# The 'window problem' for series of complex exponentials[1]

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## Abstract

Under a suitable sparsity condition on the exponents $\Lambda = \{\lambda_k = \tau_k + i\sigma_k\}$, it is shown that the individual terms $\mathbf{c}_T = \{c_k\, e^{i\lambda_k T}\}$ can be obtained from observation of the $L^2$ function $f(t) = \sum c_k\, e^{i\lambda_k t}$ through the 'window' $t \in [0, \delta]$ — with an $\ell^2$ estimate (uniform for such $\Lambda$) asymptotically as $T, \delta \to 0$. Some applications are given to control theory for partial differential equations.

Key Words: *exponential series, uniform estimate, window problem, asymptotic, distributed parameter control.*

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# 1. Introduction

For any fixed exponent sequence $\Lambda = \{\lambda_k = \tau_k + i\sigma_k\}$ in $\mathbb{C}_+$ (i.e., with[5] $\sigma_k \geq 0$), consider the set $\mathcal{M} = \mathcal{M}(\Lambda)$ of all complex functions $f$ expressible as finite sums of the form

$$f(t) = \sum_k c_k e^{i\lambda_k t} \tag{1.1}$$

for $t \in \mathbb{R}$. The 'window problem' of the title refers to the extraction of the sequence of individual terms

$$\mathbf{c}_T = \left(c_k e^{i\lambda_k T}\right)$$

(for some specified $T \geq 0$) from observation of $f$ 'through a window': restricting $t$ to a small interval $(0, \delta)$. Under appropriate hypotheses on $\Lambda$, i.e., assuming a 'separation condition' which we here express in the form:

$$\#\{\lambda \in \Lambda : 0 < |\lambda - \lambda_*| \leq r\} \leq \nu(r) \qquad \text{for each } \lambda_* \in \Lambda, \tag{1.2}$$

for a suitable function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$, the operator

$$\mathbf{C} = \mathbf{C}_\delta^T : f \mapsto \mathbf{c}_T : \mathcal{M}_\delta \to \ell^2 \tag{1.3}$$

will be continuous from $\mathcal{M}_\delta :=$[closure of $\mathcal{M}$ in $L^2(0, \delta)$] for any $\delta > 0$ and any $T > 0$.

For real $\{\lambda_k\}$ we would be considering 'nonharmonic Fourier series' in (1.1) while for purely imaginary $\{\lambda_k\}$ we would have Dirichlet series $\sum_k c_k e^{-\sigma_k t}$ (cf. [13]) and consideration of the Müntz-Szász Theorem for polynomials $\sum c_k x^{\sigma_k}$ (cf., e.g., [2]) on setting $x = e^{-t}$.

Our object is to verify continuity and to estimate the norm $\|\mathbf{C}_\delta^T\|$ — with especial concern for the asymptotics $\delta, T \to 0$, noting that $\|\mathbf{C}_\delta^T\|$ must blow up as $\delta \to 0$ and, since we are considering classes of sequences admitting unbounded $\{\sigma_k\}$, must also blow up as $T \to 0$.

[5] It is obviously sufficient to ask that $\sigma_k$ be bounded below, since an invertible multiplication of $f$ in (1.1) by $e^{-\sigma t}$ has the effect of shifting $\Lambda$ by $i\sigma$. The choice of lower bound 0 for $\sigma_k$ is purely for expository convenience, but note also (5:1.1).

**THEOREM 1:** *Given $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing with $\nu(s)/s^2$ integrable, let $\Lambda$ be any sequence in $\mathbb{C}_+$ satisfying the condition (1.2). Then, for any $\delta > 0$ and any $T > 0$ the map $\mathbf{C}_\delta^T$ defined by (1.3) is continuous: $\mathcal{M}_\delta \to \ell^2$ and we have an estimate*

$$\log \|\mathbf{C}_\delta^T\| \le Q := Q_1(\delta) + Q_2(T) + Q_* \tag{1.4}$$

*uniformly for such $\{\Lambda\}$, with $Q_1(\cdot)$, $Q_2(\cdot)$, and the constant $Q_*$ defined in terms of $\nu(\cdot)$ and a suitably chosen auxiliary function $\gamma(\cdot)$.*

The paper by W.A.J. Luxemburg and J. Korevaar [10], considered similar questions for $\Lambda$ 'close to imaginary' and with $T = \delta$, showing the (uniform) continuity of $\mathbf{C}$ although with no concern for the asymptotics. The papers [17] and [18] adapted the methods of [10], with a somewhat differently expressed separation condition for the sequence $\Lambda$, to consider real $\lambda_k$ and estimate $\|\mathbf{C}_\delta^0\|$ as $\delta \to 0$. [For real $\lambda_k$ the norm is independent of $T$. We note from those papers that the computation gives $Q_1(\delta) = \mathcal{O}(1/\delta)$ for a quadratically growing sequence ($\lambda_k \sim \pm ck^2$) and an example by Korevaar included in [17] indicates that this is sharp; more generally, it was shown in [18] that this becomes $\mathcal{O}(\delta^{-1/[p-1]})$ when $\lambda_k \sim \pm ck^p$ with $1 < p < \infty$.] The object of this paper is to extend that analysis to the consideration of complex exponent sequences $\lambda_k = \tau_k + i\sigma_k$ with $\sigma_k \ge 0$, for which the normalization implied here with $T > 0$ will be particularly appropriate.

We may note that much of our personal motivation for this investigation comes from the relation between exponential series such as (1.1) and considerations of control theory for distributed parameter systems; see [1] for a treatment of the theory and application of exponential families in this context. The context suggests an interpretation of $\mathbf{C}_\delta^T$ as related to an observation problem: *observing* some functional on the solution of a partial differential equation over a time interval $(0, \delta)$ in order to *predict* the solution state at a time $T$; see Section 6.

Nonharmonic Fourier series (of the form (1.1) with real $\lambda_k$) suffice for consideration of controllability/observability issues for the wave equation and for undamped rod [6] or plate [14], [7] equations. However, we note that treatment of the heat equation [11] involves such expansions with pure imaginary $\lambda_k$, i.e., Dirichlet series. Because of the smoothing associated with the heat equation, so the solution semigroup is compact, it is then important that one is ‘predicting’ the solution state at a time $T > 0$: typically one takes $T$ to be $\delta$, the end of the observation interval — as in [10], which was motivated by this problem through [11]. For the one-dimensional heat equation with $T = \delta$ (where we have $\lambda_k \sim ik^2$ for $k = 1, 2, \ldots$), we will obtain as in [15] an estimate $\mathcal{O}(1/\delta)$ for $\log \|\mathbf{C}_\delta^\delta\|$: an example by Güichal [4] shows that this is sharp. More general complex exponent sequences arise for consideration of damping for a plate model and we will also show how a control-theoretic result (cf., Hansen [5]) can be easily obtained for such problems (with an asymptotic estimate) by using the Theorem above.

## 2. Preliminaries

For the next sections we first fix $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ as in Theorem 1, i.e., such that

$$(2.1)\qquad \begin{array}{ll} (i) & \nu : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is nondecreasing with } \nu \equiv 0 \text{ on some } [0, r_0], \\ (ii) & \displaystyle\int_0^\infty [\nu(s)/s^2]\, ds < \infty. \end{array}$$

This is equivalent to the hypotheses on $\nu(\cdot)$ we already imposed in Theorem 1 since the requirement in (i) that $\nu$ vanish on some $[0, r_0)$ is actually redundant: in any case, by (ii) there must be $r_0 > 0$ with $\nu(r_0) < 1$ so $|\lambda - \lambda_*| \geq r_0$ for all pairs $\lambda, \lambda_* \in \Lambda$, i.e., the sequence must be uniformly separated and we can take $\nu$ vanishing on $[0, r_0]$ for this $r_0$. The integrability condition (ii) is also closely related to the standard condition that $\sum_k 1/|\lambda_k|$ be convergent — indeed, writing $\hat{\nu}(r)$ for the left hand side of (1.2), one has

$$\sum \left\{ \frac{1}{|\lambda - \lambda_*|} : \lambda_* \neq \lambda \in \Lambda \right\} = \int_{r_0}^\infty \frac{d\hat{\nu}(r)}{|r|} = \int_{r_0}^\infty \frac{\hat{\nu}(r)}{r^2}\, dr$$

by an integration by parts, noting that $\hat{\nu}(r)/r \leq \nu(r)/r \to 0$ at $\infty$ as in Lemma 1-(iv) below. In this section we introduce the class $\Omega$ of functions $\omega(\cdot)$ satisfying

$$(2.2)\qquad \begin{array}{ll} (i) & \omega : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous, increasing, and unbounded,} \\ (ii) & \omega(s)/s^2 \text{ is decreasing, with } \displaystyle\int_0^\infty [\omega(s)/s^2]\, ds =: B_\omega < \infty \end{array}$$

and provide some technical lemmas which we will need later. The last of these, designated a ‘Theorem’ (and almost the same as the construction which forms the heart of [10], [17], [18]), provides the construction of a 'mollifier function' $P(\cdot)$ with relevant properties.

**LEMMA 1:**
*(i)* *If $\omega \in \Omega$, then $\omega(s)/s \to 0$ as $s \to 0, \infty$, but is not integrable at $\infty$.*
*(ii)* *If $\omega \in \Omega$, then $\sup_{s>0}\{\omega(s)/s\} \le B_\omega < \infty$ and $0 \le \omega'(s) \le 2B_\omega$.*
*(iii)* *If $\gamma \in \Omega$ also increases rapidly enough that*

$$A_1 := \int_0^\infty s e^{-\gamma(s)}\, ds < \infty, \tag{2.3}$$

*then we have, for any $\mu > 0$,*

$$\int_0^\infty e^{-\gamma(\mu s)}\, d\tilde{\nu}(s) \le 2B_{\tilde{\nu}} B_\gamma A_1 \mu \tag{2.4}$$

*for any nondecreasing $\tilde{\nu}$ with $\tilde{\nu}(r)/r \le B_{\tilde{\nu}} < \infty$,*
*(iv)* *For $\nu(\cdot)$ satisfying (2.1), we have $\sup\{\nu(s)/s\} \le B_\nu$ with $\nu(s)/s \to 0$ as $s \to \infty$. Further, setting*

$$\vartheta(s) := 2\int_{r_0}^\infty \frac{\nu(r)}{r}\frac{s^2}{s^2+r^2}\, dr = -\int_0^\infty \nu(r)\, d\left[\log\left(1+\frac{s^2}{r^2}\right)\right]. \tag{2.5}$$

*we have $\vartheta \in \Omega$ and*

$$\vartheta(s) \le 2\left(\frac{R}{r_0}\right)^2 \int_R^\infty \frac{\nu(r)}{r}\frac{s^2}{s^2+r^2}\, dr \tag{2.6}$$

*for any $R > r_0$.*

PROOF: For $\omega \in \Omega$, we have $\omega(s)/s \ge \omega(a)/s$ on $[a, \infty)$ so $\omega(s)/s$ cannot be integrable at $\infty$. Since $\omega$ is increasing, we have $\omega(s)/s = \int_s^\infty \omega(s)/r^2\, dr < \int_s^\infty \omega(r)/r^2\, dr$ so $\omega(s)/s \to 0$ as $s \to \infty$. Similarly, we have $\omega(s)/s = 2\int_s^{2s} \omega(s)/r^2\, dr \le 2\int_s^{2s} \omega(r)/r^2\, dr \le 2\int_0^{2s} \omega(r)/r^2\, dr \to 0$ as $s \to 0$. From the above, we have $\omega(s)/s \le B_\omega := \int_0^\infty \omega(r)/r^2\, dr$. [We remark that this argument also applies to $\nu$ as in Theorem 1: even if $\nu$ is not in $\Omega$ we do have $\nu(s)/s \to 0$ and $\nu(s)/s \le B_\nu < \infty$.] Since $\omega(s)/s^2$ is decreasing, we have $0 \ge [\omega(s)/s^2]' = [\omega' - 2\omega(s)/s]/s^2$ so $\omega' \le 2\omega(s)/s \le 2B_\omega$. This completes the proof of (i), (ii) as well as part of (iv).

For (iii), an integration by parts shows that

$$\begin{aligned}\int_0^R e^{-\gamma(\mu s)}\,d\tilde\nu(s) &= \left(\frac{\tilde\nu(s)}{s}se^{-\gamma(\mu s)}\right)\Bigg|_0^R + \int_0^R \frac{\tilde\nu(s)}{s}[\mu\gamma'(\mu s)]se^{-\gamma(\mu s)}\,ds\\ &\le B_{\tilde\nu}Re^{-\gamma(\mu R)} + \int_0^R B_{\tilde\nu}\left[\mu 2B_\gamma\right]\,se^{-\gamma(\mu s)}\,ds\end{aligned}$$

since the boundary term at 0 vanishes and $\gamma' \le 2B_\gamma$. Now (2.4) follows by going to the limit as $R\to\infty$ through a sequence such that $Re^{-\gamma(\mu R)}\to 0$, possible by (2.3). In particular, setting $\mu = 1$ and $\tilde\nu(s)\equiv s$ in (2.4) gives

$$A_0 := \int_0^\infty e^{-\gamma(s)}\,ds < \infty. \tag{2.7}$$

For (iv), note that continuity and unboundedness of $\vartheta$ follow, e.g., from the Monotone Convergence Theorem, while the correct monotonicity in $s$ of $\vartheta(s)$ and of $\vartheta(s)/s^2$ follow immediately from the form of (2.5). The integrability follows on interchange of integration to get

$$\int_0^\infty \frac{\vartheta(s)}{s^2}\,ds = \pi\int_0^\infty \frac{\nu(r)}{r^2}\,dr. \tag{2.8}$$

Finally, (2.6) follows from comparison of the integrals over $[r_0, R]$ and $[R,\infty)$ after replacing $\nu(r)$ by $\nu(R)$ and noting that $t\log(1+1/t)$ is increasing, say, from $t = r_0^2/s^2$ to $t = R^2/s^2$. ∎

Given any $\omega\in\Omega$, we introduce functions $\beta, q : \mathbb{R}_+\to\mathbb{R}_+$ defined by

$$\begin{aligned}\beta(s) &:= \frac{1}{s} + 2\left[\frac{\omega(s)}{s} + \int_s^\infty \frac{\omega(r)}{r^2}\,dr\right],\\ q(s) &:= \begin{cases} 0 & \text{for } 0\le s\le 1,\\ -\displaystyle\int_1^s r^2\,d\frac{\omega(r)}{r^2} & \text{for } s\ge 1,\\ \qquad = 2\displaystyle\int_1^s \frac{\omega(r)}{r}\,dr - \omega(s) + \omega(1) & \end{cases}\end{aligned} \tag{2.9}$$

using integration by parts to obtain the second expression for $q$ when $s\ge 1$.

**LEMMA 2:** *For $\omega\in\Omega$ the functions $\beta, q$ of (2.9) are continuous and*

*(i)* $\beta : (0,\infty) \to (0,\infty)$ *is decreasing with* $\beta(s) \to \infty, 0$ *as* $s \to 0, \infty$*; hence there is a continuous decreasing inverse function* $\beta^{(-1)} : (0,\infty) \to (0,\infty)$*,*
*(ii)* $q : [1,\infty) \to \mathbb{R}_+$ *is increasing with* $q(s) \to \infty$ *as* $s \to \infty$*; hence there is a continuous increasing inverse* $q^{(-1)} : \mathbb{R}_+ \to [1,\infty)$*; further,* $q(s)/s \to 0$ *as* $s \to \infty$ *and, finally,*

$$\int_s^\infty \frac{dq(r)}{r^2} = \frac{\omega(s)}{s^2} \quad \textit{for } s \geq 1. \tag{2.10}$$

PROOF: The continuity of $\beta$ is clear and we need only note Lemma 1-(i) to see that $\beta(s) \to 0$ as $s \to \infty$. Since $\omega(s)/s^2$ is decreasing, the first expression for $q$ in (2.9) shows that $q(\cdot)$ is increasing from $q(1) = 0$ and, indeed, that

$$q(s) \geq \; q(\tilde{s}) - \left[\omega(r)/r^2\right]_{\tilde{s}}^{s} \;\geq\; \omega(\tilde{s}) - \tilde{s}^2[\omega(s)/s^2]$$

for $s > \tilde{s}$, whence $q(s) \to \infty$ as $s \to \infty$ (as we may first choose $\tilde{s}$ so $\omega(\tilde{s})$ is arbitrarily large and then the last term goes to 0). The second expression gives

$$\frac{q(s)}{s} \leq \frac{2}{s}\int_1^s \frac{\omega(r)}{r}\,dr$$

so, as the integrand goes to 0, we have $q(s)/s \to 0$ as $s \to \infty$. An integration by parts, using the information that $\beta(\infty) = 0$, then gives for $\beta$ the alternative formula:

$$\beta(s) = 1/s + 2\int_s^\infty (1/r)\,dq(r)$$

with $dq > 0$ so $\beta(\cdot)$ is decreasing and, since the second term here is positive, we must have $\beta(s) \to \infty$ as $s \to 0$. The integrability at $\infty$ of $dq(r)/r$ certainly implies integrability of $dq(r)/r^2 = -d[\omega(r)/r^2]$, giving (2.10). ∎

At this point we can define the functions appearing in (1.4) in Theorem 1: given choices of $\omega, \gamma$ as above, we set

$$\begin{aligned} Q_1(\delta) &:= \max\{\tfrac{1}{2}, \omega(\beta^{(-1)}(\delta/2))\} \\ Q_2(T) &:= \min_\mu \left\{\psi(T - 2B_\gamma\mu) - \tfrac{1}{2}\log\mu : 0 < 2\mu \leq \min\{1, T/B_\gamma\}\right\} \end{aligned} \tag{2.11}$$

where

$$(2.12)\qquad \begin{aligned}&\psi(\hat{T}) := \sup_{\sigma>0}\{h(\sigma) - \hat{T}\sigma\} \quad \text{with}\\ &\quad h(\sigma) := \vartheta(\sigma) + (2\log 2)\, q(\sigma) + (\hat{c}/\hat{x})\, \sigma\beta(\sigma) \qquad \text{using}\\ &\quad \hat{x} = \tfrac{1}{2}(1-e^{-2}), \quad \hat{c} := |\log(1-\hat{x})| = -\log\tfrac{1}{2}(1+e^{-2}).\end{aligned}$$

Note that Lemma 2-(i) ensures that $Q_1(\delta)$ is defined. The behavior of $Q_1$ for small $\delta > 0$ depends only on the tail of $\omega$ (large $s$) with $Q_1 \to \infty$ as $\delta \to 0$. As $\vartheta \in \Omega$ we have $\vartheta(\sigma)/\sigma \to 0$ and Lemma 2 then ensures that $h(\sigma)/\sigma \to 0$ so $h(\sigma) < \hat{T}\sigma$ for $\hat{T} > 0$ and large $\sigma > 0$, whence $\psi$ is finite. [One might further remark on the relation of (2.12) to the Legendre-Fenchel dual.] Thus $Q_2$ is well-defined. Since $h(\sigma) \to \infty$ as $\sigma \to \infty$ so $\psi(\hat{T}) \to \infty$ as $\hat{T} \to 0$, we have $Q_2(T) \to \infty$ as $T \to 0$; compare Remark 2, below.

**THEOREM 2:** *For any $\omega \in \Omega$ and any $\delta > 0$ (for simplicity we only consider $\delta \le 2$) there exists an entire function $P = P_\delta(\cdot;\omega)$ such that*

*(i) $P(0) = 1$ and $e^{-i(\delta/2)z}P(z)$ is of exponential type $\delta/2$ with $|P(z)| \le 1$ on the upper half-plane $\mathbb{C}_+$ (in particular, for real $z$),*

*(ii) $P(is)$ is real and positive for real $s \ge 0$ with*

$$(2.13)\qquad P(is) \ge e^{-[(2\log 2)\, q_+(s) + (\hat{c}/\hat{x})\, s\beta(s)]}$$

*where $q_+ := \max\{q, 0\}$ with $q, \beta$ as in (2.9) and $\hat{c}, \hat{x}$ as in (2.12),*

*(iii) $|P(s)| \le e^{Q_1(\delta)}\, e^{-\omega(|s|)}$ for real $s$ with $Q_1$ as in (2.11).*

As already noted, a version of this theorem was proved by W.A.J. Luxemburg and J. Korevaar [10] for the case of $\Lambda$ in a complex sector and this was later modified by T.I. Seidman [17] and by T.I. Seidman and M.S. Gowda [18] to obtain more explicit estimates in the case of real $\Lambda$, giving the asymptotics as $\delta \to 0$. Here we follow the scheme of the proof in [18], making such modifications as are necessary for the present more general setting.

PROOF: For any $\alpha \ge 1$ (to be chosen for (2.17) later; we will also assume that $q(\alpha) \ge 1/2$), we set

$$(2.14)\qquad z_j := q(\alpha) + j/2, \quad a_j := 1/q^{(-1)}(z_j)$$

so $j = 2[q(1/a_j) - q(\alpha)]$; this index $j = 0, 1, \ldots$ is unrelated to the index of $\Lambda$. Each $a_j$ is a continuous decreasing function of the choice of $\alpha$. For $s \ge 0$ we then define

$$n = n(s) := \begin{cases} 0 & \text{if } 0 \le s \le \alpha \\ \min\{j : a_j \le 1/s\} & \text{for } s \ge \alpha. \end{cases} \tag{2.15}$$

Note that $a_j \le 1/s$ just when $z_j = q(1/a_j) \ge q(s)$, i.e., just when $j/2 \ge q(s) - q(\alpha)$. Thus, (2.15) gives $n(s) = \lceil 2[q(s) - q(\alpha)] \rceil$ for $s \ge \alpha$ which ensures, with $q(\alpha) \ge 1/2$, that $n(s) \le 2q_+(s)$ for all $s \ge 0$. With $n = n(s)$, noting that $1/q^{(-1)}(z)$ is decreasing, we have the integral comparison

$$\begin{aligned} \sum_n^\infty a_j &= a_n + \sum_{n+1}^\infty \frac{2[z_{j+1} - z_j]}{q^{(-1)}(z_j)} < a_n + \int_{z_n}^\infty \frac{2\,dz}{q^{(-1)}(z)} \\ &< \frac{1}{s} + \int_{q(s)}^\infty \frac{2\,dz}{q^{(-1)}(z)} = \frac{1}{s} + 2\int_s^\infty \frac{dq(r)}{r} =: \beta(s). \end{aligned} \tag{2.16}$$

Note that (2.16) is independent of the choices of $s \ge \alpha > 1$ and, in particular, for $s = \alpha$ (so $n(s) = 0$) we consider $\hat{\delta}(\alpha) := 2\sum_0^\infty a_j < 2\beta(\alpha)$. This can be made arbitrarily small by taking $\alpha$ large while, on the other hand, $\hat{\delta} > 2a_0 = 2/\alpha$ so $\hat{\delta}(1) > 2 \ge \delta$. By the continuity of $\alpha \mapsto \hat{\delta}$ (which follows, e.g., from the Dominated Convergence Theorem), we can thus choose $\alpha = \alpha_*(\delta)$ to get exactly $\hat{\delta}(\alpha) = \delta$, i.e.,

$$\sum_0^\infty a_j = \delta/2 \qquad \text{for } \alpha = \alpha_*(\delta) < \beta^{(-1)}(\delta/2). \tag{2.17}$$

We now define $P(\cdot)$ in terms of the sequence $(a_j)$ by an infinite product:

$$P(z) := e^{i(\delta/2)z} \prod_{j=0}^\infty \cos(a_j z) = \prod_{j=0}^\infty \tfrac{1}{2}\left(1 + e^{2ia_j z}\right). \tag{2.18}$$

Since convergence of $\sum a_j$ as in (2.16), (2.17) implies that of $\sum |\cos(a_j z) - 1|$ uniformly on bounded sets in $\mathbb{C}$, this infinite product converges to an entire function. Clearly $P(0) = 1$ and for $z \in \mathbb{C}_+$ one has $|e^{2ia_j z}| \le 1$ so $|P(z)| \le 1$. Since $|\cos(az)| \le e^{a|z|}$ for $a > 0$ and all $z \in \mathbb{C}$, the product $e^{-i(\delta/2)z}P(z)$ is then of exponential type $\sum a_j = \delta/2$, as desired, completing the proof of (i).

For pure imaginary $z = is$, it is clear from (2.18) that $P(is)$ is real and positive. For $s \ge 0$ we take $n = n(s)$ and split the product into those factors for which $j < n$ and those with $j \geq n$. Each factor in the second subproduct has the form $(1 - x_j)$ with

$$0 < x_j := \tfrac{1}{2}\left(1 - e^{-2a_j s}\right) \quad \begin{cases} = \tfrac{1}{2}\displaystyle\int_0^{2a_j s} e^{-r}\,dr & \leq a_j s \\ \leq \tfrac{1}{2}\left(1 - e^{-2}\right) & =: \hat{x} \end{cases}$$

Using the concavity of 'log', we have

$$\tfrac{1}{2}\left(1 + e^{-2a_j s}\right) = e^{\log(1-x_j)} \geq e^{-\hat{c}(x_j/\hat{x})}$$

with $\hat{c}$ as in (2.12) and it follows that

$$\begin{aligned} \prod_{j=n}^{\infty} \tfrac{1}{2}\left(1 + e^{-2a_j s}\right) &\geq \exp\left[-(\hat{c}/\hat{x})\sum_{n(s)}^{\infty} x_j\right] \geq \exp\left[-(\hat{c}/\hat{x})\sum_{n(s)}^{\infty} a_j s\right] \\ &\geq e^{-(\hat{c}/\hat{x})s\beta(s)}. \end{aligned}$$

For each factor in the first subproduct we have $\frac{1}{2}\left(1 + e^{-2a_j s}\right) \geq \frac{1}{2}$ and, as there are $n = n(s)$ such terms, that product is bounded below by $2^{-n}$ and so by $e^{-(2\log 2)\,q_+(s)}$ since $n(s) \leq 2q_+(s)$ always. Multiplying the lower bounds for these two subproducts gives (2.13).

Finally, to see (iii) we first observe that

$$0 < \cos r \leq e^{-\frac{1}{2}r^2} \quad \text{for real } r \text{ with } |r| \leq 1$$

and then (for real $s$) that $|P(s)|$ is the even function $\prod_0^{\infty} |\cos(a_j s)|$ so

$$|P(s)| \leq \prod_{n(s)}^{\infty} |\cos(a_j s)| \leq \exp\left[-\tfrac{1}{2}\sum_{n(s)}^{\infty}(a_j s)^2\right]. \tag{2.19}$$

For $s \geq \beta^{(-1)}(\delta/2) \geq \alpha_*$, we again make an integral comparison to get

$$\begin{aligned} \sum_{n(s)}^{\infty} |a_j|^2 &\geq \int_{q(s)}^{\infty} \frac{2\,dz}{[q^{(-1)}(z)]^2} - \int_{q(s)}^{z_n} \frac{2\,dz}{[q^{(-1)}(z)]^2} \\ &\geq 2\int_s^{\infty} \frac{dq(r)}{r^2} - \frac{1}{2}\cdot\frac{2}{s^2} = \frac{2\omega(s) - 1}{s^2}, \end{aligned} \tag{2.20}$$

using (2.10). Then (2.19) gives $|P(s)| \leq e^{(1/2)-\omega(s)}$ for such $s$. We have

$$\log|P(s)| \leq 0 \leq \omega(\beta^{(-1)}(\delta/2)) - \omega(s) \quad \text{for } 0 \leq s \leq \beta^{(-1)}(\delta/2)$$

so, combining the cases, we have (iii), and the proof is complete. ■

## 3. The sequence $\Lambda$

For this section we fix the consideration of a sequence $\Lambda \subset \mathbb{C}_+$ and construct, with estimates, a sequence $\{g_j\}$ biorthogonal in $L^2(0,\delta)$ to $\left\{e^{i\lambda_j t}\right\}$.

**LEMMA 3:** *Let $\Lambda = \{\lambda_k\} \subset \mathbb{C}_+$ satisfy (1.2) subject to (2.1). Then each $F_j = F_j(\cdot;\Lambda)$, given by*

$$F_j(z) := \prod_{k\neq j}\left[1-\left(\frac{z-\lambda_j}{\lambda_k-\lambda_j}\right)^2\right]. \tag{3.1}$$

*for $z \in \mathbb{C}$, is an entire function of exponential type* $0$ *and satisfies*

$$F_j(\lambda_k) = \delta_{j,k} \qquad \text{and} \qquad |F_j(\lambda_j+z)| \le e^{\vartheta(|z|)} \quad \text{for } z \in \mathbb{C} \tag{3.2}$$

*with $\vartheta$ given by* (2.5).

Proof: Fix $j$. We know that $\sum 1/(\lambda_k-\lambda_j)^2$ is absolutely convergent so the infinite product (3.1) converges uniformly on bounded sets in $\mathbb{C}$ whence $F_j$ is an entire function. The interpolation condition: $F_j(\lambda_k) = \delta_{j,k}$ is immediate from the form of (3.1) and we need only verify the exponential inequality. Writing $\hat{\nu}_j(r)$ for the left hand side of (1.2) with $\lambda_* = \lambda_j$, we have

$$\begin{aligned}\log|F_j(\lambda_j+z)| \; &\le \sum_{k\neq j}\log\left[1+\frac{|z|^2}{|\lambda_k-\lambda_j|^2}\right]\\ &= \int_0^\infty \log\left[1+\frac{|z|^2}{r^2}\right]\,d\hat{\nu}_j(r) = 2\int_0^\infty \frac{\hat{\nu}_j(r)}{r}\,\frac{|z|^2}{|z|^2+r^2}\,dr\\ &\le 2\int_0^\infty \frac{\nu(r)}{r}\,\frac{|z|^2}{|z|^2+r^2}\,dr =: \vartheta(|z|)\end{aligned}$$

which gives (3.2). Finally, $F_j$ is of exponential type 0 since $\vartheta(s)/s \to 0$. ■

The next lemma is the heart of our argument.

**LEMMA 4:** *Let $\gamma \in \Omega$ with $A_0 := \int_0^\infty e^{-\gamma} < \infty$. For $\delta > 0$ consider the mollifier $P(\cdot)$ constructed in Theorem 2 in terms of $\omega := \vartheta + \gamma$ using (2.5). Now, with (3.1), define functions $G_j$ on $\mathbb{C}$ by*

$$G_j(z) := F_j(z)\frac{P(z-\tau_j)}{P(i\sigma_j)} \tag{3.3}$$

*and then functions $g_j$ on $\mathbb{R}$ by*

$$g_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G_j(s)}\, e^{ist}\, ds. \tag{3.4}$$

*We will then have:*

*(i) for real $s$, each $G_j$ satisfies the estimate:*

$$|G_j(s+\tau_j)| \le \left[e^{Q_1(\delta)} e^{h(\sigma_j)}\right] e^{-\gamma(|s|)} \tag{3.5}$$

*with $Q_1$ as in (2.11) and $h$ as in (2.12),*

*(ii) for each $j,k$ we set $\Delta = \Delta_{jk} := |\tau_k - \tau_j|/2$ and have*

$$|\langle G_j, G_k\rangle| \le 4A_0\, e^{2Q_1(\delta)} e^{h(\sigma_j)} e^{h(\sigma_k)} e^{-\gamma(\Delta)} \tag{3.6}$$

*so, in particular, each $G_j$ is in $L^2(\mathbb{R})$ and, as a function on $\mathbb{C}$,*

*(iii) each $G_j$ is entire with $e^{-i(\delta/2)z}G_j(z)$ of exponential type $\delta/2$,*

*(iv) each $g_j$ is an $L^2$ function with support in $[0,\delta]$ and the sequence $\{g_j\}$ is biorthogonal to the exponentials:*

$$\langle g_j, e^{i\lambda_k t}\rangle = \delta_{j,k}. \tag{3.7}$$

PROOF: From (3.3) we have $G_j(s+\tau_j) = F_j(\lambda_j + [s - i\sigma_j])\, P(s)/P(i\sigma_j)$. From Theorem 2-(ii,iii) and the definition of $h(\cdot)$ in (2.11) we have

$$\left|\frac{P(s)}{P(i\sigma_j)}\right| \le \frac{e^{Q_1(\delta)} e^{-\omega(|s|)}}{e^{\vartheta(\sigma_j) - h(\sigma_j)}}.$$

From (3.2) of Lemma 3 we have $|F_j(\lambda_j + [s - i\sigma_j])| \le e^{\vartheta(|s-i\sigma_j|)}$ and from the form of (2.5) and the fact that $|s - i\sigma_j|^2 = s^2 + (\sigma_j)^2$, we have $\vartheta(|s - i\sigma_j|) \le \vartheta(|s|) + \vartheta(\sigma_j)$. Since $\vartheta - \omega = -\gamma$, combining these just gives (3.5).

With given $j, k$, we use (3.5) to get

$$\begin{aligned} |\langle G_j, G_k\rangle| &\le \int_{-\infty}^{\infty} |G_j(t)||G_k(t)|\, dt \\ &\le e^{2Q_1(\delta)} e^{h(\sigma_j)} e^{h(\sigma_k)} \int_{-\infty}^{\infty} e^{-\gamma(|t-\tau_j|)} e^{-\gamma(|t-\tau_k|)}\, dt. \end{aligned}$$

Extending $\gamma \in \Omega$ as an even function on $\mathbb{R}$ for convenience and setting $s = t - \frac{1}{2}(\tau_j + \tau_k)$, the integral here becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\gamma(s+\Delta)} e^{-\gamma(s-\Delta)}\, ds &= 2\int_{0}^{\infty} e^{-\gamma(s+\Delta)} e^{-\gamma(s-\Delta)}\, ds \qquad \text{(by symmetry)} \\ &\le 2e^{-\gamma(\Delta)} \int_{0}^{\infty} e^{-\gamma(s-\Delta)}\, ds \qquad (\text{since } \gamma \nearrow) \\ &\le 2e^{-\gamma(\Delta)} \int_{-\infty}^{\infty} e^{-\gamma(r)}\, dr = 4e^{-\gamma(\Delta)} A_0, \end{aligned}$$

which gives (3.6). Taking $k = j$ in this, we have $G_j \in L^2(\mathbb{R})$.

By Lemma 3 and Theorem 2-(i) we have $F_j$ entire of exponential type 0 and $e^{-i(\delta/2)z} P(z - \tau_j)$ entire of exponential type $\delta/2$ so the product is entire of exponential type $\delta/2$: we have (iii). Then $\overline{G_j(\bar{z})}$ has corresponding properties so, by the Paley-Wiener Theorem, its inverse Fourier transform $g_j$, given by (3.4), is in $L^2$ with support in $[0, \delta]$ and satisfying

$$G_j(z) := \langle g_j, e^{izt}\rangle := \int_{-\infty}^{\infty} \overline{g_j(t)} e^{izt}\, dt, \tag{3.8}$$

$$\langle g_j, g_k\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{g}_j(t)} \hat{g}_k(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_j(t)\, \overline{G_k(t)}\, dt. \tag{3.9}$$

By (3.2) and our definition (3.3), we have

$$G_j(\lambda_k) = \delta_{j,k} \tag{3.10}$$

which is precisely the biorthogonality property (3.7). This completes the proof of the lemma. ∎

## 4. Estimating $\|\mathbf{C}_\delta^T\|$

At this point we fix the function $\nu$ of (1.2) satisfying (2.1) and a function $\gamma \in \Omega$ satisfying (2.3). [For example, one could take $\gamma = \varepsilon\vartheta$ (if (2.3) would then hold) or take $\gamma(s) := (2+\varepsilon)\log_+ s$.] Then, with $\vartheta$ obtained from $\nu$ by (2.5), we set $\omega := \vartheta + \gamma$ to obtain the function $\omega$ we will use, as in the previous section, for construction of the mollifier. We are now ready to restate and prove our principal result: the estimation of $\|\mathbf{C}_\delta^T\|$ for small $\delta, T$.

**THEOREM 1:** *Given $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (2.1), let $\Lambda$ be any sequence in $\mathbb{C}_+$ satisfying the corresponding condition (1.2):*

$$\#\{\lambda \in \Lambda : 0 < |\lambda - \lambda_*| \le r\} \le \nu(r) \qquad \text{for each } \lambda_* \in \Lambda,$$

*Then, for any $\delta > 0$ and any $T > 0$ the map*

$$\mathbf{C} = \mathbf{C}_\delta^T : f = \sum_k c_k e^{i\lambda_k t} \mapsto \mathbf{c}_T = \left(c_k e^{i\lambda_k T}\right) : \mathcal{M}_\delta \to \ell^2$$

*will be well-defined and continuous: for any $\gamma \in \Omega$ satisfying (2.3) we have, uniformly for such $\{\Lambda\}$, the estimate:*

$$\|\mathbf{C}_\delta^T\| = \|\mathbf{C}_\delta^T(\Lambda)\| \le C_* e^{Q_1(\delta)+Q_2(T)} \tag{4.1}$$

*with $Q_1(\cdot)$, $Q_2(\cdot)$ defined in terms of $\gamma(\cdot)$ and $\omega := \vartheta + \gamma$ by (2.11) and with*

$$C_*{}^2 := \frac{4}{\pi} B_\gamma B_\nu A_0 A_1 \quad \text{so} \quad Q_* := \log\left(2\sqrt{\frac{B_\gamma B_\nu A_0 A_1}{\pi}}\right) \text{ in (1.4).}$$

Proof: For $f \in \mathcal{M} = \mathcal{M}(\Lambda)$, $\mathbf{c}_T = \mathbf{C}_\delta^T f$, we have $\|\mathbf{c}_T\|^2 = \sum_k \left|c_k e^{i\lambda_k T}\right|^2$ so, using (3.7),

$$\|\mathbf{c}_T\|^2 = \sum_k \langle g_k, f\rangle\, \overline{c_k}\, \left|e^{i\lambda_k T}\right|^2 = \langle w, f\rangle \qquad \left(w := \sum_k c_k \left|e^{i\lambda_k T}\right|^2 g_k\right).$$

Note that $w$ has support in $[0,\delta]$ if each $g_k$ does, so these inner products are equally valid over $[0,\delta]$ or over $\mathbb{R}$. [We do this for finite sums, taking $f \in \mathcal{M}$, to avoid convergence issues; the result will then extend by continuity to $f \in \mathcal{M}_\delta$ by the nature of the estimates obtained.]

We now consider the operator $\mathbf{\Gamma}$ acting on $\ell^2$ by the infinite matrix $(\Gamma_{j,k})$ given by

$$\Gamma_{j,k} = \Gamma^T_{j,k} := e^{i\lambda_j T}\,\overline{e^{i\lambda_k T}}\,\langle g_j, g_k\rangle \tag{4.2}$$

(which reduces to (3.9) when $T = 0$). With a bit of manipulation we obtain

$$\|w\|^2 = \sum_j \left[\sum_k \Gamma_{j,k}\left(c_k e^{i\lambda_k T}\right)\right]\overline{(c_j e^{i\lambda_j T})} = \langle \mathbf{\Gamma}\mathbf{c}_T, \mathbf{c}_T\rangle$$

so $\|w\|^2 \le \|\mathbf{\Gamma}\|\,\|\mathbf{c}_T\|^2$. We then have

$$\|\mathbf{c}_T\|^2 = \langle w, f\rangle \le \|w\|\|f\| \le \|\mathbf{\Gamma}\|^{1/2}\|\mathbf{c}_T\|\|f\|$$

which shows that $\|\mathbf{C}^T_\delta\| \le \|\mathbf{\Gamma}\|^{1/2}$.

The Hermitian symmetry of $(\Gamma_{j,k})$ means that $\mathbf{\Gamma}$ is self-adjoint on $\ell^2$, whence the norm $\|\mathbf{\Gamma}\|$ is just the spectral radius. By the Gershgorin Theorem (or, equivalently, observing that $(\Gamma_{j,k})$ also acts as an operator on $\ell^\infty$) the spectrum is bounded by $\sup_j \{\sum_k |\Gamma_{j,k}|\}$ whence, as $\left|e^{i\lambda_j T}\right| = e^{-\sigma_j T}$, etc.,

$$\|\mathbf{C}^T_\delta\|^2 \le \sup_j \left\{ e^{-\sigma_j T}\sum_k e^{-\sigma_k T}|\langle g_j, g_k\rangle| \right\}, \tag{4.3}$$

which we will estimate using (3.9) and (3.6). For $0 \le \mu \le 1/2$ we have

$$\begin{aligned}\gamma(\mu|\lambda_k - \lambda_j|) &\le \gamma(\mu|\tau_k - \tau_j| + \mu|\sigma_k - \sigma_j|)\\ &\le \gamma(\Delta) + 2B_\gamma\mu|\sigma_k - \sigma_j|\end{aligned}$$

whence $-\gamma(\Delta) \le 2B_\gamma\mu(\sigma_j + \sigma_k) - \gamma(\mu|\lambda_k - \lambda_j|)$ in (3.6), (3.9) so

$$e^{-\sigma_j T}e^{-\sigma_k T}\,|\langle g_j, g_k\rangle| \le \frac{4}{2\pi}A_0\,e^{2Q_1(\delta)}e^{2\psi(T-2B_\gamma\mu)}e^{-\gamma(\mu|\lambda_k-\lambda_j|)}. \tag{4.4}$$

Summing over $k$, we get

$$\begin{aligned}\sum_k e^{-\gamma(\mu|\lambda_k-\lambda_j|)} &= \int_0^\infty e^{-\gamma(\mu r)}\,d\hat{\nu}_j(r)\\ &= \int_0^\infty [\hat{\nu}_j(r)\,\gamma'(\mu r)\mu]\;e^{-\gamma(\mu r)}\,dr\\ &\le \int_0^\infty \left[\frac{\nu(r)}{r}\,2B_\gamma\mu\right]\;re^{-\gamma(\mu r)}\,dr\\ &\le 2B_\gamma B_\nu\int_0^\infty (\mu r)e^{-\gamma(\mu r)}\,dr = 2B_\gamma B_\nu A_1/\,\mu\end{aligned} \tag{4.5}$$

and, as this last is independent of $j$, (4.3) and (4.4) then give

$$\|\mathbf{C}_\delta^T\| \le \frac{2}{\sqrt{\pi}} \sqrt{B_\gamma B_\nu A_0 A_1}\, e^{Q_1(\delta)}\, \frac{e^{\psi(T-2B_\gamma \mu)}}{\sqrt{\mu}} \tag{4.6}$$

which we may optimize over $\mu$ to get (4.1), as desired. [It would, of course, be possible to further optimize (4.6) over the choice of $\gamma$ (for given $\nu(\cdot), \delta$), but we do not pursue this.] ■

# 5. Remarks and examples

## 5:1. Comparison with [18]

The results of [18] apply to classes of sequences $\{\Lambda\}$ which do not involve complex exponents and *a fortiori* do not permit, as in Theorem 1 here, the possibility of unbounded $\{\sigma_k\}$ — while, on the other hand, those results provide an estimate which does not blow up as $T \to 0$. For comparison, we consider $\Lambda = \{\lambda_k = \tau_k + i\sigma_k\}$ with bounded imaginary part:

$$0 \le \sigma_k \le \sigma^* \qquad (\text{all } k), \tag{5:1.1}$$

obviously generalizing the restriction $\sigma_k \equiv 0$ in [18]; compare footnote[4]. In [18] the separation condition was given in the form:

$$|\lambda_k - \lambda_j| \ge \psi_m \qquad \text{for } |k-j| \ge m, \tag{5:1.2}$$

which there presumed a lineal ordering of $\Lambda$ along $\mathbb{R}$. Geometrically, the condition (5:1.2) just means that no interval of length $\psi_m$ can contain more than $m$ of the exponents $\lambda_k$, which now implies (1.2) if we would take

$$\nu(r) := 2m \quad \text{for } \psi_m \le r < \psi_{m+1} \qquad (m = 0, 1, \ldots). \tag{5:1.3}$$

The determining sequence $\{\psi_m\}_1^\infty$ of (5:1.2) was required in [18] to satisfy

$$\psi_0 := 0 < \psi_1 \le \psi_2 \le \cdots \qquad \text{with } \sum_{m=1}^\infty \frac{1}{\psi_m} < \infty \tag{5:1.4}$$

and the choice (5:1.3) of $\nu$ then gives

$$\int_0^\infty \frac{\nu(s)}{s^2}\,ds = \sum_{m=1}^\infty \int_{\psi_m}^{\psi_{m+1}} \frac{2m}{s^2}\,ds = 2\sum_{m=1}^\infty \frac{1}{\psi_m}$$

so (5:1.4) just provides the hypotheses on $\nu$ for Theorem 1. What we observe at this point is that the 'sup' in (2.12) was taken over all $\sigma > 0$ because it was needed for $\sigma = \sigma_j$ in getting (3.5) and $\sigma_j$ was otherwise unrestricted. If we are now imposing (5:1.1), then $\psi(\cdot)$ can be redefined in (2.12) by taking the 'sup' only over the compact interval $[0, \sigma^*]$ — which gives $\psi(\hat{T})$ bounded uniformly in $\hat{T} \geq 0$ and so $Q_2(T)$ bounded uniformly in $T \geq 0$. Finally, we observe that $Q(\delta)$ was defined in [18] in relation to

$$\Psi(s) := 2\sum_{m=1}^\infty \log\left[1 + \frac{s^2}{\psi_m^2}\right]$$

exactly (to within an additive constant, independent of $\delta$) as we have here defined $Q_1(\delta)$ in relation to $\vartheta(s)$, given by (2.5), and an elementary computation shows that $\vartheta(s) \equiv \Psi(s)$ if we are using (5:1.3) in (2.5). Thus, our present result is truly a generalization of that of [18] for this situation on making this redefinition in (2.12) to take advantage of the imposition of (5:1.1).

**REMARK 1:** Quite generally, we note a partial converse for our results: when $\{\sigma_k\}$ is unbounded, the norm *must* blow up as $T \to 0$.

One way to see this is to note that $\|\mathbf{C}_\delta^T\| \geq \sup_k\{\left|e^{-\sigma_k T}\right| / \|e^{i\lambda_k t}\|\}$ and then that $\sup_{\sigma>0}\{\sqrt{2\sigma}e^{-\sigma T}\} \sim 1/\sqrt{T} \to \infty$. Alternatively, if $\Lambda$ is unbounded it contains an unbounded subsequence $\Lambda'$ with $\sum {\sigma_k}^{-1} < \infty$ and we can then show that $\|\mathbf{C}_\delta^T(\Lambda')\| \to \infty$ which implies the same for $\mathbf{C}_\delta^T(\Lambda)$. Define

$$\mathbf{F} : \mathbf{c}_0 \mapsto f = \sum_k c_k e^{i\lambda_k t} : \mathcal{M}'_\delta = \mathcal{M}_\delta(\Lambda') \subset L^2(0,\delta)$$

for any sequence $\mathbf{c}_0 = (c_1, \ldots) \in \ell^2$. Then, as $\|e^{i\lambda_k t}\|^2 = \left[1 - e^{-2\sigma_k\delta}\right]/(2\sigma_k)$ in $L^2(0,\delta)$, we see that $\sum \|e^{i\lambda_k t}\|^2 < \infty$ so $\mathbf{F}$ is continuous and, indeed, compact. On the other hand, a bound on $\|\mathbf{C}_\delta^T(\Lambda')\|$ as $T \to 0$ would imply a bound on $\mathbf{c}_T \to \mathbf{c}_0$ and so invertibility of $\mathbf{F}$, which is impossible for compact $\mathbf{F}$.

## 5:2. Some examples

We may consider as examples exponent sequences distributed along a ray in $\mathbb{C}_+$ like powers:

$$\lambda_k = a + ck^p \qquad (k = 0, 1, \ldots) \tag{5:2.1}$$

for some constants $a, c \in \mathbb{C}_+$ with $c \neq 0$ and some $p > 1$. Obviously, the exponents are densest near $\lambda_0 = a$ so the left hand side $\hat{\nu}(r)$ in (1.2) will be largest when $\lambda_* \approx (\lambda_k + \lambda_0)/2$ with $\lambda_k - \lambda_0 \approx 2r$ so $k \approx [2r/|c|]^{1/p}$, which gives $\hat{\nu}(r) \approx [2r/|c|]^{1/p}$; similarly, the minimal separation $r_0$ is between $\lambda_0$ and $\lambda_1$, giving $r_0 = |c|$. Thus we may take

$$\nu(r) = \begin{cases} 0 & \text{for } 0 \leq r < r_0 := |c| \\ C_\nu r^{1/p} & \text{for } r \geq r_0, \;\; \text{with } C_\nu := (2/|c|)^{1/p} \end{cases} \tag{5:2.2}$$

which gives (1.2) for this $\Lambda$ and satisfies (2.1). Using this in (2.5) gives $\vartheta(s) \sim C_\vartheta s^{1/p}$ as $s \to \infty$ with

$$C_\vartheta := 2C_\nu \int_0^\infty \frac{r^{1/p}\, dr}{r(1+r^2)} = C_\nu\, \pi \, \csc \frac{\pi}{2p} \tag{5:2.3}$$

(and $\vartheta \leq C'_\vartheta s^2$ with $C'_\vartheta := 2^{1+1/p}/|c|^2(2-1/p)$ giving the behavior for small $s$). [We may note that essentially the same behavior occurs when one has only asymptotic equivalence of $\lambda_k$ to (5:2.1).] If $\Lambda$ were the union of $m$ such sequences, then the determination of $r_0$ must be verified separately, but we could certainly use the bound $\nu(r) = m(2r/\min\{|c|\})^{1/p}$ in (1.2).

For expository simplicity we just take $\gamma = \varepsilon\vartheta$ here (as at the beginning of Section 4) which gives $\omega(s) \sim C_\omega s^{1/p}$ for large $s$. Using this in (2.9), we get $q(s) \sim C_q s^{1/p}$ and $\beta(s) \sim C_\beta s^{1/p-1}$ as $s \to \infty$ so $\beta^{(-1)}(\delta) \sim (C_\beta/\delta)^{p/(p-1)}$ as $\delta \to 0$ whence (2.11) gives $Q_1(\delta) = \omega(\beta^{(-1)}(\delta/2)) \sim C_1/\delta^{1/(p-1)}$, as in [18]. Since the terms comprising $h$ in (2.12) are of the same order, we have $h(s) \sim C_h s^{1/p}$ as $s \to \infty$ and this gives $\psi(\hat{T}) \sim C_\psi/\hat{T}^{1/(p-1)}$ as $\hat{T} \to 0$. Since this dominates $\log \mu$, we take $\mu \to 0$ as $T \to 0$ in the optimization defining $Q_2$ and get $Q_2(T) \sim C_2/T^{1/(p-1)}$ as $T \to 0$. In this case, at least, the two terms in the exponent in (4.1) are of the same order with respect to their arguments. For the setting $T = \delta$ of [10] this gives

$$\log \|\mathbf{C}_T^T\| \sim CT^{-1/(p-1)} \tag{5:2.4}$$

for the problem of [11], in which $\Lambda$ is the spectrum of a Sturm–Liouville operator so we have this distribution with $c = i\pi/\ell$ and $p = 2$; this is $\log \|\mathbf{C}_T^T\| \sim C/T$ as in [15] and [4]. [Note that it would also have been possible, much as was done for (5:2.3), to obtain the various coefficients $C_\beta, \ldots$ recursively as we proceeded above, had we wished to obtain not only the order but some explicit estimate of the constant $C$ in (5:2.4) as well.]

If one specifically considers Dirichlet series, i.e., taking $\lambda_k = i\sigma_k$ with $0 \leq \sigma_1 < \sigma_2 < \ldots$, then (1.1) becomes

$$f(t) = \sum_{k=1}^{\infty} c_k e^{-\sigma_k t} \qquad (0 < t \leq \delta) \tag{5:2.5}$$

(or, equivalently, setting $e^{-\delta} \leq x := e^{-t} < 1$ one has $\hat{f}(x) = \sum_k c_k x^{\sigma_k}$). Results on the determinability of coefficients are available (cf., e.g., [13] and [2]) when $\sum_k 1/\sigma_k < \infty$ — i.e., under weaker conditions than we have been imposing through (1.2) or (5:1.2) so the force of Theorem 1 is in the uniformity with respect to these classes of sequences and in the estimate (4.1) we have obtained:

$$\sum_{k=1}^{\infty} e^{-2\sigma_k T} |c_k|^2 \leq C_*^2 e^{2[Q_1(\delta)+Q_2(T)]} \int_0^\delta \left| \sum_{k=1}^{\infty} c_k e^{-\sigma_k t} \right|^2 dt, \tag{5:2.6}$$

bounding the asymptotics as $\delta \to 0$ and/or $T \to 0$.

**REMARK 2:** We conclude this subsection with the comment that it is precisely the asymptotic behavior of $\nu(\cdot)$ at infinity which determines the asymptotics of our estimate (4.1) as $\delta \to 0$ or $T \to 0$.

Suppose we were to have $\nu$ and $\tilde{\nu}$, each satisfying (2.1), with $\nu = \mathcal{O}(\tilde{\nu})$ as $r \to \infty$. Tracking through the definitions (2.5), (2.9), (2.11), (2.12) — much as was done to get (5:2.4) above from (5:2.2) — we see that (for a suitable choice of $\gamma$ in taking $\omega = \vartheta + \gamma$ and $\tilde{\omega} = \tilde{\vartheta} + \gamma$) one has:

$$\begin{array}{rl} \text{at } \infty: & \vartheta = \mathcal{O}(\tilde{\vartheta}), \quad \omega = \mathcal{O}(\tilde{\omega}), \quad \beta = \mathcal{O}(\tilde{\beta}), \quad q = \mathcal{O}(\tilde{q}), \quad h = \mathcal{O}(\tilde{h}), \\ \text{near } 0: & \beta^{(-1)}(\delta) \leq K\tilde{\beta}^{(-1)}(\delta/K), \quad \psi(\hat{T}) \leq K\tilde{\psi}(\hat{T}/K) \qquad \text{for some } K. \end{array}$$

It then follows from (2.11) that, for some $K$,

$$Q_1(\delta) \leq K\tilde{Q}_1(\delta/K), \quad Q_2(T) \leq K\tilde{Q}_2(T/K) \tag{5:2.7}$$

as $\delta, T \to 0$ in Theorem 1; of course, if $\tilde{\nu}$ corresponds to a $k^p$ distribution as in (5:2.1) above, then this becomes the more usual $Q_1 = \mathcal{O}(\tilde{Q}_1)$, $Q_2 = \mathcal{O}(\tilde{Q}_2)$. Indeed, looking more closely shows that if $\nu \sim \hat{\nu}$ at $\infty$ we may take $K \to 1$ near 0 here which confirms our assertion that the asymptotics of (1.4) are just determined by the asymptotic behavior of $\nu(s)$ for large $s$ — although one also needs some $r_0 > 0$ as in (2.1-$i$), i.e.,

$$|\lambda - \lambda'| > r_0 \quad \text{for } \lambda, \lambda' \in \Lambda \ \text{ with } \lambda \neq \lambda'. \tag{5:2.8}$$

## 5:3. A related estimation

Suppose, rather in contrast to (5:1.2) which is uniform over $\mathbb{C}_+$, that one were to have available an *asymptotic* lower bound $\Delta_M(\cdot)$ for possible '$M$-clustering' of the sequence $\Lambda$:

$$\#\left[\Lambda \bigcap \{z \in \mathbb{C}_+ : |z - \tilde{z}| < \Delta_M(R)\}\right] \leq M \quad \text{if } |\tilde{z}| \geq R, \tag{5:3.1}$$

for some nondecreasing function $\Delta_M$ with $\Delta_M(0) > 0$, i.e., one cannot have more than $M$ elements of $\Lambda$ in any disk $\tilde{\mathcal{D}}(\tilde{z})$ of radius $\Delta_M(|\tilde{z}|)$, depending on its location,

$$\tilde{\mathcal{D}}(\tilde{z}) := \{z \in \mathbb{C} : |z - \tilde{z}| < \Delta_M(|\tilde{z}|)\} \qquad (\tilde{z} \in \mathbb{C}_+).$$

From this information — (5:3.1), for some fixed $M$, together with (5:2.8) — we can obtain the uniform estimate (1.2). [The case of interest is $\Delta_M(r) = o(r)$ as $r \to \infty$ and we assume this.]

**LEMMA 5:** *Suppose the set $\Lambda$ satisfies (5:3.1) for large $R$ and some fixed $M$; suppose also that there is some uniform lower bound $r_0$ for the separation of elements of $\Lambda$. Then $\Lambda$ satisfies (1.2) with*

$$\nu(r) = \mathcal{O}\left(\int_0^{2r} \frac{s\,ds}{[\Delta_M(s)]^2}\right) \qquad \text{as } r \to \infty. \tag{5:3.2}$$

*Further, this gives uniformity of (1.2) for a family $\{\Lambda\}$ of sequences if the sparsity condition (5:3.1) is uniform over the family.*

PROOF: For given $r > 0$, we can always find (depending on $r$) a set of centers $\{\zeta_j : j = 1, \dots, N(r)\}$ such that the set of disks $\{\tilde{\mathcal{D}}(\zeta_j)\}$ covers the '$2r$'-semidisk:

$$\mathcal{S}(r) := \{z \in \mathbb{C}_+ : |z| \le 2r\} \subset \bigcup_{j=1}^{N(r)} \tilde{\mathcal{D}}(\zeta_j).$$

One way to do this covering is to proceed incrementally: assuming $\mathcal{S}(r)$ has been covered by $N(r)$ suitable disks, cover an additional semi-annulus by evenly spacing disks of radius $\Delta_M(r)$ about $\sqrt{2}\Delta_M(r)$ apart (taking about $2\pi r/\sqrt{2}\Delta_M(r)$ disks) so the incremental semi-annulus has width $\sqrt{2}\Delta_M(r)$. Roughly, this would give $dN/dr \sim \pi r/\Delta^2$, although one may not expect attainability of precisely this constant.

We then claim that, for any disk of radius $r$ (i.e., $\hat{\mathcal{D}} := \{z \in \mathbb{C}_+ : |z - \hat{z}| \le r\}$), one has a covering $\hat{\mathcal{D}} \subset \bigcup_j \tilde{\mathcal{D}}(z_j)$ using $N(r)$ disks $\tilde{\mathcal{D}}(z_j)$. To see this, note first that $[\Lambda \bigcap \hat{\mathcal{D}}] \subset \mathcal{S}(r)$ if $|\hat{z}| \le r$ so we may then take $z_j := \zeta_j$. When $|\hat{z}| > r$, we set $u := \hat{z}/|\hat{z}|$, $\zeta := (|\hat{z}| - r)u \in \mathbb{C}_+$ and then take $z_j := \zeta - iu\zeta_j$ ($j = 1, \dots, N(r)$). It is not hard to see that this choice gives $\hat{\mathcal{D}} \subset [\zeta - iu\mathcal{S}(r)]$ and also $|z_j| > |\zeta_j|$ so $\Delta_M(|z_j|) \ge \Delta_M(|\zeta_j|)$ whence this is again a suitable covering. Noting that each of the disks $\tilde{\mathcal{D}}(z_j)$ contains at most $M$ elements of $\Lambda$, we may take

$$\nu(r) := MN(r) \tag{5:3.3}$$

and have (1.2). [As (5:3.1) is only known to be valid for $R > R_0$, we modify this by adding a bound $N_* \ge \#\{\lambda \in \Lambda : |\lambda| \le R_0\}$ (which can be obtained from $r_0$, assumed known) to the right hand side of (5:3.3) and then proceed as before without changing the asymptotics.] We may then integrate the earlier estimate for $dN/dr$ and combine this with (5:3.3) to get (5:3.2) as desired. ■

If we have no information about $\Lambda$ beyond (5:3.1), then we are considering a set of exponents distributed over $\mathbb{C}_+$ with a density roughly like $M/\pi[\Delta_M]^2$. If we are to use such a $\Lambda$ in (1.1), then we must verify (2.1-*ii*) from (5:3.2): one easily sees that this is equivalent to integrability on $\mathbb{R}_+$ of $[\Delta_M]^{-2}$ so, for example, our theory applies if (5:3.1) holds with $\Delta_M(R) \sim R^\alpha$ (or with $\Delta_M(R) \sim \sqrt{R}[\log R]^\alpha$) for some $\alpha > 1/2$.

If, on the other hand, we were to know that $\Lambda$ lies on a ray: $s \mapsto [a + cs]$ as in (5:2.1) (or on/near some curve with an asymptotic direction given by $c \in \mathbb{C}_+$), then this information could be combined with (5:3.1) by a modification of the argument for Lemma 5: we observe that, as there and also in the treatment of (5:2.1), it is sufficient to bound the number of exponents in an initial segment of length $2r$ (i.e., for $0 \le s \le 2r/|c|$) which we can do by covering that segment with disks of radius $\Delta_M(R)$. Since 'covering' now means covering segment length (rather than area of $\mathcal{S}$), the bound (5:3.2) now becomes, by a similar analysis,

$$\nu(r) \sim M \int_0^{2r} \frac{dR}{\Delta_M(R)} \tag{5:3.4}$$

for large $r$. [This, of course, appears independent of $a, c$ (provided one bounds $a$), since length along the ray or curve will always be asymptotically equivalent to distance $R$ from the origin.]

As an example, for the situation of (5:2.1) we get $\Delta \approx |\lambda_k - \lambda_{k-1}| \approx |c|pk^{p-1}$ and $R \approx |c|k^p$ so $\Delta_1(R) \sim |c|^{1/p} p R^{1-1/p}$ from which (5:3.4) gives, asymptotically, the same result as (5:2.2). More generally, consider $\lambda_k \approx \varphi(k)$ for a suitable function $\varphi : \mathbb{R}_+ \to \mathbb{C}_+$ so $\Delta_1 \approx \inf\{|\varphi'(s) : |\varphi(s)| \ge r\}$. If $\psi(s) := |\varphi(s)|$ is increasing with $\psi' \ge a|\varphi'|$ for large $s$ and some $a > 0$, then the increase in $|\lambda_k|$ is comparable to the separation and, if $|\varphi'|$ would be increasing, we would have $\Delta_1(r) \approx |\varphi'(s)|$ with $|\varphi(s)| \approx r$, whence $\Delta_1 \approx |\varphi' \circ |\varphi|^{(-1)}|$ asymptotically. Using this in (5:3.4), one gets simply

$$\nu(r) \sim (const)|\varphi|^{(-1)}(2r), \tag{5:3.5}$$

which, of course, coincides with (5:2.2) for the case (5:2.1).

# 6. Applications to system theory

## 6:1. An abstract system

We now consider an abstract (autonomous, linear) 'observation system'

$$z := \mathbf{b} \cdot y \text{ on } (0, \delta) \text{ with } \dot{y} + \mathbf{A}y = 0, \tag{6:1.1}$$

in which we observe an output $z$ and seek an operator giving the state at some time $T$ for the ODE, i.e.,

$$\boldsymbol{\mathcal{O}} : z(\cdot) \mapsto y(T). \tag{6:1.2}$$

Note that (6:1.1) is essentially 'formal': $y(0)$ is unknown and, indeed, we are not even requiring that $y$ have a well-defined value at $t = 0$ while $\mathbf{b}$ need not actually be in the dual of the state space $Y$ but need only act continuously on solutions of $\dot{y} + \mathbf{A}y = 0$, i.e., on the range of the semigroup generated by $-\mathbf{A}$. It is not difficult to verify that $\mathbf{U} = \boldsymbol{\mathcal{O}}^*$ then provides a nullcontrol for the adjoint system:

$$
\begin{aligned}
-\dot{x} + \mathbf{A}^* x &= -u(t)\mathbf{b}, \quad x(T) = \xi \\
&\text{gives } x(0) = 0 \text{ by taking } u(\cdot) = \mathbf{U}\xi \text{ on } (0, \delta).
\end{aligned}
\tag{6:1.3}
$$

We refer to (existence and) boundedness of $\boldsymbol{\mathcal{O}}$ as *continuous observability* for the system (6:1.1) — obviously dependent on $T, \delta, \mathbf{b}, \mathbf{A}$ — and to the existence of a bounded $\mathbf{U}$ giving (6:1.3) as *exact nullcontrollability* for the adjoint system; these properties are equivalent 'by duality' in that one can take $\mathbf{U} = \boldsymbol{\mathcal{O}}^*$. For $T = \delta$ it is equivalent to replace (6:1.3) by

$$
\begin{aligned}
\dot{x} + \mathbf{A}^* x &= -u(t)\mathbf{b}, \quad x(0) = \xi \\
&\text{gives } x(T) = 0 \text{ for } u(t) = [\mathbf{U}\xi](T - t) \text{ on } (0, T).
\end{aligned}
\tag{6:1.4}
$$

If $\mathbf{A}$ has a basis of eigenvectors $\{\eta_k\}$ with corresponding eigenvalues $\{\alpha_k\}$, then one has formal expansions

$$
y(t) = \sum_k \hat{c}_k e^{-\alpha_k t} \eta_k, \qquad z(t) = \sum_k c_k e^{-\alpha_k t}
\tag{6:1.5}
$$

with $c_k = \beta_k \hat{c}_k$ where $\beta_k := \mathbf{b} \cdot \eta_k$. We then have, formally,

$$
\boldsymbol{\mathcal{O}} z := y(T) = \sum_k \left( c_k e^{-\alpha_k T'} \right) \left[ \frac{e^{-\alpha_k (T - T')}}{\beta_k} \eta_k \right]
\tag{6:1.6}
$$

for any $0 < T' \leq T$, provided no $\beta_k = 0$. We recognize the sequence of coefficients $(c_k e^{-\alpha_k T'})$ as $\mathbf{C}_\delta^{T'} z$ for $\Lambda = \{\lambda_k\}$ with $\lambda_k := i\alpha_k$ and then note that (6:1.6) gives boundedness of $\boldsymbol{\mathcal{O}}$ if $\{i\alpha_k\}$ satisfies (1.2), subject to (2.1), and provided

$$
\begin{aligned}
&\text{either} \quad \left( e^{-\alpha_k (T - T')} \|\eta_k\| / \beta_k \right) \in \ell^2 \\
&\text{or} \quad \left( e^{-\alpha_k (T - T')} / \beta_k \right) \in \ell^\infty \text{ and } \{\eta_k\} \text{ is a Riesz basis for } Y
\end{aligned}
\tag{6:1.7}
$$

for some choice of $T' \in (0, T]$.

Given a family of operators $\{\mathbf{A}\}$ for (6:1.1), we say that they are *uniformly* observable (for given $T, \delta, \mathbf{b}$) if $\{\boldsymbol{\mathcal{O}} = \boldsymbol{\mathcal{O}}(\mathbf{A}, \cdots)\}$ is uniformly bounded. This will clearly be the case if (6:1.7) holds uniformly and $\Lambda = \Lambda(\mathbf{A})$ satisfies (1.2) for each $\mathbf{A}$ of the family, with the same $\nu(\cdot)$ satisfying (2.1). Equivalently, we say that (6:1.3) is *uniformly nullcontrollable* if the family of corresponding nullcontrol operators $\{\mathbf{U} = \mathbf{U}(\mathbf{A}, \cdots)\}$ is uniformly bounded.

## 6:2. Boundary control of the heat equation

Consider a system governed by the homogeneous heat equation

$$\begin{aligned} &\rho u_t = (p u_x)_x - q u \qquad (0 < x < \ell) \\ &u_x\Big|_{x=0} \equiv 0, \quad u\Big|_{x=\ell} \equiv 0 \end{aligned} \tag{6:2.1}$$

with initial conditions: $u\Big|_{t=0} = u_0(\cdot)$. Here, $\rho, p, q$ are bounded functions with $\rho, p > 0$ bounded away from 0: physically, $\rho$ is heat capacity, $p$ is a diffusion coefficient and $q$ gives the rate of heat transfer to the environment along the rod. If the initial state $u_0(\cdot)$ is unknown, we may wish to determine the internal state — say, at some later time $T$ — by observation of the temperature $u\Big|_{x=0}$ at, e.g., the insulated end.[6] This amounts to seeking an operator

$$\boldsymbol{\mathcal{O}} : u(\cdot, 0) \mapsto u(T, \cdot) : L^2(0, \delta) \to L^2(0, \ell), \tag{6:2.2}$$

much as in (6:1.2). The relevant eigenpairs $\{\eta_k,\, \alpha_k\}$ are here given by the Sturm-Liouville problem

$$\begin{aligned} -(p\eta')' + q\eta = \alpha\rho\eta \quad &\text{on } (0, \ell) \\ \eta'(0) = 0 = \eta(\ell) \end{aligned} \tag{6:2.3}$$

for which it is standard that the eigenvalues $\{\alpha_k\}$ are real and distinct and that the eigenfunctions $\{\eta_k\}$ are real and are orthogonal (with respect to the $\rho$-weighted inner product for $L^2(0, \ell)$) and so may be taken as orthonormal.

[6] Comparing with (6:1.1), we note that this boundary observation $\mathbf{b} : u \mapsto u\Big|_{x=0}$ will not act continuously on the nominal state space $Y = L^2(0, \ell)$, but does act on $\mathcal{D}(\mathbf{A})$.

It is also standard that $\beta_k = \eta_k(0) \geq m > 0$ for use in (6:1.5) and (6:1.7) and that $\{\alpha_k\}$ is quadratically distributed:

$$0 < \alpha_k \sim c(k + \tfrac{1}{2})^2 \tag{6:2.4}$$

with $c$ easily computed from $\rho, p$ by integration over $(0, \ell)$. It follows that $\Lambda := \{i\alpha_k\}$ satisfies (1.2) with $\nu(r) \sim \sqrt{r}$, as in (5:2.2) for $p = 2$. As in the previous subsection, it then follows that $\boldsymbol{\mathcal{O}}$ is a well-defined bounded operator with $\log \|\boldsymbol{\mathcal{O}}\| = \mathcal{O}(1/T)$ as in [15], [4]. This means that the heat equation (6:2.1) is continuously observable (with an asymptotic estimate as $\delta = T \to 0$) and that the adjoint problem (controlling the input flux at $x = 0$ as a nonhomogeneous boundary condition) is exactly nullcontrollable, using controls in $L^2(0, \delta)$ for arbitrarily small $\delta > 0$. Such observability has, of course, long been known (e.g., since [11]) and, as in [11], this 1-dimensional argument gives corresponding results by separation of variables when, e.g., $q$ has the form $q_1(x) + q_2(y)$ in the 2-d heat equation on a rectangle (or similarly for 3-d):

$$u_t = \Delta u - qu \tag{6:2.5}$$

with separable boundary conditions and observation along one of the sides; compare (6:1.4).

**REMARK 3:** Taking $\rho \equiv 1 \equiv p$ for expository simplicity, we show, subject to a uniform bound $|q| \leq M$, that the *family* of observation problems

$$\begin{array}{cc} u_t = u_{xx} - qu & (0 < x < \ell) \\ u_x\big|_{x=0} \equiv 0, \quad u\big|_{x=\ell} \equiv 0 & u\big|_{t=0} = \text{??} \\ \boldsymbol{\mathcal{O}} = \boldsymbol{\mathcal{O}}_q : u(\cdot, 0) =: z \mapsto u(T, \cdot) & \end{array} \tag{6:2.6}$$

is *uniformly* observable (whence, also, we have uniform nullcontrollability for the corresponding family of adjoint boundary control problems).

From our theory above, it is sufficient for this to show that (5:2.8) and (asymptotically, say, for $r, |\lambda| \geq R_* > 0)$) (1.2) hold, uniformly for such $q(\cdot)$. We note that $\Lambda = \Lambda_q$ is obtained with $\lambda = i\alpha$ where $\alpha = \alpha(\mathbf{A}_q)$ is an eigenvalue for the Sturm-Liouville operator $\mathbf{A}_q : y \mapsto -y'' + qy$ (with the BC: $y'(0) = 0 = y(\ell)$ specifying the domain). By the Courant Minmax Theorem, these eigenvalues depend monotonically on the corresponding quadratic form and so on $q(\cdot)$, whence $\alpha_k(\mathbf{A}_{(-M)}) \leq \alpha_k(\mathbf{A}_q) \leq \alpha_k(\mathbf{A}_M)$, comparing to the extremes $q \equiv \pm M$. We can explicitly compute $\alpha_k(\mathbf{A}_{\pm M}) = [(k+1/2)\pi/\ell]^2 \pm M$ for $k = 0, \ldots$ so we get

$$\left|\alpha_k(\mathbf{A}_q) - \left[\left(k + \frac{1}{2}\right)\frac{\pi}{\ell}\right]^2\right| \leq M.$$

It follows from this that (uniformly in $q$) we have (1.2) with $\nu(r) \sim (\ell/\pi)\sqrt{2r}$ as $r \to \infty$. To verify (5:2.8), we first note that we may restrict consideration to a finite range of $k$ and then recall Theorem 1 of [16], asserting that each $\alpha_k(\mathbf{A}_q)$ is continuously dependent on $q$ if $q$ is topologized by weak convergence in $H^{-1}(0,\ell)$. Since the set $\{q \in L^\infty(0,\ell) : |q| \leq M\}$ is compact in $H^{-1}(0,\ell)$, the minimum separation $|\alpha_k(\mathbf{A}_q) - \alpha_{k-1}(\mathbf{A}_q)|$ for this range of $k$ is attained for some admissible $q$ — and cannot vanish, since these are necessarily all simple eigenvalues. Thus, this minimum separation is bounded below by some $r_0 > 0$, uniformly with respect to admissible $q$ as desired.

It can be shown that a similar result holds for the more general setting of (6:2.1) with constraints $0 < M_- \leq \rho, p \leq M_+$ (and possibly with different homogeneous boundary conditions). [We must note, for this, the validity of extending Theorem 1 of [16], now using weak-* convergence in $L^\infty(0,\ell)$ for $p$; a proof of that will appear elsewhere.] Such an extension of our present remark would, for example, generalize a recent result by Lopez and Zuazua [9] involving homogenization of a rapidly oscillating coefficient: $\rho(x/\varepsilon)u_t = u_{xx}$ as $\varepsilon \to 0$.

## 6:3. Vibrational control with structural damping

Linear vibrational dynamics, in mechanics, correspond to a Hamiltonian system with quadratic total energy

$$H = H(p,q) := \tfrac{1}{2}\langle \mathbf{M}^{-1}p, p\rangle + \tfrac{1}{2}\langle \mathbf{Q}q, q\rangle \tag{6:3.1}$$

($\mathbf{M}$ =“mass” and $\mathbf{Q}$ positive definite), typically obtained by linearizing around a stable equilibrium so $\frac{1}{2}\langle \mathbf{Q}q, q\rangle$ approximates the behavior of a potential well: $\mathbf{Q} = \delta^2\psi/\delta q^2$ at $q = 0$, where the potential $\psi$ has a strict (local) minimum. From (6:3.1) together with the standard Hamiltonian formalism: $\dot q = H_p$, $\dot p = -H_q$ one then obtains the dynamics: $\mathbf{M}\ddot q + \mathbf{Q}q = 0$ on writing $\mathbf{M}\ddot q$ for $\dot p$. These dynamics preserve the energy $H$ but dissipation is introduced by the modification: $\dot p = -[H_q + \mathbf{D}\mathbf{M}^{-1}]$ for some positive selfadjoint operator $\mathbf{D}$ (noting that that gives $\dot{H} = -\langle H_p, \mathbf{D}\mathbf{M}^{-1}p\rangle = -\langle \dot{q}, \mathbf{D}\dot{q}\rangle \leq 0$) which leads to: $\mathbf{M}\ddot{q} + \mathbf{D}\dot{q} + \mathbf{Q}q = 0$. This can be put in the form: $\dot{y} + \mathbf{A}y = 0$ of (6:1.1) on taking, e.g.,

$$(6{:}3.2)\quad y = \begin{pmatrix} \mathbf{M}^{-1/2}p \\ \mathbf{Q}^{1/2}q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \hat{\mathbf{D}} & \mathbf{E} \\ -\mathbf{E}^* & \mathbf{0} \end{pmatrix}, \text{ with } \begin{array}{l} \mathbf{E} := \mathbf{Q}^{1/2}\mathbf{M}^{-1/2} \\ \hat{\mathbf{D}} := \mathbf{M}^{-1/2}\mathbf{D}\mathbf{M}^{-1/2} \end{array}$$

The operator $\mathbf{A}_0$ (corresponding to the undamped setting with $\mathbf{D} = 0$) is skew adjoint, which would then make $\Lambda$ real in (6:1.5); more generally, since $\hat{\mathbf{D}}$ is again positive the eigenvalues of $\mathbf{A}$ for the damped setting have positive real parts so one still has $\Lambda \subset \mathbb{C}_+$, as is appropriate for our analysis.

For continuum mechanics we will have

$q = u(\cdot) =$ [pointwise deviation from the equilibrium configuration on $\Omega$]

and $\mathbf{Q}$ a differential operator for functions on $\Omega$. For simplicity, assume uniform density $\rho \equiv 1$ so $\mathbf{M}$ is the identity: $p = u_t$ and $\mathbf{E} = \mathbf{Q}^{1/2}$, $\hat{\mathbf{D}} = \mathbf{D}$ in (6:3.2). The damped dynamics (without forcing) are then given by the partial differential equation

$$(6{:}3.3)\qquad u_{tt} + \mathbf{D}u_t + \mathbf{Q}u = 0 \quad \text{ on } \Omega$$

with suitable homogeneous boundary conditions at $\Gamma := \partial\Omega$, corresponding to the domain of $\mathbf{Q}$ so $\mathbf{Q}$ is selfadjoint (and positive definite) with respect to pivoting on the inner product of $L^2(\Omega)$. We will also be taking observation and control as acting at (part of) the boundary of $\Omega$.

The operator $\mathbf{Q}$ is given by the stress-strain response of the material and, following Euler, we take the stress (material distortion) to be proportional to the linearized curvature $\Delta u$ so

$$\tfrac{1}{2}\langle \mathbf{Q}u, u\rangle = \tfrac{1}{2}\int_\Omega a\,(\Delta u)^2\, d\Omega$$

approximating the potential $\psi$. Thus, taking $a \equiv 1$ for simplicity, we have $\mathbf{Q} = \Delta^2$ — although this will give $\mathbf{Q}^{1/2} = -\Delta$ only for special choices of boundary conditions as in (6:3.8) below. [This choice of $\mathbf{Q}$ implicitly assumes the stress is dominated by flexion – neglecting material distortions associated with tension, shear, or torsion which might otherwise be included.]

In general the dissipation operator need not be closely related to the operator $\mathbf{Q}$, but for our example we consider a form of *structural damping.* This represents 'internal friction' within the material itself, so it is not unreasonable to expect the vibrational modes to dissipate energy independently, i.e., for $\mathbf{D}$ and $\mathbf{Q}$ to have a common (orthonormal) basis of eigenfunctions as we henceforth assume. [This is consistent with the experimental observation, for certain composite materials, that the modes are damped at rates (asymptotically) proportional to their vibrational frequencies — indeed, the model $\mathbf{D} = 2\kappa\mathbf{Q}^{1/2}$ was studied in [12]; see also [3], [5].]

We begin by considering a one-dimensional setting: a damped uniform rod of length $\ell$, clamped at both ends (so $u, u_x = 0$ at $x = 0, \ell$) with observation of $u_{xx}\Big|_{x=0}$. The eigenpair equation for $\mathbf{Q} = d^4/dx^4$ is then

$$\eta'''' = \mu^2\eta \qquad \eta = 0 = \eta' \text{ at } x = 0, \ell. \tag{6:3.4}$$

and some computation shows that this has a nontrivial solution when ($\mu$ is real, positive, and) $[\cos(\sqrt{\mu}\ell)][\cosh(\sqrt{\mu}\ell)] = 1$. We note that $\cosh(\sqrt{\mu}\ell)$ is very large for large $\mu$ so one has $\cos(\sqrt{\mu}\ell) \approx 0$ whence (asymptotically) one has $\mu = \mu_k \approx [(k-\frac{1}{2})\pi/\ell]^2$ — quite similar, as it happens, to what one would have gotten exactly with the different boundary condition: $[\eta' = 0 = \eta'''$ at $x = 0$; $\eta = 0 = \eta''$ at $x = \ell]$. Since $\mathbf{Q}$ is selfadjoint, the corresponding eigenfunctions $\eta_k$ provide an orthonormal basis for $L^2(0, \ell)$. In this case we have $\beta_k = \eta''(0)$ for use in (6:1.5) and (6:1.7) and expect $\beta_k \sim (const)\mu_k$ (as again would be exact for the 'different boundary conditions' above). We are assuming that $\eta_k$ is also an eigenfunction for $\mathbf{D}$ and will write: $\mathbf{D}\eta_k = 2\delta_k\eta_k$ — with $\delta_k > 0$ so this is dissipative. The exponent sequence $\Lambda$ is obtained by setting $u = e^{i\lambda t}\eta(x)$ in (6:3.3), which gives a quadratic equation for $\lambda$:

$$-\lambda^2 + 2i\delta\lambda + \mu^2 = 0 \qquad (\delta = \delta_k,\ \mu = \mu_k > 0). \tag{6:3.5}$$

For the undamped case $\delta \equiv 0$ this would give: $\lambda = \pm\mu$ and it is convenient to index as

$$\lambda_k = \begin{cases} \mu_k > 0 & \text{for } k > 0 \\ -\mu_{-k} < 0 & \text{for } k < 0, \end{cases} \tag{6:3.6}$$

(omitting $k = 0$). With $\Lambda$ real here and in view of the asymptotics $\mu_k \sim ck^2$, the theory of [17], [18] then suffices to give observability for this one-dimensional undamped version of (6:3.3) as was already well-known (cf., [6] and earlier work of Krabs).

For the damped case, it is convenient to introduce $r := \delta/\mu > 0$; we will assume that we always have $\delta < \mu$ so $r < 1$ and, indeed, that $r \leq \bar{r} < 1$ for simplicity. Then (6:3.5) gives

$$\lambda = \mu\left[ir \pm \sqrt{1-r^2}\right] = \tau + i\sigma \tag{6:3.7}$$

with $\tau = \pm\mu\sqrt{1-r^2}$ and $\sigma = \mu r > 0$; we index consistently with (6:3.6) to get $\Lambda = \{\lambda_k\}$. [Since the vibrational frequency is $\tau$ and the damping rate is $\sigma$, asymptotic proportionality would just mean that $r/\sqrt{1-r^2}$ should have a limit for large $\mu$, i.e., that $r$ should have a limit ($\neq 1$). If we had taken $\mathbf{D} = 2\kappa\mathbf{Q}^{1/2}$ as suggested earlier, then we would have always $2\delta = 2\kappa\mu$ so $r = const = \kappa$ (with $\kappa < 1$) and, conversely, if $r = const$, then we must have $\mathbf{D} = 2r\mathbf{Q}^{1/2}$. We note that this certainly permits us also to consider $\mathbf{D} = \kappa\mathbf{Q}^{\alpha/2}$ for any $\alpha < 1$ and then any $\kappa > 0$, although this seems non-physical.[7]] It is easily seen from (6:3.7) that $|\lambda| = |\mu|$ in this setting so it is geometrically clear that $|\lambda_j - \lambda_k| \geq |\mu_{|j|} - \mu_{|k|}|$. If $\nu^{[\mu]}(\cdot)$ would correspond to the sequence $\{\mu_k\}$ in respect to the sparsity condition (1.2) then we clearly could take $\nu(r) = 2\nu^{[\mu]}(r)$ in (1.2) for $\Lambda$; one might have anticipated difficulty with the existence of $r_0 > 0$ in (2.1) if $r \approx 1$ were possible, but this has been obviated by our imposition of the requirement $r \leq \bar{r} < 1$. The asymptotics above for $\{\mu_k\}$ ensure (2.1) so Theorem 1 applies and observability follows for arbitrarily small $T > 0$ — with a norm blowup as $T \to 0$ exponential in $\mathcal{O}(1/T)$, corresponding to (5:2.4).

We finally turn to consideration of an Euler plate with structural damping of this form, governed by the partial differential equation

$$\begin{aligned} u_{tt} - 2\kappa\Delta u_t + \Delta^2 u = 0 &\quad \text{on } \Omega = (0,1)^2 \\ u_\nu = 0 = (\Delta u)_\nu &\quad \text{on } \Gamma := \partial\Omega. \end{aligned} \tag{6:3.8}$$

We will treat the observation problem

$$\boldsymbol{\mathcal{O}} : z \mapsto u(T,\cdot) \qquad \text{where } z := u\Big|_{x=0} \in L^2([0,T]\times[0,1]). \tag{6:3.9}$$

The particular boundary conditions chosen here have the considerable advantage of making the problem separable and so permitting explicit computation. This is the problem considered, e.g., in [5]; compare [7], [17] for the

[7] On the other hand, if one were to take $\alpha \geq 1$ as, e.g., in the often-used Kelvin-Voigt model $\mathbf{D} = \kappa\mathbf{Q}$, then the set of exponents would have a finite limit point and our argument would fail.

undamped version of this. To avoid some expositional complications, we will assume it given that $u$ has mean 0 in (6:3.8), i.e., that this is true for the initial data $u, u_t\Big|_{t=0}$.

The eigenvalues for the Laplace operator $(-\Delta)$ on $\Omega = (0,1)^2$ with Neuman boundary conditions are $\mu_{jk} := (j^2+k^2)\pi^2$ for $j,k = 0,1,\dots$ (omitting $\mu_{00} = 0$ by our simplifying assumption, so we always have $\mu > 0$). With the damping operator $\mathbf{D} = -2\kappa\Delta$ we have $\delta = \kappa\mu$ and, as for the one-dimensional case above, (6:3.7) applies with $r = \kappa$. It will be necessary for us to partition the eigenvalues to get a family of scalar problems involving exponent sequences $\Lambda_j := \{\lambda_k^{[j]}\}$ where

$$\lambda_k = \lambda_k^{[j]} \;:= \begin{cases} a_j^+ + c^+k^2 & \text{for } k = 0,1,\dots \\ a_j^- + c^-k^2 & \text{for } k = -0,-1,\dots \end{cases}$$
$$\text{with } c^{\pm} := \left[\kappa \pm \sqrt{1-\kappa^2}\right]\pi^2,\; a_j^{\pm} := c^{\pm}j^2. \tag{6:3.10}$$

[This requires us to distinguish, as indices, between $k = +0$ and $k = -0$, which would be potentially awkward for $j = 0$ where $a_0^+ = 0 = a_0^-$ our omission of the '00' terms eliminates that.] We have the orthonormal basis for $L^2(0,1)$

$$\eta_k(x) = \left\{1 \text{ if } k = 0;\; \frac{\cos(k\pi x)}{\sqrt{2}} \text{ if } k = 1,2,\dots\right\}$$

and separation of variables in (6:3.8) gives the expansion

$$u(t,x,y) = \sum_{j=0}^{\infty}\sum_{k=\pm 0}^{\pm\infty} c_{jk}\, e^{i\lambda_k^{[j]}t}\, \eta_{|k|}(x)\eta_j(y). \tag{6:3.11}$$

The key to our treatment of (6:3.8) is reduction of the problem to a sequence of independent simpler problems, involving one $\Lambda_j$ at a time, by introducing

$$z_j(t) := \langle z(t,\cdot), \eta_j\rangle = \sum_{k=\pm 0}^{\pm\infty} c_{jk}\, e^{i\lambda_k^{[j]}t} \tag{6:3.12}$$

for $j = 0,1,\dots$ With $j$ fixed, (6:3.12) is of the form (1.1) for the exponents $\Lambda_j$. Since each $\Lambda_j$ consists of the union of two copies of (5:2.1), we may (as noted following (5:2.2)) take $\nu(r) = 4r$ in (1.2) for $r > r_0$ and note that (since we have omitted $k = 0$ for $j = 0$) we may take $r_0 = 2\sqrt{1-\kappa^2}\,\pi^2$ uniformly in $j$. Thus, Theorem 1 (together with our discussion in subsection 5.2) immediately gives a uniform bound:

$$\|\mathbf{C}_T^T(\Lambda_j)\| \leq B = B(T) = e^{\mathcal{O}(1/T)} \qquad (j = 0, 1 \ldots). \tag{6:3.13}$$

By setting $t = T$ in (6:3.11) we then obtain

$$\begin{aligned}
\|u(T, \cdot\cdot)\|^2 &= \sum_{j=0}^{\infty} \left[ \sum_{k=\pm 0}^{\pm\infty} \left| c_{jk}\, e^{i\lambda_k^{[j]} T} \right|^2 \right] = \sum_{j=0}^{\infty} \|\mathbf{C}_T^T(\Lambda_j) z_j(\cdot)\|^2 \\
&\leq \sum_{j=0}^{\infty} B^2 \|z_j(\cdot)\|^2 \quad = B^2 \|z\|^2.
\end{aligned}$$

For (6:3.9), this just means that

$$\|\boldsymbol{\mathcal{O}}\| \leq B(T). \tag{6:3.14}$$

This continuity of $\boldsymbol{\mathcal{O}}$ is the principal result of [5]; the asymptotic estimate: $\log \|\boldsymbol{\mathcal{O}}\| = \mathcal{O}(1/T)$ which we have obtained by use of Theorem 1 is 'extra'. The twin keys to success here were the uniformity in (6:3.13) with respect to the independent exponent sequences $\Lambda_j$ and the orthogonality of the decomposition:

$$X = L^2(\Omega) = X_1 \oplus X_2 \oplus \cdots \qquad \text{with } X_j := \operatorname{span}_k\{\eta_j(x)\eta_k(y)\}.$$

[It would have been sufficient for this to have been a 'Riesz decomposition' (generalizing the notion of a Riesz basis): each $u \in X$ uniquely expressible as $\sum_j u_j$ with $u_j \in X_j$ and constants $c, C > 0$ such that $c\|u\|^2 \leq \sum_j \|u_j\|^2 \leq C\|u\|^2$.] Much as for (6:1.4), we then note that $\boldsymbol{\mathcal{O}}^*$ enables us to find null-controls ($w = 0 = w_t$ at $t = T$, given $w, w_t$ at $t = 0$) for the problem:

$$\begin{aligned}
&w_{tt} - 2\kappa\Delta w_t + \Delta^2 w = 0 \quad \text{on } \Omega = (0,1)^2 \\
&w_\nu = 0 \text{ on } \partial\Omega \quad (\Delta w)_\nu = \begin{cases} \varphi(= \text{control}) & \text{when } x = 0 \\ 0 & \text{else on } \partial\Omega. \end{cases}
\end{aligned} \tag{6:3.15}$$

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