# Periodic Solutions of a Nonlinear Parabolic Equation, II $^{1}$ 

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Abstract. Based on a strict coercivity estimate [8] for a class of nonlinear elliptic operators A : $u \mapsto \nabla \cdot a(\cdot,|\nabla u|) \nabla u$, it is shown that (with suitable consistency conditions on the data) the problem:

$$
\dot{u}+\mathbf{A} u=f, \quad a \nabla u \cdot \overrightarrow{\mathbf{n}}=\varphi, \quad \text { time periodicity }
$$

has a unique solution depending continuously on $f, \varphi$ and also on the nonlinear form of the diffusion coefficient $a(\cdot \cdot)$. This is then used to obtain existence by a fixpoint argument for the more general problem with $a, f$ of the form $a=a(\cdot, u,|\nabla u|), f=f(\cdot, u, \nabla u)$.

KEY WORDS: Nonlinear, partial differential equation, parabolic, periodic solutions, well posed, existence, structural stability, nonuniform ellipticity.

## 1. Introduction

The present paper is a continuation of the sequence [7], [8], [9]. It uses the results of [8] concerning elliptic operators of the form

$$
\begin{equation*}
\mathbf{A}: u \mapsto-\nabla \cdot a(\cdot,|\nabla u|) \nabla u \tag{0.1}
\end{equation*}
$$

to generalize the results of [7] on periodic solutions of

$$
\begin{equation*}
\dot{u}+\mathbf{A} u=f \tag{0.2}
\end{equation*}
$$

with Neumann boundary conditions

$$
\begin{equation*}
u_{\nu}:=a(\cdot \cdot) \nabla u \cdot \overrightarrow{\mathbf{n}}=\varphi . \tag{0.3}
\end{equation*}
$$

In comparison with [7], which considered only $a=a(|\nabla u|)$, we have now for (0.1)

$$
\begin{equation*}
a: \mathcal{Q} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+} \tag{0.4}
\end{equation*}
$$

[^0]rather than simply $a: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$; we will consider still more general dependence later.
Note that we have assumed in (0.4) that we are given a spatial region $\Omega$ (bounded in $\mathbb{R}^{m}$ with sufficiently smooth ${ }^{2}$ boundary $\partial \Omega$ ). We are also assuming that any explicit time dependence in $a, f, \varphi$ has periodicity with a common period which, with no loss of generality, we can take to be 1 ; thus, $\mathbb{P}:=\mathbb{R} / \mathbb{Z}$ is the period interval, topologically a circle. The periodicity is now "built in" by defining $f$ on $\mathcal{Q}:=\mathbb{P} \times \Omega, a$ on $\mathcal{Q} \times \mathbb{R}^{+}, \varphi$ on $\Sigma:=\mathbb{P} \times \partial \Omega$, and seeking $u$ defined on $\mathcal{Q}$. (Note that one automatically has
\[

$$
\begin{equation*}
\int_{\mathcal{Q}} \dot{v}=0 \text { for } v \text { defined on } \mathcal{Q} \tag{0.5}
\end{equation*}
$$

\]

when suitable interpretation of $\dot{v}$ is possible.)
The operators: $u \mapsto-\nabla a(|\nabla u|) \nabla u$ considered in $[7]$ involve functions $a(\cdot)$ which behave like a power as $|\nabla u| \rightarrow \infty$ :

$$
a(r) \sim r^{p-2} \text { as } r \rightarrow \infty
$$

for a fixed $p$ which determines the space in which to seek solutions; for simplicity we will assume $p \geq 2$. The principle concern of [7] was the effect of possible non-uniform ellipticity as $r \rightarrow 0$ on the well-posedness of the problem (0.2), (0.3) with periodicity. The arguments of [8] made it possible there to consider this possibility as well as other ways for the diffusion coefficient to vanish. The analysis in [8] also made it possible to demonstrate existence of solutions to elliptic equations of the form

$$
-\nabla \cdot a(\cdot, u,|\nabla u|) \nabla u=f(\cdot, u, \nabla u)
$$

with Dirichlet conditions. It was promised in [8] that we would correspondingly treat the parabolic problem with periodicity in time

$$
\begin{equation*}
\dot{u}-\nabla \cdot a(\cdot, u,|\nabla u|) \nabla u=f(\cdot, u, \nabla u) \tag{0.6}
\end{equation*}
$$

with a Neumann condition (0.3). Here we will first analyze the simpler problem (0.2), (0.3) with $\mathbf{A}$ as in (0.1) and with $f$ independent of $u$ and then will proceed to fulfil this promise by a fixpoint argument. Thus, the main results of this paper will be well-posedness for the "simple" problem

$$
\begin{gather*}
\dot{u}-\nabla \cdot a(\cdot,|\nabla u|) \nabla u=f \text { on } \mathcal{Q}, \\
a \nabla u \cdot \overrightarrow{\mathbf{n}}=\varphi \text { on } \Sigma, \quad \text { periodicity in } t \tag{0.7}
\end{gather*}
$$

and existence for the problem

$$
\begin{gather*}
\dot{u}-\nabla \cdot a(\cdot, u,|\nabla u|) \nabla u=f(\cdot, u, \nabla u), \\
a \nabla u \cdot \overrightarrow{\mathbf{n}}=\varphi \text { on } \Sigma, \quad \text { periodicity in } t \tag{0.8}
\end{gather*}
$$

under suitable hypotheses on $a, f, \varphi$.
A new difficulty which arises in connection with the problem (0.8), as compared to a corresponding problem with Dirichlet conditions, is that one has a consistency condition for the data and an element of nonuniqueness for the solution. From (0.5) we see, integrating

[^1]over $\mathcal{Q}$ and using the Divergence Theorem, that it is not possible for (0.7), to have a solution on $\mathcal{Q}$ unless $f, \varphi$ satisfy the consistency condition
\[

$$
\begin{equation*}
\int_{\mathcal{Q}} f+\int_{\Sigma} \varphi=0 . \tag{0.9}
\end{equation*}
$$

\]

The solution $u$, when it exists, cannot be unique since, on adding any constant $c$ to the solution of problem (0.7), one again has a solution. The solution can be made unique by imposing some auxiliary condition such as, e.g.,

$$
\begin{equation*}
\int_{\mathcal{Q}} u=0 . \tag{0.10}
\end{equation*}
$$

Neither the direct verification of (0.9) nor the imposition of (0.10) is feasible for the more general problem (0.8), and existence of a solution to that problem will be obtained by an application of Glicksberg's Theorem [5] on fixpoints of set-valued maps without imposing (0.10) to specify a solution.

We have adopted here, as in [7] (see also Chapter II of [6]), the static approach of looking in a space of periodic functions (defined on $\mathcal{Q}$ ) for a solution rather than the dynamic approach of first treating the initial value problem and then (e.g., by seeking a fixpoint of the Poincaré period map) looking for a solution which is periodic. Much as in [9], our formulation will take $\xi:=\nabla u$ in $L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ as the principal unknown.

In the treatment of (0.7) we seek a well-posedness result which also includes structural stability: not only should the solution depend continuously on $f, \varphi$ but also on the form of the nonlinear diffusion coefficient $a(\cdot \cdot)$ in that a suitable form of convergence $a_{k} \rightarrow a$ will imply convergence for the corresponding solutions; compare [9]. Such structural stability is, of course, necessary for any consideration of real applications in which, at best, $a(\cdot \cdot)$ is only known approximately (by measurement or by inference); it also is an essential ingredient in the treatment of (0.6).

The original motivation of [7] was an application involving induced eddy currents in a nonlinearly ferromagnetic material with longitudinal symmetry. The generalization to (0.7), now permits the consideration of material inhomogeneity but the further generalization to (0.8), is here motivated solely by the mathematical interest.

## 2. The Coercivity Estimate

We begin by recalling from [8] the analysis of elliptic operators of the form (0.1). Note that we have assumed $\Omega$ bounded in $\mathbb{R}^{m}$ with $\partial \Omega$ sufficiently smooth and, for simplicity, will assume $p \geq 2$. We have set $\mathbb{P}:=\mathbb{R} / \mathbb{Z}$ (the period "interval", topologically a circle) and then $\mathcal{Q}:=\mathbb{P} \times \Omega, \Sigma:=\mathbb{P} \times \partial \Omega$.

Given a function $a: \mathcal{Q} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$satisfying Carathéodory conditions, we define $g(\cdot, r):=r a(\cdot, r)$ and then define

$$
\begin{equation*}
\mu(\cdot, s):=\sup \left\{\mu: g(\cdot, r)-g(\cdot, s) \geq \mu s^{p-2}(r-s) \text { for } r>s \geq 0\right\} \tag{0.11}
\end{equation*}
$$

Note that $\mu$ is nondecreasing in $s$ and $\mu(\cdot, s)$ is measurable; when $g$ is differentiable in its second variable, one easily sees that $\mu$ can conveniently be equivalently ${ }^{3}$ defined by

$$
\begin{aligned}
\mu(\cdot, s) & :=\inf \left\{r^{-(p-2)} \partial g(\cdot, r) / \partial r: r>s \geq 0\right\} \\
& =(p-1) \inf \left\{\partial g / \partial\left(r^{p-1}\right): r>s\right\}
\end{aligned}
$$

[^2]which may be easier to compute.
Now, for $\lambda \geq 0$ set
\[

$$
\begin{align*}
\sigma(\cdot, \lambda) & :=4 \inf \{s>0: \mu(\cdot, s) \geq \lambda\}, \\
N(\lambda) & :=\|\sigma(\cdot, \lambda)\|^{p}:=\int_{\mathcal{Q}} \sigma(\cdot, \lambda)^{p} . \tag{0.12}
\end{align*}
$$
\]

Clearly $\sigma(\cdot, \lambda)$ is measurable and (where finite) nondecreasing in $\lambda$ so $N(\lambda)$ is well-defined and nondecreasing in $\lambda$. If there is any $\bar{\lambda}>0$ for which $N(\bar{\lambda})<\infty$, then, by the Monotone Convergence Theorem, $N$ is continuous on $[0, \bar{\lambda})$ and, in particular, $N(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$. Our basic set of assumptions regarding $a(\cdot \cdot)$ is, then,
(i) measurability in $(t, x) \in \mathcal{Q}$ for each $r \in \mathbb{R}^{+}$ and continuity in $r$ a.e. on $\mathcal{Q}$,

$$
\begin{equation*}
0 \leq g(\cdot, r):=r a(\cdot, r) \leq g_{*}(\cdot)+C_{*} r^{p-1} \tag{ii}
\end{equation*}
$$

(iii) for some $\bar{\lambda}>0,(0.11),(0.12)$ give $\sigma(\cdot, \lambda) \in L_{+}^{p}(\mathcal{Q})$ for $0 \leq \lambda<\bar{\lambda}$.

Fixing $C_{*}, g_{*} \in L_{+}^{q}(\mathcal{Q})$, and $N_{*}(\cdot)$ nondecreasing on $[0, \bar{\lambda})$ with $N_{*}(0+)=0$, we define $\mathcal{G}_{*}=\mathcal{G}_{*}\left(C_{*}, g_{*}, N_{*}\right)$ and $\mathcal{G}$ by

$$
\mathcal{G}_{*}:=\left\{g: \begin{array}{l}
(0.13-i i, i i i) \text { hold and one has }  \tag{0.14}\\
(0.12) \text { with } N(\lambda) \leq N_{*}(\lambda)
\end{array}\right\}, \quad \mathcal{G}:=\bigcup \mathcal{G}_{*} .
$$

Note that $\mathcal{G}_{*}$ is a nondecreasing set-valued function of $\left[C_{*}, g_{*}, N_{*}\right]$ and the union in defining $\mathcal{G}=\cup \mathcal{G}_{*}$ is over all admissible choices of $\left[C_{*}, g_{*}, N_{*}\right]$ for $\mathcal{G}_{*}$. We then define (sequential) convergence $g_{k}(\cdot) \rightarrow g(\cdot)$ in $\mathcal{G}$ to mean:
(i) $\quad g_{k} \in \mathcal{G}_{*}$ for some common $\left[C_{*}, g_{*}, N_{*}\right]$
(ii) $\quad g_{k}(\cdot, r(\cdot)) \rightarrow g(\cdot, r(\cdot))$ in $L_{+}^{q}(\mathcal{Q})$ for each fixed $r(\cdot) \in L_{+}^{q}(\mathcal{Q})$.

Finally, given $g \in \mathcal{G}$ and vector fields $\xi(\cdot), \eta(\cdot) \in L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ we define

$$
\begin{equation*}
\beta(\xi, \eta)=\beta_{g}(\xi(\cdot), \eta(\cdot)):=[a(\cdot,|\xi|) \xi-a(\cdot,|\eta|) \eta] \cdot[\xi-\eta] \tag{0.16}
\end{equation*}
$$

(euclidean dot product, pointwise a.e. on $\mathcal{Q}$ ) and

$$
\begin{equation*}
B(\xi, \eta)=B_{g}(\xi, \eta):=\int_{\mathcal{Q}} \beta_{g}(\xi, \eta) \tag{0.17}
\end{equation*}
$$

Note that the assumption ( $0.13-i i)$ is just sufficient to ensure the integrability of $\beta(\xi, \eta)$ for $\xi, \eta \in L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$. Other than some minor notational changes, these definitions are just those introduced in [8]; we recall from there some important properties:
LEMMA 2.1: $\quad$ Suppose $\left\{g_{k}\right\} \subset \mathcal{G}_{*}$ for some fixed $\mathcal{G}_{*}=\mathcal{G}_{*}\left(C_{*}, g_{*}, N_{*}\right)$ as in (0.14). Then pointwise convergence:

$$
g_{k}(\cdot, \bar{r}) \rightarrow g(\cdot, \bar{r}) \text { on } \mathcal{Q} \text { for each } \bar{r} \geq 0
$$

is sufficient to ensure that $g_{k} \rightarrow g$ in the sense of (0.15).

THEOREM 2.2: Given vector fields $\xi, \eta \in L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ and $g \in \mathcal{G}$ - here $g(\cdot, r)=r a(\cdot, r))$ - one has

$$
\begin{equation*}
\beta(\xi, \eta) \geq C \mu(r / 4) \delta^{p} \tag{0.18}
\end{equation*}
$$

where, pointwise, $r(\cdot):=\max \{|\xi|,|\eta|\}$ and $\delta:=|\xi-\eta|$.

THEOREM 2.3: $\quad$ There is a nondecreasing function $\Phi: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, depending only on $N_{*}(\cdot)$, such that $\Phi(r) \rightarrow 0$ as $r \rightarrow 0$ and, for any $g \in \mathcal{G}$ (with $N=N_{g}$ bounded by the $N_{*}$ used to determine $\Phi$ ), one has

$$
\begin{equation*}
\|\xi-\eta\| \leq \Phi\left(\frac{B_{g}(\xi, \eta)}{\|\xi-\eta\|}\right) \tag{0.19}
\end{equation*}
$$

(using $L^{p}$ norms) for arbitrary vector fields $\xi, \eta \in L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$.
We note that $C=C_{p}=\min \left\{3^{-p}, \frac{1}{2} 8^{2-p}\right\}$ in (0.18) and that $\Phi=\Phi(\rho)$ in Theorem 2.3 can be obtained as the maximal solution of

$$
\begin{equation*}
\Phi^{p}=2 \inf _{\lambda}\left\{2^{p} N_{*}(\lambda)+\rho \Phi / C_{p} \lambda\right\} . \tag{0.20}
\end{equation*}
$$

For further details see [8], noting that we are considering only $p \geq 2$ here.
Given $g \in \mathcal{G}$, the inequality (0.19) is a strict coercivity estimate for the Nemytskii operator

$$
\begin{equation*}
\boldsymbol{\Gamma}: \xi \mapsto a(\cdot,|\xi|) \xi=g(\cdot,|\xi|)(\xi /|\xi|) \tag{0.21}
\end{equation*}
$$

since $B_{g}(\xi, \eta)$ is just $\langle\boldsymbol{\Gamma} \xi-\boldsymbol{\Gamma} \eta, \xi-\eta\rangle$; continuity of $\boldsymbol{\Gamma}$ follows from Krasnoselskii's Theorem on Nemytsky operators (cf. e.g., [3]) in view of (0.13-i, ii).

## 3. Formulation of the Parabolic Problem

Suppose we are given $\Omega, p$ as above and a function $a: \mathcal{Q} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$such that $g \in \mathcal{G}$ so (0.13) holds. We begin by considering a standard weak formulation of the problem (0.2), (0.3):

$$
\begin{equation*}
\int_{\Omega}[v \dot{u}+\nabla v \cdot(\boldsymbol{\Gamma} \nabla u)]=\int_{\mathcal{Q}} v f+\int_{\partial \Omega} v \varphi \tag{0.22}
\end{equation*}
$$

which is to hold a.e. on $\mathbb{P}$ for suitable test functions $v$ on $\Omega$. Indeed, one usually further integrates over $\mathbb{P}$ with $v$ a test function defined on $\mathcal{Q}$ :

$$
\begin{equation*}
\int_{\mathcal{Q}}[v \dot{u}+\nabla v \cdot(\boldsymbol{\Gamma} \nabla u)]=\int_{\mathcal{Q}} v f+\int_{\Sigma} v \varphi, \tag{0.23}
\end{equation*}
$$

but it will be convenient to consider (0.22) first, at least formally, to obtain an ordinary differential equation for the spatial mean:

$$
\begin{equation*}
z=z(t):=\int_{\Omega} u(t, \cdot) /|\Omega| \tag{0.24}
\end{equation*}
$$

Setting $v=1$ in ( 0.22 ) one obtains

$$
\begin{equation*}
\dot{z}=f^{0}:=\left[\int_{\Omega} f+\int_{\partial \Omega} \varphi\right] /|\Omega|, \tag{0.25}
\end{equation*}
$$

to hold a.e. on $\mathbb{P}$; note that ( 0.25 ) is solvable on $\mathbb{P}$ if and only if ( 0.9 ) holds.
The appearance of $\boldsymbol{\Gamma}$ in (0.23) and the coercivity condition (0.19) suggest seeking a solution $u$ for which $\nabla u$ is in $L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$, i.e., such that $u$ is in

$$
\mathscr{X}:=L^{p}(\mathbb{P} \rightarrow \mathcal{X}) \quad \text { with } \mathcal{X}:=W^{1, p}(\Omega)
$$

One easily sees that

$$
\begin{array}{rll}
\mathcal{X}=\mathbb{R} \oplus \mathcal{X}_{0} & \text { with } & \mathcal{X}_{0}:=\left\{v \in \mathcal{X}: \int_{\Omega} v=0\right\}, \\
\mathbb{X}=\mathbb{y} \oplus \mathbb{X}_{0} & \text { with } & \mathbb{X}_{0}:=L^{p}\left(\mathbb{P} \rightarrow \mathcal{X}_{0}\right), \quad \mathbb{y}:=L^{p}(\mathbb{P}) .
\end{array}
$$

Note that $\|v\|_{\mathcal{X}}:=\|\nabla v\|_{p}\left(L^{p}\right.$-norm on $\Omega$ for $\left.\nabla v\right)$ is a suitable norm for $\mathbb{X}_{0}$. This decomposition permits us to look for $u$ in the form

$$
u=z+\bar{u} \text { with } z \in \mathscr{y} \text { and } \bar{u} \in \mathscr{X}_{0}
$$

(Since $u-\bar{u}=z$ is constant on $\Omega$ for each $t \in \mathbb{P}$, we have $\nabla \bar{u}=\nabla u$ and, of course, (0.24) gives $\int_{\Omega} \bar{u}=0$ for each $t$ so having $\bar{u} \in \mathbb{X}_{0}$ is equivalent to having $\bar{u} \in \mathbb{X}$.)

From the hypotheses (0.13-i, ii), if $\nabla u$ is in $L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ then $\Gamma \nabla u$ is in $L^{q}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$; hence one considers test functions $v$ for which $\nabla v$ is in $L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$, i.e., $v \in \mathscr{X}$. Using (0.25) one sees that $\bar{u}$ must satisfy

$$
\begin{equation*}
\int_{\mathcal{Q}}[v \dot{\bar{u}}+\nabla v \cdot(\boldsymbol{\Gamma} \nabla \bar{u})]=\int_{\mathcal{Q}} v \bar{f}+\int_{\partial \Omega} v \varphi \tag{0.26}
\end{equation*}
$$

with $\bar{f}:=f-f^{0}$. Note that this definition of $\bar{f}$ gives

$$
\begin{equation*}
\int_{\Omega} \bar{f}+\int_{\partial \Omega} \varphi=0 \quad \text { a.e. on } \mathbb{P} \tag{0.27}
\end{equation*}
$$

and, since $=\partial_{t}$ is a closed (unbounded) operator: $\mathbb{X}^{\boldsymbol{Z}}$ to $\mathbb{X}_{0}^{*}=L^{q}\left(\mathbb{P} \rightarrow \mathcal{X}_{0}^{*}\right)$, one has

$$
\int_{\Omega} \bar{u}=0 \text { a.e. on } \mathbb{P} \text {. }
$$

The effect of this with (0.27) is to make (0.26) invariant under addition to $v$ of a spatially constant function. Thus, having ( 0.26 ) for all $v \in \mathscr{X}_{0}$ is equivalent to having ( 0.26 ) for all $v \in \mathbb{X}$ and we take $\mathscr{X}_{0}$ as our space of suitable test functions.

Looking at the right hand side of $(0.26)$ for $v \in \mathbb{X}_{0}$, we see that the data $f, \varphi$ must be such that (0.25) makes sense and that

$$
\begin{equation*}
\psi: v \mapsto\left[\int_{\mathcal{Q}} v \bar{f}+\int_{\partial \Omega} v \varphi\right] \quad\left(\bar{f}:=f-f^{0}\right) \tag{0.28}
\end{equation*}
$$

is in $\mathbb{X}_{0}^{*}$; we will wish to topologize $\boldsymbol{\psi}$ in $\mathbb{X}_{0}^{*}$. From the Sobolev Embedding Theorem [1] and then using duality, we have

$$
\begin{array}{rll}
\mathcal{X}=W^{1, p}(\Omega) & \hookrightarrow L^{\tilde{p}}(\Omega) & \text { with } 1 / \tilde{p}=\max \{0,1 / p-1 / m\} \\
L^{\tilde{q}}(\Omega) & \hookrightarrow \mathcal{X}_{0}^{*} &  \tag{0.29}\\
\mathcal{F}_{0}:=L^{q}\left(\mathbb{P} \rightarrow L^{\tilde{q}}(\Omega)\right) & \hookrightarrow \mathbb{X}_{0}^{*} & \text { with } 1 / \tilde{q}+1 / \tilde{p}=1 .
\end{array}
$$

(continuous embeddings). Similarly, for the trace $\tau:\left.v \mapsto v\right|_{\partial \Omega}$ and its adjoint one has [1] continuity of

$$
\begin{aligned}
& \tau: \mathcal{X} \rightarrow L^{\hat{p}}(\partial \Omega) \quad \text { with } \hat{p}:=\left\{\frac{m-1}{m-p} p \text { if } p<m ; \infty \text { else }\right\} \\
& \tau^{*}: \mathscr{F}_{1}:=L^{q}\left(\mathbb{P} \rightarrow L^{\hat{p}}(\partial \Omega)\right) \rightarrow \mathscr{X}_{0}^{*} \quad \text { with } 1 / \hat{q}+1 / \hat{p}=1 .
\end{aligned}
$$

We assume, then, that the data satisfy

$$
\begin{equation*}
[f, \varphi] \in \mathscr{F}_{0} \times \mathscr{F}_{1}=L^{q}\left(\mathbb{P} \rightarrow L^{\tilde{q}}(\Omega) \times L^{\hat{q}}(\partial \Omega)\right) \tag{0.30}
\end{equation*}
$$

with $\tilde{q}, \hat{q}$ as above. Setting ${ }^{4}$

$$
\begin{equation*}
\mathbb{y}^{*}:=L^{q}(\mathbb{P}), \quad \mathbb{y}_{0}^{*}:=\left\{y \in \mathbb{y}^{*}: \int_{\mathbb{P}} y=0\right\} \tag{0.31}
\end{equation*}
$$

we note that (0.30) ensures that the consistency condition (0.9) is meaningful and that imposing it gives $f^{0} \in \mathbb{D}_{0}^{*}$. Thus, the map:

$$
\begin{aligned}
{[f, \varphi] } & \mapsto\left[f^{0}, \boldsymbol{\psi}\right] \\
\left\{[f, \varphi] \in \mathscr{F}_{0} \times \mathscr{F}_{1}:(0.9)\right\} & \rightarrow \mathcal{X}_{0}^{*} \times \mathbb{X}_{0}^{*}
\end{aligned}
$$

is well-defined and continuous. Assuming (0.30) with (0.9) and (0.31), we turn to $\left[f^{0}, \boldsymbol{\psi}\right] \in$ $\mathbb{D}_{0} \times \mathbb{X}_{0}^{*}$ as the principal formulation of the data. With this, the problem becomes

$$
\begin{align*}
& \text { (i) } \quad \int_{\mathcal{Q}}[v \dot{\bar{u}}+\nabla v \cdot(\boldsymbol{\Gamma} \nabla \bar{u})]=\boldsymbol{\psi} v \quad \text { for all } v \in \mathbb{X}_{0}  \tag{0.32}\\
& \text { (ii) } \quad \dot{z}=f^{0} \text { on } \mathbb{P},
\end{align*}
$$

given $\left[f^{0}, \boldsymbol{\psi}\right] \in \mathbb{D}_{0}^{*} \times \mathbb{y}_{0}^{*}$.
It will be convenient to make one final reformulation of $(0.32-i)$. It has already been noted that

$$
\partial_{t}: \mathscr{X}_{0} \rightarrow \mathscr{X}_{0}^{*}
$$

is a (densely defined) closed operator. One easily sees (working initially with smooth functions) that $\partial_{t}$ is skew adjoint in view of the periodicity inherent in the consideration of functions defined on $\mathcal{Q}$. If we then define an operator $\mathbf{A}: \mathbb{X}_{0} \rightarrow \mathbb{X}_{0}^{*}$ by

$$
\begin{equation*}
\mathbf{A} u: v \mapsto \int_{\mathcal{Q}} \nabla v \cdot(\boldsymbol{\Gamma} \nabla u)=\int_{\mathcal{Q}} a(\cdot,|\nabla u|) \nabla u \cdot \nabla v, \tag{0.33}
\end{equation*}
$$

we see that the continuity of $\boldsymbol{\Gamma}$ in (0.21) is just equivalent to continuity of the operator $\mathbf{A}: \mathscr{X}_{0} \rightarrow X_{0}^{*}$ and that

$$
\begin{gather*}
{[\mathbf{A} u-\mathbf{A} v](u-v)=\int_{\mathcal{Q}}[\boldsymbol{\Gamma} \nabla u-\boldsymbol{\Gamma} \nabla v] \cdot[\nabla u-\nabla v]}  \tag{0.34}\\
=B_{g}(\nabla u, \nabla v)
\end{gather*}
$$

for $u, v \in \mathbb{X}_{0}$. The equation (0.32-i) now takes the equivalent form

$$
\begin{equation*}
\left(\partial_{t}+\mathbf{A}\right) \bar{u}=\boldsymbol{\psi} \text { with } \bar{u} \in \mathcal{D}\left(\partial_{t}\right) \subset \mathscr{X}_{0} . \tag{0.35}
\end{equation*}
$$

[^3]Our final formulation of the "simple" problem (0.7) is then the system (0.35), (0.32-ii) for $\bar{u}, z-\operatorname{giving} u=\bar{u}+z$.

We turn next to the more involved problem corresponding to (0.6) with (0.3), i.e., (0.8). Suppose, for $w \in \mathbb{X}$, we set

$$
\begin{align*}
\text { (i) } \quad a_{w}(\cdot, r) & :=a(\cdot, w(\cdot), r) \quad\left(r \in \mathbb{R}^{+}\right),  \tag{i}\\
& g_{w}(\cdot, r)  \tag{0.36}\\
\text { (ii) } \quad f_{w}(\cdot) & :=f(\cdot, w(\cdot), \nabla w(\cdot), r)
\end{align*}
$$

on $\mathcal{Q}$. We cannot simply replace $a, f$ in (0.7) by $a_{w}, f_{w}$ as we might expect in approaching (0.8) as a fixed point problem: in general the consistency condition (0.9) would not be satisfied. Thus, from $f_{w}$ as in $(0.36-i i)$ and the given $\varphi \in \mathscr{F}_{1}$, we use a slight modification ${ }^{5}$ of our earlier definitions of $f^{0}, \bar{f}, \boldsymbol{\psi}$ and set

$$
\begin{align*}
f_{w}^{0} & :=\left[\left(\int_{\Omega} f_{w}+\int_{\partial \Omega} \varphi\right)-\left(\int_{\mathcal{Q}} f_{w}+\int_{\Sigma} \varphi\right)\right] /|\Omega| \\
\bar{f}_{w} & :=f_{w}-\left[\int_{\Omega} f_{w}+\int_{\partial \Omega} \varphi\right] /|\Omega|  \tag{0.37}\\
\boldsymbol{\psi}_{w} & :=v \mapsto\left[\int_{\mathcal{Q}} v \bar{f}_{w}+\int_{\Sigma} v \varphi\right] .
\end{align*}
$$

It is then possible to consider the system

$$
\begin{align*}
& \text { (i) }\left(\partial_{t}+\mathbf{A}_{w}\right) \bar{u}=\boldsymbol{\psi}_{w} \text { with } \bar{u} \in \mathcal{D}\left(\partial_{t}\right) \subset \mathbb{X}_{0} \\
& \text { (ii) } \dot{z}_{0}=f_{w}^{0} \text { with } \int_{\mathbb{P}} z_{0}=0 \tag{0.38}
\end{align*}
$$

where $\mathbf{A}_{w}$ is defined from $a_{w}$ as in (0.1). From our earlier analysis of (0.35) it is clear that $u$ will satisfy (0.8) if and only if
(i) $\left[f_{u}, \varphi\right]$ satisfies (0.9) so (0.37) reduces to (0.25), etc.
(ii) $\left[\bar{u}, z_{0}\right]$ satisfies (0.38) with $w=u$,
(iii) $\left(u-\left[\bar{u}+z_{0}\right]\right)$ is constant on $\mathcal{Q}$.

The final aspect of our construction - relating the nonuniqueness of the constant in (0.39$i i i)$ to the requirement $(0.39-i i)$ - will be deferred to the existence proof of the next section but we describe here the sets of functions $a, f$ for which our argument works. ${ }^{6}$

We are assuming that $\Omega, p$, and $\varphi \in \mathscr{F}_{1}$ are fixed; we further introduce parameters $\vartheta, \gamma$ with

$$
\begin{align*}
& 0 \leq \gamma \leq 1, \gamma<p-1 \text { so } \bar{q}: p / \gamma>q, \\
& 1 / p+1 / q=1=1 / \bar{p}+1 / \bar{q} ; \quad \vartheta<1, \vartheta \leq \gamma . \tag{0.40}
\end{align*}
$$

Now define $\mathscr{F}_{0}^{\prime}$ as the set of $f$ such that $f: \mathcal{Q} \times \mathbb{R} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and a growth condition

$$
\begin{equation*}
|f(\cdot, s, \xi)| \leq f_{*}(\cdot)+C\left(|s|^{\vartheta}+|\xi|^{\gamma}\right) \tag{0.41}
\end{equation*}
$$

[^4]with $f_{*} \in L_{+}^{\bar{p}}(\mathcal{Q})$. Given $f \in \mathscr{F}_{0}^{\prime}$ and any $w \in \mathscr{X}$, we define $H_{w}: \mathbb{R} \rightarrow \mathbb{R}$ by
\[

$$
\begin{equation*}
H_{w}(c):=\int_{\mathcal{Q}} f_{w+c}+\int_{\Sigma} \varphi:=\int_{\mathcal{Q}} f(\cdot, w(\cdot)+c, \nabla w(\cdot))+\int_{\Sigma} \varphi . \tag{0.42}
\end{equation*}
$$

\]

By Krasnosel'skii's Theorem, one sees that the functional $[w, c] \mapsto H_{w}(c)$ is continuous from $\mathscr{X} \times \mathbb{R}$ to $\mathbb{R}$. We will assume that $f \in \mathscr{F}_{0}^{\prime}$ is such that (0.42) gives

$$
\begin{equation*}
\text { there are } \kappa_{0}, \kappa_{1} \text { such that } \tag{0.43}
\end{equation*}
$$

$$
\begin{equation*}
c \geq \kappa_{0}+\kappa_{1}\|w\|_{\nrightarrow \mathcal{W}} \Rightarrow H_{w}(-c) \leq 0 \leq H_{w}(c) \tag{i}
\end{equation*}
$$

(ii) for each $w \in \mathbb{X}$ the function $H_{w}(\cdot)$ is nondecreasing.

With $\Omega, p$ as above, we define $\boldsymbol{A}$ as the set of functions $a$ such that
$a: \mathcal{Q} \times \mathbb{R} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$satisfies Carathéodory conditions
(i) and a growth condition

$$
\begin{equation*}
\left.0 \leq a(\cdot, s, r) r \leq g_{*}(\cdot)+C|s|+r\right)^{p-1} \quad \text { with } g_{*} \in L_{+}^{q}(\mathcal{Q}) \tag{0.44}
\end{equation*}
$$

(ii)
for each $w \in \mathscr{X}$ one has $g_{w} \in \mathcal{G}$ (as in (0.14));
write $\mu_{w}, \sigma_{w}, N_{w}$ for the functions as in (0.11), (0.12).
For any $a \in \mathcal{A}$ and $\nu \geq 0$ one can define $\bar{N}_{\nu}: \mathbb{R}^{+} \rightarrow[0, \infty]$ by

$$
\bar{N}_{\nu}(\lambda):=\sup \left\{N_{w}(\lambda):\|w\|_{\notin} \leq \nu\right\} .
$$

This need not be finite but is certainly nondecreasing both in $\nu$ and in $\lambda$. We will further assume that $a \in \mathcal{A}$ is such that, using the same $\gamma$ as in (0.39), (0.40), one has ${ }^{7}$

$$
\begin{equation*}
\text { for each } \nu>0 \text { one has } \bar{N}_{\nu}(\cdot) \text { finite on an interval }[0, \bar{\lambda}) \tag{i}
\end{equation*}
$$ and $N_{\nu}(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$,

there is a function $\Lambda: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$such that

$$
\begin{equation*}
\left.\Lambda(\nu)=o\left(\nu^{p-1-\gamma}\right), \quad \bar{N}_{\nu}(1 / \Lambda)\right)=o\left(\nu^{p}\right) \text { as } \nu \rightarrow \infty . \tag{ii}
\end{equation*}
$$

## 4. Results

In this section we state and prove the main results of the paper: Theorems 4.1 and 4.2, giving well-posedness for (0.7) and existence for (0.8).

It is of course, because we are working with a periodicity condition in time (rather than an initial condition) that the approach can be as similar as it is to that used in [8] for the elliptic problem. Indeed, the most significant differences come from the shift here to Neumann conditions rather than considering Dirichlet conditions as in [8]. We split the problem into a system so as to seek $\bar{u}$ in the space $\mathscr{X}_{0}$ for which $\xi:=\nabla \bar{u}$ determines $\bar{u}$ and we impose ( 0.43 ) to handle the consistency condition.

Without further mention, we assume that $p$ is fixed $(2 \leq p<\infty)$ and that $\Omega$ is a given bounded region in $\mathbb{R}^{m}$ with boundary $\partial \Omega$ smooth enough to justify ( 0.29 ). Thus $\mathcal{G}$ and sequential convergence in $\mathcal{G}$ are defined by (0.14), (0.15) with this $\Omega$ (hence $\mathcal{Q}$ ) and this $p$.

[^5]THEOREM 4.1: Suppose a set of data $[a, f, \varphi]$ is given with $a(\cdot)$ as in (0.13) and $[f, \varphi] \in \mathscr{F}_{0} \times \mathscr{F}_{1}$ as in (0.30) satisfying the consistency condition (0.9). Then there exists a weak solution $u \in \not X:=L^{p}\left(\mathbb{P} \rightarrow W^{1, p}(\Omega)\right)$ of the problem (0.7) in the sense that $u=\bar{u}+z$ satisfies (0.32). This solution is unique to within an additive constant; we impose the auxiliary condition (0.10) to ensure uniqueness. Further, if $\left\{\left[a_{k}, f_{k}, \varphi_{k}\right]: k=1,2, \ldots\right\}$ is any sequence of data sets with $g_{k} \rightarrow g$ in the sense $\mathcal{G}$ (corresponding to $a_{k}, a$ as in (0.13-ii), $f_{k} \rightarrow f$ in $\mathscr{F}_{0}$ and $\varphi_{k} \rightarrow \varphi$ in $\mathscr{F}_{1}$, then the corresponding solutions $u_{k}$ converge in $\mathbb{X}$ to $u$.

THEOREM 4.2: Let $a(\cdot \cdot)$ be given in $\mathcal{A}$, satisfying (0.44), (0.45); let $\varphi$ be given in $\mathscr{F}_{1}$ and let $f(\cdot \cdot)$ be given in $\mathscr{F}_{0}^{\prime}$, satisfying (0.41), (0.43). Then there exists a weak solution $u$ of (0.8) in $\mathbb{X}$.

Proof of Theorem 4.1: Formulating the problem as (0.32), we note that the equations are decoupled and can be considered separately. For existence for ( $0.32-i$ ), re-written as (0.35), we may appeal to a result of Browder's [4], which we recall in a conveniently specialized form:

Let $\mathbf{L}$ be a closed, densely defined, skew-adjoint linear operator from a reflexive Banach space $\mathcal{V}$ to its dual $\mathcal{V}^{*}$ and let $\mathbf{A}: \mathcal{V} \rightarrow \mathcal{V}^{*}$ be coercive and maximal monotone. Then $(\mathbf{L}+\mathbf{A})$ is maximal monotone and surjective to $\mathcal{V}^{*}$.

The linear problem (0.32-ii), is trivial. We have a continuous (indeed, compact) linear map:

$$
\begin{equation*}
f^{0} \mapsto z: \mathbb{D}_{0}^{*} \rightarrow \mathbb{D}_{0}:=\left\{y \in \mathbb{y}:=L^{p}(\mathbb{P}): \int_{\mathbb{P}} y=0\right\} \tag{0.46}
\end{equation*}
$$

to obtain the unique mean-zero solution; all other solutions of ( $0.32-i i$ ) are obtained by adding arbitrary constants to the solution specified by (0.46). For ( 0.35 ) we have (from (0.34) and the skew-adjointness of $\partial_{t}$ )

$$
\begin{equation*}
\left[\left(\partial_{t}+\mathbf{A}\right) u-\left(\partial_{t}+\mathbf{A}\right) v\right](u-v)=B_{g}(\nabla u, \nabla v) \tag{0.47}
\end{equation*}
$$

which shows both monotonicity and coercivity of the operator

$$
\begin{equation*}
\left(\partial_{t}+\mathbf{A}\right): X_{0} \supset \mathcal{D}\left(\partial_{t}\right) \rightarrow \mathscr{X}_{0}^{*} \tag{0.48}
\end{equation*}
$$

in view of Theorem 2.3. Since $\mathbf{A}: \mathbb{X}_{0} \rightarrow \mathbb{X}_{0}^{*}$ is continuous and $\partial_{t}$ is closed, skew-adjoint, and densely defined, Browder's theorem [4] applies to give existence of a solution $\bar{u} \in \mathbb{X}_{0}$ for $(0.35)=(0.32-i)$. If one had two solutions $\bar{u}, \bar{v}$ of (0.35), then (0.47) gives $B(\nabla \bar{u}, \nabla \bar{v})=0$ and application of Theorem 2.3 gives $\nabla \bar{u}=\nabla \bar{v}$. For $\bar{u}, \bar{v} \in \mathbb{X}_{0}$, this means $\bar{u}=\bar{v}$ so the solution $\bar{u}$ of ( $0.32-i$ ) is unique. Adding, we obtain $u=\bar{u}+z$ as the unique solution of (0.7), (0.10).

Next consider a sequence of such problems (0.7), (0.10), determined by $\left\{\left[a_{k}, f_{k}, \varphi_{k}\right]\right\}$ with solutions $u_{k}$, decomposed as above into $u_{k}=\bar{u}_{k}+z_{k}$ with $\bar{u}_{k} \in \mathbb{X}_{0}, z_{k} \in \mathbb{y}_{0}$. For each $k$ we
obtain $\left[f_{k}^{0}, \boldsymbol{\psi}_{k}\right]$ from $\left[f_{k}, \varphi_{k}\right]$ by (0.25), (0.28) and let $\boldsymbol{\Gamma}_{k}, \mathbf{A}_{k}$ be operators defined using $a_{k}$; abusing notation slightly, we write $B_{k}(\cdot \cdot)$ for the corresponding forms. The hypotheses of this theorem give $g_{k} \rightarrow g$ in the sense of (0.15) - in particular, (0.13) holds with $C_{*}, g_{*}$ and $N_{*} \geq\|\sigma\|^{p}$ fixed - and $\left[f_{k}, \varphi_{k}\right] \rightarrow[f, \varphi]$ in $\mathscr{F}_{0} \times \mathscr{F}_{1}$ so $\left[f_{k}^{0}, \boldsymbol{\psi}_{k}\right] \rightarrow\left[f_{0}, \boldsymbol{\psi}\right]$ in $\mathbb{y}_{0}^{*} \times \mathbb{X}_{0}^{*}$. From the continuity of (0.46) it is immediate that $z_{k} \rightarrow z$ in $\boldsymbol{y}_{0}$ (which we now view as embedded in $\mathbb{X}$ ) so we need only show convergence $\bar{u}_{k} \rightarrow \bar{u}$ in $\mathbb{X}_{0} \hookrightarrow \mathbb{X}$.

Setting $\xi_{k}:=\nabla u_{k}, \xi:=\nabla u$ in $L^{p}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$, we have observed that the $L^{p}$-norm $\left\|\xi_{k}-\xi\right\|_{p}$ is equivalent on $\mathbb{X}_{0}$ to $\left\|u_{k}-u\right\|_{\nVdash}$. We have, then,

$$
\begin{aligned}
\left\|\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right\|_{\mathbf{x}_{0}^{*}}\left\|\xi_{k}-\xi\right\|_{p} & \geq\left[\left(\partial_{t}+\mathbf{A}\right) u_{k}-(\partial+\mathbf{A}) u\right]\left(u_{k}-u\right) \\
& =\left[\mathbf{A}_{k} u_{k}-\mathbf{A}_{k} u\right]\left(u_{k}-u\right)+\left[\mathbf{A}_{k} u-\mathbf{A} u\right]\left(u_{k}-u\right) \\
& =B_{k}\left(\xi_{k}, \xi\right)+\left\langle\mathbf{\Gamma}_{k} \xi-\mathbf{\Gamma} \xi, \xi_{k}-\xi\right\rangle
\end{aligned}
$$

so

$$
\begin{align*}
B_{k}\left(\xi_{k}, \xi\right) /\left\|\xi_{k}-\xi\right\|_{p} & \leq\left\|\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right\|+\left\|\boldsymbol{\Gamma}_{k} \xi-\boldsymbol{\Gamma} \xi\right\|_{q} \\
& =\left\|\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right\|+\left\|g_{k}(\cdot,|\xi|)-g(\cdot,|\xi|)\right\|_{q} \tag{0.49}
\end{align*}
$$

since, pointwise a.e. on $\mathcal{Q}$, we have $\left(\boldsymbol{\Gamma}_{k} \xi-\boldsymbol{\Gamma} \xi\right)=\left(g_{k}-g\right) \xi /|\xi|$.
Since we already know that $\boldsymbol{\psi}_{k} \rightarrow \boldsymbol{\psi}$ in $\mathbb{X}_{0}^{*}$, we see that the right hand side of (0.49) goes to 0 by $(0.13-i i)$ on setting $r(\cdot):=|\xi(\cdot)| \in L_{+}^{p}(\mathcal{Q})$. Applying Theorem 2.3 and noting that we may fix $\Phi=\Phi_{*}$ in (0.19) independently of $k$ since $N_{*}(\cdot)$ is fixed, (0.49) gives

$$
\left\|\xi_{k}-\xi\right\|_{p} \leq \Phi_{*}\left(B_{k}\left(\xi_{k}, \xi\right) /\left\|\xi_{k}-\xi\right\|\right) \rightarrow 0
$$

Hence, $\bar{u}_{k} \rightarrow \bar{u}$ in $X_{0}$ whence $u_{k} \rightarrow u$ in $\not X$.
Before proceeding to the proof of Theorem 4.2 we introduce the space $\mathcal{U}:=\mathcal{D}\left(\partial_{t}\right):=$ $\left\{u \in \mathscr{X}_{0}: \dot{u} \in \mathscr{X}_{0}^{*}\right\}$ with the norm

$$
\|u\|_{\mathcal{U}}:=\|\nabla u\|_{p}+\|\dot{u}\|_{x_{0}^{*}},
$$

essentially the graph norm of $\partial_{t}: \mathbb{X}_{0} \rightarrow \mathbb{X}_{0}^{*}$. We showed, above, existence of a unique solution $\bar{u} \in \mathcal{U}$ of (0.35) and now obtain a bound.

LEMMA 4.3: Consider (0.35) with A obtained, as above, from a(..) giving $g \in \mathcal{G}_{*}$ - i.e., (0.14) with $\|\sigma(\cdot)\|^{p} \leq N_{*}(\cdot)$. Then there is a bound, depending only on $\mathcal{D}_{*}$ and the $\mathbb{X}_{0}^{*}$-norm of $\boldsymbol{\psi}$, for the $\mathcal{U}$-norm of the solution $\bar{u}$ of (0.35).

Proof: $\quad$ Setting $\xi:=\nabla \bar{u}$, one has

$$
\begin{aligned}
B(\xi, 0) & :=\langle\boldsymbol{\Gamma} \xi-\boldsymbol{\Gamma} 0, \xi-0\rangle=\langle\boldsymbol{\Gamma} \xi, \xi\rangle \\
& =\left[\mathbf{A} \bar{u} \bar{u} \bar{u}=\left[\left(\partial_{t}+\mathbf{A}\right) \bar{u}\right] \bar{u}=\boldsymbol{\psi} \bar{u}\right. \\
& \leq\|\boldsymbol{\psi}\|\|\bar{u}\|=\|\boldsymbol{\psi}\|\|\xi\|,
\end{aligned}
$$

using the $L^{p}(\mathcal{Q})-$ norm of $\xi$ and the $\mathbb{X}_{0}$-norm of $\bar{u}$ and the corresponding $\mathbb{X}_{0}^{*}$-norm for $\boldsymbol{\psi}$. From (0.19) we have then

$$
\|\bar{u}\|=\|\xi\| \leq \Phi(B(\xi, 0) /\|\xi\|) \leq \Phi(\|\boldsymbol{\psi}\|) .
$$

Next, one easily sees that the norm of $\mathbf{A} \bar{u}$ in $\mathbb{X}_{0}^{*}$ is just the norm of $\mathbf{\Gamma} \xi$ in $L^{q}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ which, in turn, is just the norm of $g(\cdot,|\xi|)$ in $L^{q}(\mathcal{Q})$. Thus, re-writing the equation (0.32-i) as $\dot{\bar{u}}=\boldsymbol{\psi}-\mathbf{A} \bar{u}$, one obtains an inequality for the $\mathbb{X}_{0^{*}}^{*}$-norm of $\dot{\bar{u}}$ : we have

$$
\begin{aligned}
\|\dot{\bar{u}}\| & \leq\|\boldsymbol{\psi}\|+\|g(\cdot,|\xi|)\|_{q} \\
& \leq\|\boldsymbol{\psi}\|+\left\|g_{*}\right\|_{q}+C_{*}\|\xi\|^{p-1}
\end{aligned}
$$

using the growth condition (0.13-ii), and this bounds $\bar{u}$ in $\mathcal{U}$ as desired.

Proof of Theorem 4.2: We will prove existence by suitably applying Glicksberg's generalization [5] of the Schauder Fixpoint Theorem:

Let $\mathbf{T}$ map points of a compact, convex subset $\mathcal{K}$ of a complete topological vector space $\mathcal{Z}$ to nonempty convex subsets of $\mathcal{K}$; assume the graph of $\mathbf{T}$ is closed. Then there exists at least one fixpoint $x \in \mathcal{K}$ such that $x \in \mathbf{T} x$.
Following the previous discussion we take

$$
\begin{aligned}
\mathcal{Z} & :=\mathbb{R} \oplus \mathbb{y}_{0} \oplus \mathscr{X}_{0} \quad \text { with the norm } \\
\|w\|_{\mathcal{Z}} & :=\left[\left|\int_{\mathcal{Q}} w\right|^{p}+\int_{\mathbb{P}}\left|w-\int_{\Omega} w\right|^{p}+\int_{\mathcal{Q}}|\nabla w|^{p}\right]^{1 / p} .
\end{aligned}
$$

We then construct $\mathbf{T}_{0}: \mathcal{Z} \rightarrow \mathbb{y}_{0}+\mathscr{X}_{0}$ by

$$
\begin{align*}
w \mapsto\left[a_{w}, f_{w}\right] & \mapsto\left[a_{w}, f_{w}^{0}, \boldsymbol{\psi}_{w}\right] \text { as in (0.36), (0.37) }  \tag{0.50}\\
& \mapsto \mathbf{T}_{0} w:=\bar{u}+z_{0} \text { as in (0.38) }
\end{align*}
$$

where we are using the specified Neumann data $\varphi$. Finally, we define $\mathbf{T} w$ from $\tilde{u}:=\mathbf{T}_{0} w$ by

$$
\begin{equation*}
\mathbf{T} w:=\left\{\tilde{u}+c: H_{\tilde{u}}(c)=0,|c| \leq \kappa_{0}+\kappa_{1}\|\tilde{u}\|\right\} \tag{0.51}
\end{equation*}
$$

with $\kappa_{0}, \kappa_{1}$ as in ( $0.43-i$ ). The argument falls naturally into four parts:
(A) Show $\mathbf{T}$ is well-defined on $\mathcal{Z}$ and that a fixpoint of $\mathbf{T}$ solves the problem,
(B) Find a convex set $\mathcal{B} \subset \mathcal{Z}$ invariant under $\mathbf{T}$, i.e., such that $\mathbf{T \mathcal { B }}:=\{u \in \mathbf{T} w: w \in$ $\mathcal{B}\} \subset \mathcal{B}$,
(C) Show that the graph of $\mathbf{T}$ is closed,
(D) Show that $\mathbf{T B}$ is precompact in $\mathcal{Z}$ so, setting $\mathcal{K}:=[$ closed convex hull of $\mathbf{T B}]$, one has $\mathcal{K}$ compact, convex and $\mathbf{T} \mathcal{K} \subset \mathcal{K}$
from which the result is then immediate.
(A) Definition: The hypotheses $(0.44),(0.45)$ for $a(\cdot \cdot)$ and ( 0.41 ), ( 0.43 ) for $f(\cdot \cdot)$ ensure that $a_{w}$ satisfies the hypotheses of Theorem 4.1 regarding $a$ and that $f_{w} \in \mathscr{F}_{0}$ for each $w \in \mathcal{Z}$. Using (0.37) one obtains $\left[f_{w}^{0}, \boldsymbol{\psi}_{w}\right]$ in $\mathbb{y}_{0}^{*} \times \mathbb{X}_{0}^{*}$ from $\left[f_{w}, \varphi\right]$ so, as in Theorem 4.1, there is a
unique solution $z_{0}+\bar{u}=: \tilde{u}=: \mathbf{T}_{0} w$ in $\mathbb{y}_{0} \oplus \mathscr{X}_{0}$. The hypothesis (0.43-i) ensures that the right hand side of ( 0.51 ), defining $\mathbf{T} w$ is a nonempty convex set, as desired.

In (0.51) $\tilde{u}:=\mathbf{T}_{0} w$ is given by (0.32), with $f^{0}, \boldsymbol{\psi}$ modified slightly from what would be needed for ( 0.8 ) - as noted in the footnote ${ }^{4}$. This modification is nugatory for $w$ such that $\left[f_{w}, \varphi\right]$ satisfies (0.9) and we note that the definitions (0.42), (0.51) just ensure that this is the case for any $w$ in the range of $\mathbf{T}$. Hence, for a fixpoint $w$ of $\mathbf{T}$ one necessarily has consistency so ( 0.37 ) gives $f_{w}^{0}$ from $\left[f_{w}, \varphi\right]$ exactly as in ( 0.25 ) whence $\tilde{u}$ satisfies (the formulation (0.32) of)

$$
\begin{equation*}
\dot{\tilde{u}}-\nabla \cdot a(\cdot, w,|\nabla \tilde{u}|) \nabla \tilde{u}=f(\cdot, w, \nabla w) \tag{0.52}
\end{equation*}
$$

with (0.3), (0.10). Since $(u-\tilde{u})$ is constant on $\mathcal{Q}$ for any $u \in \mathbf{T} w$, one has $\dot{u}=\dot{\tilde{u}}$ and $\nabla u=\nabla \tilde{u}$. In particular, for $u=w(=$ fixpoint: $w \in \mathbf{T} w)$, the definition (0.52) becomes (0.8); compare (0.39).
(B) Invariance: We will construct $\mathcal{B}$ in the form

$$
\mathcal{B}=\mathcal{B}(\alpha, \beta):=\left\{w \in \mathcal{Z}:\|\nabla w\|_{p} \leq \alpha,\|w\|_{p} \leq \beta\right\}
$$

showing that this is invariant for suitable $\alpha, \beta$. Note that $\nu:=\|w\|_{\mathcal{Z}} \leq \alpha+\beta$. For convenience, we set $\mathbf{T}_{0}:=\mathbf{T}_{1} \oplus \mathbf{T}_{2}$ with

$$
\mathbf{T}_{1}: w \mapsto f_{w}^{0} \mapsto z_{0}, \quad \mathbf{T}_{2}: w \mapsto\left[a_{w}, \boldsymbol{\psi}_{w}\right] \mapsto \bar{u}
$$

noting that $u \in \mathbf{T} w\left(u=c+z_{0}+\bar{u}\right)$ gives

$$
\begin{array}{ll}
\bar{\alpha}:=\|\nabla u\|_{p} & =\left\|\nabla\left(\mathbf{T}_{2} w\right)\right\|, \\
\bar{\beta}:=\|u\|_{p} & \leq|\Omega|\|c \mid+\| z_{0}\|+\| \bar{u} \| \leq C\left(\left\|\mathbf{T}_{1} w\right\|+\bar{\alpha}\right)
\end{array}
$$

since $|c| \leq \kappa_{0}+\kappa_{1}\left\|\mathbf{T}_{0} w\right\|$.
For $\bar{u}=\mathbf{T}_{2} w$, set $\xi:=\nabla \bar{u}$ so we have (0.49) with $\Phi=\Phi_{\nu}$ obtained from $\bar{N}_{\nu}$ as in (0.20). Bounding the infimum in (0.20) by the choice $\lambda=1 / \Lambda(\nu)$, as in ( $0.45-i i$ ) and noting that (0.41) gives

$$
\rho_{\nu}:=\sup \left\{\left\|\boldsymbol{\psi}_{w}\right\|:\|w\|_{\mathcal{Z}} \leq \nu\right\}=\mathcal{O}\left(\nu^{\gamma}\right)
$$

we have, for any $w \in \mathcal{B}$ (so $\nu \leq \alpha+\beta$ ) and $\bar{u}=\mathbf{T}_{2} w$,

$$
\begin{aligned}
\bar{\alpha} & =\|\xi\| \leq \Phi_{\nu}\left(\rho_{\nu}\right)=: \tilde{\Phi}_{\nu} \\
\tilde{\Phi}_{\nu}^{p} & \leq 2^{p+1} \bar{N}_{\nu}(1 / \Lambda(\nu))+\left(2 / C_{p}\right) \rho_{\nu} \tilde{\Phi}_{\nu} \Lambda(\nu) \\
& =o\left(\nu^{p}\right)+\tilde{\Phi}_{\nu}\left(o\left(\nu^{p-1}\right)\right)
\end{aligned}
$$

from which we conclude that

$$
\begin{equation*}
\bar{\alpha} \leq \tilde{\Phi}_{\nu}=o(\nu)=o(\alpha+\beta) \tag{0.53}
\end{equation*}
$$

as $\nu \rightarrow \infty(\alpha+\beta \rightarrow \infty)$.
From the growth condition (0.41) one can bound $f_{w}$ in $L^{q}(\mathcal{Q})$, whence also $f_{w}^{0}$ since $f_{w} \mapsto f_{w}^{0}$ is affine with $\varphi$ fixed:

$$
\left\|f_{w}^{0}\right\| \leq A+B \alpha+C \beta^{\vartheta}
$$

since $\gamma \leq 1$. Thus, as the map: $f^{0} \mapsto z$ defined by (0.38-ii) is bounded and linear, one has

$$
\begin{align*}
\left\|\mathbf{T}_{1} w\right\| & =\mathcal{O}(\alpha)+o(\beta) \\
\bar{\beta} & =\mathcal{O}(\alpha)+o(\beta)+\mathcal{O}(\bar{\alpha}) \tag{0.54}
\end{align*}
$$

as $\alpha, \beta \rightarrow \infty$. For some fixed $C$ and for $\varepsilon$ arbitrarily small, one can combine (0.53), (0.54) to obtain

$$
\begin{equation*}
\bar{\alpha} \leq \varepsilon \alpha+\varepsilon \beta, \quad \bar{\beta} \leq C \alpha+\varepsilon \beta \tag{0.55}
\end{equation*}
$$

for $\alpha, \beta$ large enough. One easily sees that, taking $\varepsilon$ to be the smaller root of the quadratic $(1-\varepsilon)^{2}=C \varepsilon($ so $0<\varepsilon<1)$, one can take $\alpha, \beta$ such that $\beta / \alpha=(1-\varepsilon) / \varepsilon=C(1-\varepsilon)$ in (0.55) with $\alpha, \beta$ large enough to obtain (0.55) from (0.53), (0.54) These choices give

$$
\bar{\alpha} \leq \varepsilon \alpha+\varepsilon \beta=\alpha, \quad \bar{\beta} \leq C \alpha+\varepsilon \beta=\beta,
$$

which just gives the invariance of $\mathcal{B}$.
(C) Closed Graph: We first show the continuity of $\mathbf{T}_{0}$. Note that $w \mapsto f_{w}$ is continuous by Krasnoselskii's Theorem, so boundedness of the affine map: $f_{w} \mapsto f_{w}^{0} \mapsto z_{0}$ gives continuity of $\mathbf{T}_{1}: \mathcal{Z} \rightarrow \mathbb{y}_{0}$. Now suppose $w_{k} \rightarrow w$ in $\mathcal{Z}$. With a self-explanatory notation we have

$$
\begin{align*}
{\left[\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right]\left(\bar{u}_{k}-\bar{u}\right) } & =\left[\left(\partial_{t}-\mathbf{A}_{k}\right) \bar{u}_{k}-\left(\partial_{t}-\mathbf{A}\right) \bar{u}\right]\left(\bar{u}_{k}-\bar{u}\right) \\
& =\left\langle\boldsymbol{\Gamma}_{k} \xi_{k}-\boldsymbol{\Gamma} \xi, \xi_{k}-\xi\right\rangle  \tag{0.56}\\
& =B_{k}\left(\xi_{k}, \xi\right)+\left\langle\boldsymbol{\Gamma}_{k} \xi-\boldsymbol{\Gamma} \xi, \xi_{k}-\xi\right) .
\end{align*}
$$

Thus

$$
\begin{aligned}
\frac{B_{k}\left(\xi_{k}, \xi\right)}{\left\|\xi_{k}-\xi\right\|_{p}} & \leq\left\|\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right\|_{\mathbb{X}_{0}^{*}}+\left\|\boldsymbol{\Gamma}_{k} \xi-\boldsymbol{\Gamma} \xi\right\|_{q} \\
& =\left\|\boldsymbol{\psi}_{k}-\boldsymbol{\psi}\right\|+\left\|g\left(\cdot, w_{k},|\xi|\right)-g(\cdot, w,|\xi|)\right\|_{q}=: \delta_{k}
\end{aligned}
$$

which goes to 0 as $w_{k} \rightarrow w$ by Krasnoselskii's Theorem and the continuity of the map: $f_{w} \mapsto \boldsymbol{\psi}_{w}: \mathcal{Z} \rightarrow \mathbb{X}_{0}^{*}$. Thus

$$
\left\|\mathbf{T}_{2} w_{k}-\mathbf{T}_{2} w\right\|=\left\|\xi_{k}-\xi\right\|_{p} \leq \bar{\Phi}_{\nu}\left(\delta_{k}\right) \rightarrow 0
$$

with $\bar{\Phi}_{\nu}$ obtained, as in (0.20), from $\bar{N}_{\nu}$ with $\nu$ a bound for $\left\{w_{k}\right\}$ in $\mathcal{Z}$.
Finally, the continuity of $[\tilde{u}, c] \mapsto H_{\tilde{u}}(c):\left(\mathbb{y}_{0} \oplus \mathbb{X}_{0}\right) \times \mathbb{R} \rightarrow \mathbb{R}$ ensures that, if $u_{k}=$ $\tilde{u}_{k}+c_{k} \in \mathbf{T} w_{k}$ converges to $u=\tilde{u}+c$ while $w_{k} \rightarrow w$, then $\tilde{u}_{k} \rightarrow \tilde{u}=\mathbf{T}_{0} w$ and $c_{k} \rightarrow c$ whence $H_{k}\left(c_{k}\right) \rightarrow H_{\tilde{u}}(c)$ so $u \in \mathbf{T} w$ - i.e., the graph of $\mathbf{T}$ is closed in $\mathcal{Z} \times \mathcal{Z}$.
(D) Compactness: We must work slightly to obtain both invariance and compactness for $\mathcal{K}$. Note, first, that $\mathbf{T}_{2} \mathcal{B}$ is bounded in $\mathcal{U}$ by Lemma 4.3 and so is precompact in $L^{p}(\mathcal{Q})$ by the Aubin Compactness Theorem [2]. Also, $\mathbf{T}_{1} \mathcal{B}$ is precompact in $\mathbb{y}_{0}$ as $\left\{f_{w}^{0}: w \in \mathcal{B}\right\}$ is bounded in $\mathbb{D}_{0}^{*}$ and the linear map: $f_{w}^{0}=\dot{z}_{0} \mapsto z_{0}=\mathbf{T}_{1} w: \mathbb{y}_{0}^{*} \rightarrow \mathbb{y}_{0}$ is compact. Since $\mathbf{T B} \subset[-\nu, \nu] \oplus \mathbf{T}_{1} \mathcal{B} \oplus \mathbf{T}_{2} \mathcal{B}(\nu$ is a bound on $\mathcal{B}$ in $\mathcal{Z}$-norm $)$, this gives $\mathbf{T B}$ precompact in $L^{p}(\mathcal{Q})$. Thus,

$$
\mathcal{B}_{1}:=[\text { closed convex hull of } \mathbf{T B} \text { in } \mathcal{Z}]
$$

is precompact in $L^{p}(\mathcal{Q})$ and $\mathcal{B}_{1} \subset \mathcal{B}$ gives $\mathbf{T} \mathcal{B}_{1} \subset \mathbf{T} \mathcal{B} \subset \mathcal{B}_{1}$ so $\mathcal{B}_{1}$ is also convex, closed, and invariant. Similarly, setting

$$
\mathcal{K}:=\left[\text { closed convex hull of } \mathbf{T} \mathcal{B}_{1} \text { in } \mathcal{Z}\right],
$$

we have $\mathcal{K}$ convex, closed, and invariant; we will show $\mathcal{K}$ is precompact, hence compact. (Note that if we restrict $\mathbf{T}$ to $\mathcal{K}$ the graph will be closed in $\mathcal{K} \times \mathcal{K}$.)

Suppose $\left\{w_{k}\right\}$ is any sequence in $\mathcal{B}_{1}$ and $u_{k} \in \mathbf{T}_{k} w$ with $u_{k}=c_{k}+z_{k}+\bar{u}_{k}$. We may assume, extracting a subsequence if necessary, that
(i) $w_{k} \rightharpoonup \bar{w}$ (weak convergence in $\left.\mathcal{B}_{1} \subset \mathcal{Z}\right)$,
(ii) $w_{k} \rightarrow \bar{w}$ in $L^{p}(\mathcal{Q})$,
(iii) $f_{k}:=f\left(\cdot, w_{k}, \nabla w_{k}\right) \rightharpoonup \bar{f}\left(\right.$ weak convergence in $\left.L^{q}(\mathcal{Q})\right) .{ }^{8}$

As earlier, defining $g_{k}(\cdot, r):=g\left(\cdot, w_{k}, r\right)$, we have $g_{k} \rightarrow \bar{g}$ in the sense of $\mathcal{G}$ where $\bar{g}(\cdot, r):=$ $g(\cdot, \bar{w}, r)$; in particular, for fixed $r(\cdot) \in L^{p}(\mathcal{Q})$ one has $\left\|g_{k}(\cdot, r)-\bar{g}(\cdot, r)\right\|_{q} \rightarrow 0$ by duality, since the embedding $\mathcal{U} \rightarrow L^{p}(\mathcal{Q}) \rightarrow \mathcal{U}^{*}$. Thus (iii) above gives $\boldsymbol{\psi}_{k} \rightarrow \boldsymbol{\psi}$ in $\mathcal{U}^{*}$. Now let $\bar{u}$ be the (unique) solution in $\mathcal{D}\left(\partial_{t}\right) \subset \mathbb{X}_{0}$ of the equation $\left(\partial_{t}+\overline{\mathbf{A}}\right) \bar{u}=\overline{\boldsymbol{\psi}}$. Returning to (0.56), one has

$$
\begin{aligned}
B_{k}\left(\xi_{k}, \bar{\xi}\right) & =\left[\boldsymbol{\psi}_{k}-\overline{\boldsymbol{\psi}}\right]\left(\bar{u}_{k}-\bar{u}\right)-\left\langle\boldsymbol{\Gamma}_{k} \bar{\xi}-\overline{\mathbf{\Gamma}} \bar{\xi}, \xi_{k}-\bar{\xi}\right\rangle \\
& \leq\left[\left\|\boldsymbol{\psi}_{k}-\overline{\boldsymbol{\psi}}\right\|_{\mathcal{U}^{*}}\left\|\bar{u}_{k}-\bar{u}\right\|_{\mathcal{U}}+\left\|g_{k}(\cdot,|\bar{\xi}|)-\bar{g}(\cdot|\bar{\xi}|)\right\|_{q}\left\|\xi_{k}-\bar{\xi}\right\|_{p}\right.
\end{aligned}
$$

so $B_{k}\left(\xi_{k}, \bar{\xi}\right) \rightarrow 0$ since $\left\{\bar{u}_{k}\right\}$ is bounded in $\mathcal{U}$ by Lemma 4.3 . If, for some subsequence, $\left\|\xi_{k}-\bar{\xi}\right\| \rightarrow 0$ we are done: this means $\bar{u}_{k} \rightarrow \bar{u}$ in $\mathscr{X}_{0}$. On the other hand, if one could have $\left\|\xi_{k}-\bar{\xi}\right\|$ bounded away from 0 , then application of (0.19) with $\Phi=\bar{\Phi}_{\nu}$ would give $\xi_{k} \rightarrow \bar{\xi}$ in $L^{q}\left(\mathcal{Q} \rightarrow \mathbb{R}^{m}\right)$ so, in any case, (a subsequence of) $\left\{\mathbf{T}_{2} w_{k}\right\}$ is convergent in $\mathbb{X}_{0}$. This completes the argument that $\mathbf{T} \mathcal{B}_{1}$ is precompact, since each factor in $\mathbb{R} \times \mathbb{D}_{0} \times \mathscr{X}_{0}$ is precompact. Since we took $\mathcal{K}$ to be the closed convex hull of $\mathbf{T} \mathcal{B}_{1}$, we have $\mathcal{K}$ compact in $\mathcal{Z}$. Thus the Glicksberg Fixpoint Theorem [5] applies, as desired, to give existence of a solution.

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[^0]:    ${ }^{1}$ Adv. in Math. Sci. \& Appl. 9, pp. 539-555 (1999).

[^1]:    ${ }^{2}$ The regularity of $\partial \Omega$ is to be adequate to justify the trace and extension theorems employed. We take [1] as a general reference for the relevant hypotheses and results.

[^2]:    ${ }^{3}$ Part of the reason for restricting our attention here to $p \geq 2$ is that this convenient equivalence fails for $p<2$.

[^3]:    ${ }^{4}$ This should be read here as $\left(\boldsymbol{y}^{*}\right)_{0}$ rather than as $\left(\boldsymbol{y}_{0}\right)^{*}$, although $\boldsymbol{X}_{0}^{*}$ means $\left(\boldsymbol{X}_{0}\right)^{*}$.

[^4]:    ${ }^{5}$ Note that only the definition of $f^{0}$ has changed from (0.25) and even that is unchanged if $f$ already satisfies the consistency condition (0.9).
    ${ }^{6}$ These hypotheses can be compared with the corresponding hypotheses $(3.11),(3.12)$ of $[8]$; the only real novelty here is ( $0.43-i$ ).

[^5]:    ${ }^{7}$ We will only need that: for any weakly convergent sequence $\left\{w_{k}\right\}$ one has $N_{*}(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$ where $N_{*}(\lambda):=\sup \left\{N_{w}(\lambda): w \in\left\{w_{k}\right\}\right\}$. However, it can easily be shown that this is actually equivalent to the apparently stronger condition (0.45-i).

[^6]:    ${ }^{8}$ It need not be true that $\bar{f}=f_{\bar{w}}$ so, while we prove that $\left\{\mathbf{T}_{2} w_{k}\right\}$ is convergent, there is no suggestion that the limit is $\mathbf{T}_{2} \bar{w}$.

