), \tag{21.3d}$$

Note the sign flip in going from (5.7c) to (21.3b).

Even though (21.3) is independent of the $\boldsymbol{d}_k$, the boundary conditions for (21.3) are typically not. Indeed, (7.1c) implies that $n_k(1,t) = \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{d}_k(1,t)$, etc., with the $\boldsymbol{d}_k(1,t)$ found as the solution of an initial-value problem for (21.1) (or, equivalently, for (5.7a)) for $s = 1$. Thus the uncoupling of (21.3) from (21.1) need not be complete. This difficulty would be more pronounced if equal forces were prescribed at each end of the rod, in which case $\boldsymbol{n}(0,t) = -\boldsymbol{n}(1,t)$, but, in general, the components are not equal: $n_k(0,t) = \boldsymbol{d}_k^1(1,t) \cdot \boldsymbol{d}_l^2(1,t) n_l(1,t)$.

Other versions of the Galerkin approximations. Some form of a componential version of the governing equations must be used for numerical treatments. But as we now observe, its use in Part IV could complicate the analysis. The componential equations (21.3c,d) are clearly more complicated than their vectorial counterparts (5.7d,e). But (21.3c,d) has the virtue that $\rho \mathbf{J}$ is independent of t. Thus we can replace the Galerkin approximations (13.1) with others in which the basis functions are taken to be orthonormal with respect to the natural weights ρA and $\rho \mathbf{J}$, and thereby obtain approximate ordinary differential equations that are uncoupled in the time derivatives. In particular, for the boundary conditions (7.1), we can take Galerkin approximations of the form

$$\mathbf{p}^{\mathsf{N}}(s,t) := \sum_{\mathsf{a}=1}^{\mathsf{N}} \mathsf{P}_{\mathsf{a}}(t) \mathbf{p}_{\mathsf{a}}(s), \qquad \mathbf{w}^{\mathsf{N}}(s,t) := \sum_{\mathsf{a}=1}^{\mathsf{N}} \mathsf{W}_{\mathsf{a}}(t) \mathbf{w}_{\mathsf{a}}(s) \tag{21.4}$$

where the scalar-valued functions $\mathsf{P}_{\mathsf{a}}, \mathsf{W}_{\mathsf{a}}$ are to be determined and where $\{\mathbf{p}_{\mathsf{a}}\}$, $\{\mathbf{w}_{\mathsf{a}}\}$, $\mathsf{a} = 1, 2, \dots$, are each complete independent sets of given functions in $H^1(0,1)$ with $\mathbf{p}_{\mathsf{a}}(0) = \mathbf{o}$, $\mathbf{w}_{\mathsf{a}}(0) = \mathbf{o}$. To get the desired orthonormality, we choose the $\{\mathbf{p}_{\mathsf{a}}\}$ to be the normalized eigenfunctions satisfying

$$\begin{gathered} \mathbf{p}_{ss} + \lambda(\rho A)(s) \mathbf{p} = \mathbf{o} \quad \text{on } (0,1), \quad \mathbf{p}(0) = \mathbf{o}, \quad \mathbf{p}_s(1) = \mathbf{o}, \\ \int_0^1 \rho A \, \mathbf{p}_{\mathsf{a}} \cdot \mathbf{p}_{\mathsf{b}} \, ds = \delta_{\mathsf{ab}} \end{gathered} \tag{21.5}$$

corresponding to eigenvalues $\{\lambda_{\mathsf{a}}\}$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ (each distinct eigenvalue having multiplicity 3 by virtue of the vectorial character of (21.5)) and take $\{\mathbf{w}_{\mathsf{a}}\}$ to be the normalized eigenfunctions for

$$\begin{gathered} \mathbf{w}_{ss} + \mu(\rho \mathbf{J})(s) \cdot \mathbf{w} = \mathbf{o} \quad \text{on } (0,1), \quad \mathbf{w}(0) = \mathbf{o}, \quad \mathbf{w}_s(1) = \mathbf{o}, \\ \int_0^1 \mathbf{w}_{\mathsf{a}} \cdot \rho \mathbf{J} \cdot \mathbf{w}_{\mathsf{b}} \, ds = \delta_{\mathsf{ab}} \end{gathered} \tag{21.6}$$

corresponding to eigenvalues $\{\mu_{\mathsf{a}}\}$ with $0 < \mu_1 \le \mu_2 \le \cdots$. (The properties of the eigenfunctions are consequences of Sturm-Liouville theory. Should ρA and $\rho\mathbf{J}$ only be piecewise continuous, the eigenfunction solutions of these problems interpreted in an appropriately generalized sense have second derivatives that need only be piecewise continuous.)

If we define $\mathbf{v}^{\mathsf{N}}$ and $\mathbf{u}^{\mathsf{N}}$ to be solutions of (21.3a,b) modified by having all the variables bear the superscript N, then the Galerkin approximations constructed as in Sec. 13 have the standard form

(21.7)
$$d_t \mathsf{P}_{\mathsf{a}} = \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{d}_k^{\mathsf{N}} p_{\mathsf{a}k}(1) + \int_0^1 \{-\mathbf{n}^{\mathsf{N}} \cdot \partial_s \mathbf{p}_{\mathsf{a}} + [\mathbf{u}^{\mathsf{N}} \times \mathbf{n}^{\mathsf{N}} - \mathbf{w}^{\mathsf{N}} \times (\rho A \mathbf{p}^{\mathsf{N}})] \cdot \mathbf{p}_{\mathsf{a}}\}\, ds,$$

(21.8)
$$d_t \mathsf{W}_{\mathsf{a}} = \bar{\boldsymbol{m}}(t) \cdot \boldsymbol{d}_k^{\mathsf{N}} w_{\mathsf{a}k}(1) + \int_0^1 \{-\mathbf{m}^{\mathsf{N}} \cdot \partial_s \mathbf{w}_{\mathsf{a}} + [\mathbf{u}^{\mathsf{N}} \times \mathbf{m}^{\mathsf{N}} + \mathbf{v}^{\mathsf{N}} \times \mathbf{n}^{\mathsf{N}} - \mathbf{w}^{\mathsf{N}} \times (\rho \mathbf{J} \cdot \mathbf{w}^{\mathsf{N}})] \cdot \mathbf{w}_{\mathsf{a}}\}\, ds.$$

The virtues of these kinematically uncoupled ordinary differential equations are counterbalanced by the added complexity in the governing equations.

In the study of hyperbolic conservation laws of mechanics, weak forms like (7.10a,b) of compatibility equations play a central role on par with the weak forms of the momentum equations. With some effort, we could give the componential versions of the compatibility equations the same status as the momentum equations in the Faedo-Galerkin method: Instead of defining $\mathbf{v}^{\mathsf{N}}$ and $\mathbf{u}^{\mathsf{N}}$ as above, we would seek Galerkin approximations for them in the form

$$\text{(21.9)} \quad \mathbf{v}^{\mathsf{N}}(s,t) := \mathbf{v}_{\dagger}^{\mathsf{N}}(t) + \sum_{\mathsf{a}=1}^{\mathsf{N}} \mathsf{V}_{\mathsf{a}}(t) \mathbf{v}_{\mathsf{a}}(s), \qquad \mathbf{u}^{\mathsf{N}}(s,t) := \mathbf{u}_{\dagger}^{\mathsf{N}}(t) + \sum_{\mathsf{a}=1}^{\mathsf{N}} \mathsf{U}_{\mathsf{a}}(t) \mathbf{u}_{\mathsf{a}}(s)$$

with
(21.10)
$$\mathbf{v}_{\mathsf{a}} = \frac{\partial_s \mathbf{p}_{\mathsf{a}}}{\sqrt{\lambda_{\mathsf{a}}}}, \quad \mathbf{u}_{\mathsf{a}} = \frac{\partial_s \mathbf{w}_{\mathsf{a}}}{\sqrt{\mu_{\mathsf{a}}}} \qquad \text{so that} \qquad \int_0^1 \mathbf{v}_{\mathsf{a}} \cdot \mathbf{v}_{\mathsf{b}}\, ds = \delta_{\mathsf{ab}}, \quad \int_0^1 \mathbf{u}_{\mathsf{a}} \cdot \mathbf{u}_{\mathsf{b}}\, ds = \delta_{\mathsf{ab}}.$$

Here the vector-valued functions $\mathbf{v}_{\dagger}^{\mathsf{N}}, \mathbf{u}_{\dagger}^{\mathsf{N}}$ and the scalar-valued functions $\mathsf{V}_{\mathsf{a}}, \mathsf{U}_{\mathsf{a}}$ are to be determined, with the vector-valued functions allowed to accommodate the boundary conditions (7.1c). (Such an accommodation seems necessary to get the requisite estimates.) We set

$$\text{(21.11)} \qquad \mathbf{n}^{\mathsf{N}}(s,t) := \hat{\mathbf{n}}(\mathbf{u}^{\mathsf{N}}(s,t), \mathbf{v}^{\mathsf{N}}(s,t), \mathbf{u}_t^{\mathsf{N}}(s,t), \mathbf{v}_t^{\mathsf{N}}(s,t), s), \quad \text{etc.}$$

To handle the mechanical boundary conditionat $s = 1$, we substitute our constitutive equations into the boundary conditions (7.1c) and then replace the resulting system with the approximation
(21.12)
$$\hat{n}_k(\mathbf{u}_{\dagger}^{\mathsf{N}}, \mathbf{v}_{\dagger}^{\mathsf{N}}, \tfrac{d}{dt}\mathbf{u}_{\dagger}^{\mathsf{N}}, \tfrac{d}{dt}\mathbf{v}_{\dagger}^{\mathsf{N}}, 1) = \bar{\boldsymbol{n}}(t) \cdot \boldsymbol{d}_k^{\mathsf{N}}, \quad \hat{m}_k(\mathbf{u}_{\dagger}^{\mathsf{N}}, \mathbf{v}_{\dagger}^{\mathsf{N}}, \tfrac{d}{dt}\mathbf{u}_{\dagger}^{\mathsf{N}}, \tfrac{d}{dt}\mathbf{v}_{\dagger}^{\mathsf{N}}, 1) = \bar{\boldsymbol{m}}(t) \cdot \boldsymbol{d}_k^{\mathsf{N}},$$

which we supplement with initial conditions

$$\mathbf{v}_\dagger^{\mathsf{N}}(0) = \mathbf{v}^\circ(1), \qquad \mathbf{u}_\dagger^{\mathsf{N}}(0) = \mathbf{u}^\circ(1). \tag{21.13}$$

Hypothesis 6.2 implies that we can put (21.12) into standard form. In particular, we can solve the finite-dimensional system $\hat{\mathbf{n}}(\mathbf{u}, \mathbf{v}, \dot{\mathbf{u}}, \dot{\mathbf{v}}, 1) = \mathbf{n}$, $\hat{\mathbf{m}}(\mathbf{u}, \mathbf{v}, \dot{\mathbf{u}}, \dot{\mathbf{v}}, 1) = \mathbf{m}$ uniquely for $(\dot{\mathbf{u}}, \dot{\mathbf{v}})$ in terms of the other variables: $\dot{\mathbf{u}} = \dot{\mathbf{u}}^\sharp(\mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{n})$, $\dot{\mathbf{v}} = \dot{\mathbf{v}}^\sharp(\mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{n})$. Thus (21.12) is equivalent to

$$\begin{aligned} \tfrac{d}{dt}\mathbf{v}_\dagger^{\mathsf{N}} &= \dot{\mathbf{v}}^\sharp(\mathbf{u}_\dagger^{\mathsf{N}}, \mathbf{v}_\dagger^{\mathsf{N}}, \bar{\boldsymbol{m}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}, \bar{\boldsymbol{m}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}, \bar{\boldsymbol{n}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}), \\ \tfrac{d}{dt}\mathbf{u}_\dagger^{\mathsf{N}} &= \dot{\mathbf{u}}^\sharp(\mathbf{u}_\dagger^{\mathsf{N}}, \mathbf{v}_\dagger^{\mathsf{N}}, \bar{\boldsymbol{m}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}, \bar{\boldsymbol{m}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}, \bar{\boldsymbol{n}}(t)\cdot \boldsymbol{d}_k^{\mathsf{N}}). \end{aligned} \tag{21.14}$$

The Galerkin approximations for the compatibility equations are readily shown to be the kinematically uncoupled system

$$\frac{d}{dt}\mathsf{V}_{\mathsf{a}} = \sqrt{\lambda_{\mathsf{a}}}\mathsf{P}_{\mathsf{a}} + \int_0^1 (\mathbf{u}^{\mathsf{N}} \times \mathbf{p}^{\mathsf{N}} - \mathbf{w}^{\mathsf{N}} \times \mathbf{v}^{\mathsf{N}}) \cdot \mathbf{v}_{\mathsf{a}}\, ds - \dot{\mathbf{v}}^\sharp \cdot \int_0^1 \mathbf{v}_{\mathsf{a}}\, ds, \tag{21.15}$$

$$\frac{d}{dt}\mathsf{U}_{\mathsf{a}} = \sqrt{\mu_{\mathsf{a}}}\mathsf{W}_{\mathsf{a}} + \int_0^1 (\mathbf{u}^{\mathsf{N}} \times \mathbf{w}^{\mathsf{N}}) \cdot \mathbf{u}_{\mathsf{a}}\, ds - \dot{\mathbf{u}}^\sharp \cdot \int_0^1 \mathbf{u}_{\mathsf{a}}\, ds. \tag{21.16}$$

Unfortunately, these equations correspond to a projection of the compatibility equations

$$\mathbf{v}_t^{\mathsf{N}} = \mathbf{p}_s^{\mathsf{N}} + \mathbf{u}^{\mathsf{N}} \times \mathbf{p}^{\mathsf{N}} - \mathbf{w}^{\mathsf{N}} \times \mathbf{v}^{\mathsf{N}}, \qquad \mathbf{u}_t^{\mathsf{N}} = \mathbf{w}_s^{\mathsf{N}} - \mathbf{w}^{\mathsf{N}} \times \mathbf{u}^{\mathsf{N}} \tag{21.17}$$

onto span $\{\mathbf{v}_1, \dots, \mathbf{v}_{\mathsf{N}}\}$ and span $\{\mathbf{u}_1, \dots, \mathbf{u}_{\mathsf{N}}\}$. The treatment of the discrepancy between cross products like $\mathbf{u}^{\mathsf{N}} \times \mathbf{p}^{\mathsf{N}}$ and its projection onto span $\{\mathbf{v}_1, \dots, \mathbf{v}_{\mathsf{N}}\}$ presumably would require estimates like those of [1, Sec. 4.5] and [43, Sec. III.13].

22. Discussion. Generalizations

Monotone operators. Some of our techniques, especially those in the proofs of Theorem 11.1 and in the treatment of the Galerkin method, represent simplifications of methods used in [12]. In particular, compactness results like (11.31) and (16.1) following from our a priori estimates (which in turn follow from our constitutive hypotheses) give rise to important instances of strong convergence in Part IV. The availability of these strong convergence results, in particular, (16.4), enabled us to avoid appealing to the theory of monotone operators [26, 39] to handle difficulties with weak convergence. We do, of course, use the monotonicity available for the modified problem to obtain the estimates in Part IV.

Rougher data. At various places in our analysis, we had to assume that the boundary data are fairly smooth. We can immediately weaken such assumptions to the corresponding requirements that the data be piecewise smooth. No doubt, approximation arguments can give further weakening.

Discontinuous material behavior. If the material data ρA, ρJ_{pq}, or $\hat{\boldsymbol{\sigma}}$ happen to be merely piecewise continuous in s, then the uniform convergence of continuous functions to continuous limits used in Part IV is replaced by the uniform convergence of piecewise continuous functions to piecewise continuous limits because the loci of the jump discontinuities are fixed. All our analysis would go through.

Hölder spaces. It would be interesting to know whether comparable results can be obtained from a direct Hölder-space formulation (rather than from the use of embedding theorems in a higher-order Sobolev-space setting). Such an approach might exploit the available parabolicity to produce additional regularization. We used a Sobolev-space setting here because it is natural for energy-like estimates.

More general theories of rods. We can readily handle a more general form of our equations for rods in which the acceleration terms are mixtures of derivatives of $\boldsymbol{r}$ and $\boldsymbol{w}$. These forms arise when the base curve $\boldsymbol{r}$ is not taken to be a generalized curve of centroids of the body. See [7, Ex. 8.4.8]. Indeed, we can handle any problem of nonlinear viscoelastic of strain-rate type with just one spatial variable. These include arbitrary theories for rods and axisymmetric shells [7, Chaps. 16, 17], and semi-inverse problems (with one independent spatial variable) of the 3-dimensional theory (cf. [11]). (The equations for rods and axisymmetric shells are far richer and more complicated than those for semi-inverse problems of the 3-dimensional theory with one independent space variable.)

Many of the difficulties we have confronted for the problems treated here arise because configurations take their values in the manifold $\mathbb{E}^3 \times \mathrm{SO}(3)$ rather than in $\mathbb{R}^N$. Among these difficulties is the treatment of nonlinearities in the angular acceleration, which is typical of 3-dimensional rigid-body mechanics. Some of these difficulties do not appear for many more general rod theories, but do appear in various models for superconductivity [14], for liquid crystals [25], and in harmonic maps. It would be nice to have a general theory for the dynamics in which some variables are constrained to lie in manifolds. (cf. [23])).

Planar problems. We can immediately specialize our results to planar problems for rods. The governing equations are obtained by constraining $\boldsymbol{d}_2$, say, to lie along a fixed direction $\boldsymbol{k}$, and using the two components of the force balance perpendicular to $\boldsymbol{k}$ and the one component of the moment balance parallel to $\boldsymbol{k}$. Then the configuration of $\boldsymbol{d}_1$ and $\boldsymbol{d}_3$ is determined by the angle θ between $\boldsymbol{d}_3$ and a fixed direction in the plane perpendicular to $\boldsymbol{k}$. Then the only flexural strain is $u_2 = \theta_s$. The main simplification is that the angular acceleration term on the right-hand side of (5.2) reduces to a scalar $\rho J_{22}\theta_{tt}$ (times $\boldsymbol{k}$), which is linear in the unknown θ. See [7] for details.

Nonzero loads. Standard methods would allow us readily to extend all our analysis to nonzero $\boldsymbol{f}$ and $\boldsymbol{l}$ provided that these loads are *dead*, i.e., that they are prescribed functions of s and t. (If these loads and the boundary data are independent of t, then versions of our constitutive assumptions provide uniform a priori bounds, i.e., bounds independent of T.) A *live* load (applied to body of the rod or to its ends) is one that also depends on the motion itself. E.g., a centrifugal force at a material point depends on the distance of the material point from the axis of rotation. Other live loads are generated by hydrostatic pressure, air resistance, attached rigid bodies, and feedbacks. Our methods handle live loads as long as they deliver the same estimates.

The trouble with live loads is that they can have a very subtle interaction with constitutive equations with the consequence that they could produce blowups in finite time [7,16]. (No such blowups occur for the problem of a rod carrying a rigid body at an end [10].)

For dead loads, just as for zero loads, we would obtain existence for all time by getting estimates valid on the time interval $[0, T]$ for all real T. We use the finiteness of T merely to ensure the uniformity of our estimates, e.g., to ensure that $\gamma(T) > 0$ in (10.6) and that $\mathit{\Gamma}(T) < \infty$ in (11.3). When a blowup occurs (as a consequence of live loads), T plays the alternative and more critical role of the blowup time.

The methods of Sec. 18 can readily be extended to show well-posedness with respect to constitutive functions and dead body loads. Because there could be blowups produced by slight changes in constitutive functions or in parameters of live loads, the issue of stability is much more delicate here.

Change of phase. Since we do not require that the stored-energy function φ be convex, we can handle models accounting for change of phase. Cf. , e.g., [3,28]

Optimal control. The results of our analysis make these problems amenable to the methods of optimal control (cf. [30,31]).

Potential for the viscous response. It has sometimes been advocated on the basis of thermodynamical considerations [41,42] that the dissipative part $\boldsymbol{\sigma}^{\mathrm{D}}$ of the stress resultants should be obtainable from a potential, so that $\boldsymbol{\sigma}_{\dot{\boldsymbol{\eta}}}$ should be symmetric. We found no need for such an assumption.

Thermoviscoelastic rods. It might be possible to combine our methods with those developed for 1-dimensional thermoviscoelasticity in [19,36], inter al., to treat thermoviscoelastic rods.

Strain-gradient effects. The appropriate incorporation of strain-gradient effects, in which the resultants also depend upon derivatives of $\boldsymbol{\eta}$, would no doubt regularize solutions in a way akin to that effected by the viscosity (see [6,21,35]). There is even some evidence that appropriate versions can preclude total compressions. An objection to using strain-gradient effects is that there is scarcely any non-mathematical evidence for physically natural forms for constitutive functions accounting for these effects.

More spatial dimensions. The main difficulty in extending our methods to more spatial dimensions is that our methods for precluding total compression in Sec. 10 break down. Basically, for a problem in two spatial dimensions we can show that ζ_3 can vanish along any given material curve at most on a set of measure zero. Indeed, we can bound the integral of ψ along such a curve. (This is reminiscent of the regularity theory for nonlinear elastostatics.) But we are unable to obtain a pointwise bound.

There is a second potential difficulty: In 2- and 3-dimensional viscoelasticity of strain-rate type and in corresponding shell theories, the assumption that the Piola-Kirchhoff stress is a monotone function of appropriate strain rates is consistent with the requirement that the response be invariant under rigid motions. On the other hand, in analogy with nonlinear elasticity, one might wish to generalize this assumption by merely requiring that the dependence of this stress on the strain rate be strongly elliptic. In this case, the extensive theory of monotone operators would not be available.

Correction. In [12], the analysis analogous to that beginning with (15.16) was superficial. The treatment here shows how to correct it.

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