NORM DEPENDENCE OF THE COEFFICIENT MAP ON THE WINDOW SIZE[1]

Thomas I. Seidman[2] and M. Seetharama Gowda

Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, MD 21228, U.S.A

ABSTRACT: For sparse exponent sequences $(\lambda_k)_{-\infty}^{\infty}$, satisfying a suitable 'separation condition' defined by an auxiliary sequence $\boldsymbol{\psi}$, one has a 'coefficient map' $\mathbf{C}_\delta$ giving $(c_k)_{-\infty}^{\infty} =: \mathbf{c}$ from observation of $f = \sum_{k=-\infty}^{\infty} c_k\, e^{i\lambda_k t}$ on any arbitrarily small interval $[-\delta, \delta]$. In terms of $\boldsymbol{\psi}$, we estimate the norm of $\mathbf{C}_\delta : L^2[-\delta, \delta] \to \ell^2$, asymptotically as $\delta \to 0$. In particular, for $(\lambda_k)_{-\infty}^{\infty} \sim k^p$ $(p > 1)$ we get a bound which is exponential in $(1/\delta)^{1/(p-1)}$, generalizing an earlier result for the case $p = 2$.

KEY WORDS: *nonharmonic Fourier series, coefficient map, norm, window, asymptotic.*

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[2] e-mail: seidman@math.umbc.edu. This research has been partially supported for this author by the National Science Foundation under grant ECS-8814788 and by the U.S. Air Force Office of Scientific Research under grant AFOSR-91-0008.

1. Introduction

Let $\boldsymbol{\lambda} = \{\lambda_k\}$ be a real 'double sequence' ($k = 0, \pm 1, \pm 2, \ldots$) and let $\overset{\circ}{\mathcal{M}} = \overset{\circ}{\mathcal{M}}(\boldsymbol{\lambda})$ denote the collection of all finite sums

$$f = \sum_k c_k e^{i\lambda_k t} \tag{1.1}$$

with complex coefficients c_k. We will here think of viewing such f through a *window* $(-\delta, \delta)$ and determining the coefficients $\{c_k\}$ from this, defining a coefficient map

$$\overset{\circ}{\mathbf{C}} = \overset{\circ}{\mathbf{C}}(\boldsymbol{\lambda}) : f = \sum_k c_k e^{i\lambda_k t} \mapsto (c_k) \tag{1.2}$$

for $f \in \overset{\circ}{\mathcal{M}}$. If we now let $\mathcal{M}_\delta = \mathcal{M}_\delta(\boldsymbol{\lambda})$ be the closure of $\overset{\circ}{\mathcal{M}}$ in $L^2(-\delta, \delta)$, then it is classical that $\overset{\circ}{\mathbf{C}}$ extends from $\overset{\circ}{\mathcal{M}}$ to $\mathcal{M}_\delta$ as a continuous linear operator

$$\mathbf{C}_\delta = \mathbf{C}_\delta(\boldsymbol{\lambda}) : \mathcal{M}_\delta \mapsto \ell^2 : f = \sum_{k=-\infty}^{\infty} c_k\, e^{i\lambda_k t} \mapsto (c_k)_{-\infty}^{\infty} =: \mathbf{c} \tag{1.3}$$

provided the asymptotic density of $\boldsymbol{\lambda}$ is bounded by δ/π.

In this paper, we consider sequences $\boldsymbol{\lambda}$ satisfying sparsity conditions of the form

$$|\lambda_{k+m} - \lambda_k| \geq \psi_m \qquad (m = 1, 2, \ldots) \tag{1.4}$$

for suitable $\boldsymbol{\psi} = \{\psi_m : m = 1, 2, \ldots\}$. Noting that $m/\psi_m \to 0$ ensures that $\mathbf{C}_\delta(\boldsymbol{\lambda})$ is well defined for all $\delta > 0$, we then investigate the rapidity with which $\|\mathbf{C}_\delta(\boldsymbol{\lambda})\| \to \infty$ as $\delta \to 0$. As a by-product of this analysis, we note that our estimates are uniform over the classes of exponent sequences $\Lambda = \Lambda(\boldsymbol{\psi})$ satisfying (1.4) for particular admissible sequences $\boldsymbol{\psi}$. Our results are new in this aspect as well as in the consideration of the asymptotics as $\delta \to 0$.

It is clear that $\mathbf{C}_\delta(\boldsymbol{\lambda})$ is made up of the coefficient functionals

$$\gamma_k : \mathcal{M}_\delta = \mathcal{M}_\delta(\boldsymbol{\lambda}) \to \mathbb{C} : f \mapsto c_k$$

and that each of these functionals can be represented as

$$\gamma_k : f \mapsto c_k = \langle f, g_k \rangle \tag{1.5}$$

for some $g_k \in L^2(-\delta, \delta)$. There is some arbitrariness in the determination of g_k since (1.5) constitutes an extension of γ_k from $\mathcal{M}_\delta$ to all of $L^2(-\delta, \delta)$; this also gives an extension $\tilde{\mathbf{C}}_\delta$ of $\mathbf{C}_\delta(\boldsymbol{\lambda})$ to $L^2(-\delta, \delta)$.

Since we are working with exponentials, it is then convenient to construct the Fourier transforms to obtain g_k and we actually will work with the adjoint of $\tilde{\mathbf{C}}_\delta$,

$$\tilde{\mathbf{C}}_\delta^* : \ (a_k) \mapsto \sum_k a_k g_k : \ \ell^2 \to L^2(-\delta, \delta), \tag{1.6}$$

to estimate[3] $\|\mathbf{C}_\delta\| \le \|\tilde{\mathbf{C}}_\delta\| = \|\tilde{\mathbf{C}}_\delta^*\|$.

We will be able to treat conditions (1.4) for real sequences $\boldsymbol{\psi} = \{\psi_m : m = 1, 2, \ldots\}$ for which

$$(1.7) \qquad 0 < \psi_1 \le \psi_2 \le \ldots \quad \text{and} \quad \sum_1^\infty 1/\psi_k < \infty$$

Note that this already implies that $m/\psi_m \to 0$ which precisely corresponds to the condition that Λ have asymptotic density zero.

Our paper falls into three parts:

First, considering a sequence $\boldsymbol{\lambda}$ satisfying (1.4) subject to (1.7), we will apply an important theorem due to Luxembourg and Korevaar ([2]; Theorem 3.1) which we restate here in a relevant form:

THEOREM K-L: *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing with $\omega(t)/t^2$ integrable at ∞. Then, for any $\delta > 0$, there exists a number $Q > 0$ and an entire function $P(\cdot)$ such that:*

(*i*) *P is of exponential type δ,*

(*ii*) *P is normalized so $P(0) = 1$,*

(*iii*) *$|P(s)| \le e^Q e^{-\omega(|s|)}$ for $s \in \mathbb{R}$.*

Clearly, the constant Q in (*iii*) will depend on δ so $Q = Q(\delta) = Q(\delta; \omega)$. From this we then obtain an estimate:

$$(1.8) \qquad \|\mathbf{C}_\delta\| \le Ae^{Q(\delta)} \text{ for all } \delta > 0$$

where A is independent of δ and we use a function ω depending only on $\boldsymbol{\psi}$. So far, this is only slightly different from the treatment in [2].

Second, and this is the principal technical innovation of the paper, we extend the analysis of Theorem K-L from that of [2], specifically considering the estimation of $Q(\delta)$ in (1.8) so as to exhibit explicitly its asymptotics as $\delta \to 0$ as well as the dependence on $\boldsymbol{\psi}$ through $\omega(\cdot)$. From (1.8), this is precisely what is needed to investigate the asymptotic behavior of $\|\mathbf{C}_\delta\|$. For this estimation, we find it necessary to assume $\omega \in \Omega$ where

$$(1.9) \qquad \Omega := \left\{ \omega : \mathbb{R}_+ \to \mathbb{R}_+ : \begin{array}{l} \omega(t) \text{ is nondecreasing while} \\ \omega(t)/t^2 \text{ is decreasing and integrable at } \infty \end{array} \right\};$$

this strengthens very slightly the hypotheses above for Theorem K-L in requiring that $\omega(t)/t^2$ be decreasing.

Third, the combination of the above is applied to obtain specific growth estimates for a number of interesting special cases. In particular, we apply the general analysis to the case $\psi_m = am^p \quad (a > 0, p > 1)$ which corresponds to $\lambda_k \sim \pm ak^p$ and obtain for that case the estimate

$$(1.10) \qquad \log \|\mathbf{C}_\delta\| = \mathcal{O}([1/\delta]^{1/(p-1)}) \quad \text{as } \delta \to 0$$

[3]Note that one has the geometric characterization $\|\gamma_k\| = 1/[\text{distance from } e^{i\lambda_k t} \text{ to } \boldsymbol{\mathcal{M}}_\delta(\boldsymbol{\lambda}^k)]$ where $\boldsymbol{\lambda}^k = \boldsymbol{\lambda} \backslash \{\lambda_k\}$. This gives $\|\gamma_k\| \ge 1/\|e^{i\lambda_k t}\| = 1/\sqrt{2\delta}$, showing the uselessness of the crude estimate $\|\mathbf{C}_\delta f\|^2 = \sum_k |\langle \gamma_k, f\rangle|^2 \le \left(\sum_k \|\gamma_k\|^2\right) \|f\|^2$.

(i.e., $Q(\delta) \le \mu[1/\delta]^{1/(p-1)}$ in (1.8) for small $\delta > 0$; we also have an estimate for μ). We recall that earlier investigation in [4] of the special case

$$\lambda_k = k^2 \quad (k \ge 0), \quad \lambda_{-k} = -\lambda_k \tag{1.11}$$

resulted in an estimate $\log \|\mathbf{C}_\delta\| = \mathcal{O}(1/\delta)$ which was there shown to be sharp (by an example due to Korevaar).

2. The Interpolation Family

Assume that $\boldsymbol{\psi}$ satisfies (1.7) and that $\boldsymbol{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ is in $\boldsymbol{\Lambda}(\boldsymbol{\psi})$, i.e., satisfies the separation condition (1.4). With $\boldsymbol{\psi}$ we associate the function Ψ given by

$$\Psi(s) := 2 \sum_{m=1}^{\infty} \log\left(1 + \frac{s^2}{\psi_m^2}\right). \tag{2.1}$$

We will show in the Appendix (Lemma A.1) that this function $\Psi(\cdot)$ is, indeed, in Ω. Given $\boldsymbol{\lambda} \in \Lambda(\boldsymbol{\psi})$, we next define a family of functions $(\eta_k)_{-\infty}^{\infty}$ by first defining

$$\mu_k(z) := \prod_{j \ne k} \left(\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right) \qquad (z \in \mathbb{C}) \tag{2.2}$$

and then setting

$$\eta_k(z) := \mu_k(z)\mu_k(2\lambda_k - z) = \prod_{j \ne k} \left[1 - \left(\frac{z - \lambda_k}{\lambda_k - \lambda_j}\right)^2\right]. \tag{2.3}$$

LEMMA 2.1: *Let $\boldsymbol{\psi}$, Ψ be as above and define η_k for $k \in \mathbb{Z}$ as in (2.3). Then one has*

$$\eta_k(\lambda_j) = \delta_{j,k} \qquad (j, k \in \mathbb{Z}) \tag{2.4}$$

and each $\eta_k(\cdot)$ is an entire function of exponential type 0 with

$$|\eta_k(\lambda_k + s)| \le e^{\Psi(|s|)} \qquad \forall s \in \mathbb{R}. \tag{2.5}$$

Proof: For each k and any $N > 0$, there is some $M = M_{k,N}$ such that

$$\begin{aligned} |\mu_k(z)| &\le \left(\prod_{0<|j-k|\le N} \left|\frac{z - \lambda_j}{\lambda_k - \lambda_j}\right|\right) \exp\left[(|z| + |\lambda_k|) \sum_{|j-k|>N} \frac{1}{|\lambda_j - \lambda_k|}\right] \\ &\le M(1 + |z|)^{2N+1} \exp\left[|z| \sum_{m>N} \frac{1}{\psi_m}\right] \end{aligned}$$

for all $z \in \mathbb{C}$. This estimate ensures suitable convergence of the product in (2.2) to have μ_k entire. Further, the sum in the exponential can be made arbitrarily small by taking N large since $\{1/\psi_m\}$ is summable by assumption whence each μ_k (and so each η_k) is of exponential type 0. The property (2.4) is obvious from $\mu_k(\lambda_j) = \delta_{j,k}$. Finally, for real s we have

$$\begin{aligned} |\eta_k(\lambda_k + s)| &\leq \prod_{j \neq k} \left[1 + \left(\frac{s}{\lambda_k - \lambda_j} \right)^2 \right] \\ &\leq \prod_m \left(1 + \frac{s^2}{\psi_m^2} \right)^2 = e^{\Psi(|s|)} \end{aligned}$$

so one has (2.5) as desired. ∎

Selecting any $\gamma \in \Omega$ such that $e^{-\gamma}$ is integrable, we take $\omega = \Psi + \gamma$ which is in Ω by Lemma A.1; then, fixing $\delta > 0$, we let $P(\cdot)$ and $Q = Q(\delta)$ be as in Theorem K-L. In terms of this P, we define the family of functions

$$G_k(z) := \eta_k(z) P(z - \lambda_k) \qquad (z \in \mathbb{C}). \tag{2.6}$$

Our first principal result of this section is the following.

THEOREM 2.2: *We have:*

(i) *Each G_k is an entire analytic function of exponential type δ,*

(ii) *For $j, k \in \mathbb{Z}$ we have $G_k(\lambda_j) = \delta_{j,k} := \{1$ for $j = k$; 0 else $\}$,*

(iii) *Each G_k, considered on the reals, is in $L^1(\mathbb{R})$ with*

$$|G_k(\lambda_k + s)| \leq e^{Q(\delta)} e^{-\gamma(|s|)} \tag{2.7}$$

(iv) *Each G_k is in $L^2(\mathbb{R})$ and one has*

$$|\langle G_j, G_k \rangle| \leq \left[4e^{2Q(\delta)} \int_0^\infty e^{-\gamma(s)}\, ds \right] e^{-\gamma(\psi_m/2)} \tag{2.8}$$

for any $j = k \pm m$ (i.e., $m = |k - j|$).

Proof: The assertion (i) follows on combining Lemma 2.1 (for η_k) and Theorem K-L with $\omega = \Psi + \gamma$. As noted in Lemma 2.1, we have $\eta_k(\lambda_j) = \delta_{j,k}$; hence, since $P(0) = 1$, we have (ii). The estimate (2.7) is immediate from (2.5) combined with Theorem K-L (iii) so we have (iii).

Finally, to prove (iv) we assume, with no loss of generality, that $\lambda_j \leq \lambda_k$ and set $\lambda := (\lambda_j + \lambda_k)/2$. Note that we then have $\lambda_j = \lambda - \tau$ and $\lambda_k = \lambda + \tau$ with $\tau := (\lambda_k - \lambda_j)/2 \geq \psi_m/2$ by (1.4) so $\gamma(\tau) \geq \gamma(\psi_m/2)$. Note also that for $t \leq \lambda$ one has $t - \lambda_j =: s \leq \tau$ so

$2\tau - s \geq \tau$ and $\gamma(|2\tau - s|) \geq \gamma(\tau)$; for $t \geq \lambda$ we set $s := t - \lambda_k \geq -\tau$ and $\gamma(|2\tau + s|) \geq \gamma(\tau)$. Thus, using (2.7),

$$\begin{aligned}
|\langle G_j, G_k\rangle| &\leq \int_{-\infty}^{\infty} |G_j(t)||G_k(t)|\,dt = \int_{-\infty}^{\lambda} + \int_{\lambda}^{\infty} \\
&= \int_{-\infty}^{\tau} |G_j(\lambda_j + s)||G_k(\lambda_k - [2\tau - s])|\,ds \\
&\qquad + \int_{-\tau}^{\infty} |G_j(\lambda_j + [2\tau + s])||G_k(\lambda_k + s)|\,ds \\
&\leq e^{2Q(\delta)}\left[\int_{-\infty}^{\tau} e^{-\gamma(|s|)}e^{-\gamma(\tau)}\,ds + \int_{-\tau}^{\infty} e^{-\gamma(\tau)}e^{-\gamma(|s|)}\,ds\right] \\
&\leq e^{2Q(\delta)}e^{-\gamma(\psi_m/2)}\left[2\int_{-\infty}^{\infty} e^{-\gamma(|s|)}\,ds\right]
\end{aligned}$$

which is just (2.8). In particular, for $j = k$ this shows $G_k \in L^2(\mathbb{R})$. ■

Depending on the choice of $\gamma(\cdot)$, this construction will determine the 'constant' $Q(\delta)$ of (2.7) as a function of $\delta > 0$. Also depending on the choice of $\gamma(\cdot)$, but now not on δ, we set

$$A^2 := \frac{2}{\pi}\int_0^{\infty} e^{-\gamma(s)}\,ds\left[e^{-\gamma(0)} + 2\sum_{m=1}^{\infty} e^{-\gamma(\psi_m/2)}\right]. \tag{2.9}$$

Noting that (1.7) gives $\psi_m \geq cm$ for some $c > 0$ so $\gamma(\psi_m/2) \geq \gamma(cm)$, we may compare the sum to the integral $\int_0^{\infty} e^{-\gamma(cs)}\,ds$ and observe that the integrability of $e^{-\gamma}$ ensures finiteness of A. Our other principal result of this section is the following.

THEOREM 2.3: *Let $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}(\boldsymbol{\psi})$ for some sequence $\boldsymbol{\psi}$ satisfying (1.7). Then for any $\delta > 0$ the coefficient map $\mathbf{C}_\delta = \mathbf{C}_\delta(\boldsymbol{\lambda})$ defined by (1.3) satisfies*

$$\|\mathbf{C}_\delta\| \leq Ae^{Q(\delta)} \qquad (\delta > 0) \tag{2.10}$$

with $Q(\delta)$ as in (2.7) and A as in (2.9), independent of δ.

Proof: The argument is here quite similar to that in [4]. We use the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\mathcal{F} : g \mapsto G \text{ with } G(z) = \int_{-\infty}^{\infty} e^{-izt}g(t)\,dt \tag{2.11}$$

and note that, with a factor of 2π, this is an isometric isomorphism:

$$\langle g, \tilde{g}\rangle := \int_{-\infty}^{\infty} \overline{g(t)}\tilde{g}(t)\,dt = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(t)}\tilde{G}(t)\,dt = \tfrac{1}{2\pi}\langle G, \tilde{G}\rangle. \tag{2.12}$$

By Theorem 2.2 (iv), each G_k is in $L^2(\mathbb{R})$ so there exist functions $g_k \in L^2(\mathbb{R})$ with $G_k = \mathcal{F}g_k$ $(k \in \mathbb{Z})$. By the Paley-Wiener Theorem [3], since Theorem 2.2 (i) gives each G_k entire and of exponential type δ, the support of each g_k is contained in the 'window' $[-\delta, \delta]$.

Initially, let us consider $f \in \overset{\circ}{\mathcal{M}}$ (so $f = \sum_k c_k e^{i\lambda_k t}$ is a finite sum) and view this through the window as $f \in L^2(-\delta, \delta)$. Then, for each $k \in \mathbb{Z}$, noting that $\operatorname{supp}(g_k) \subset [-\delta, \delta]$,

$$\begin{aligned} \langle f, g_k \rangle &:= \int_{-\delta}^{\delta} \overline{\left(\sum_j c_j e^{i\lambda_j t} \right)} g_k(t)\, dt \\ &= \sum_j \overline{c_j} \int_{-\infty}^{\infty} e^{-i\lambda_j t} g_k(t)\, dt = \sum_j G_k(\lambda_j) \overline{c_j}. \end{aligned}$$

By Theorem 2.2 (ii) we thus have, as in (1.5),

$$c_k = \overline{\langle f, g_k \rangle} \qquad (k \in \mathbb{Z},\, f \in \overset{\circ}{\mathcal{M}}). \tag{2.13}$$

Now consider the Gramian matrix $\mathbf{G}$ with entries $\langle g_j, g_k \rangle$. Since we continue to consider the (fixed) function $f \in \overset{\circ}{\mathcal{M}}$ as a finite sum, we may take $\mathbf{G}$ to be a finite matrix, considering only the indices k for which $c_k \neq 0$; thus there are no convergence problems but we seek estimates independent of this restricted index set. As a Gramian matrix, $\mathbf{G}$ is positive definite so the ℓ^2-induced matrix norm $\|\mathbf{G}\|_2$ is just the largest eigenvalue of $\mathbf{G}$. Hence,

$$\|\mathbf{G}\|_2 \leq \|\mathbf{G}\|_\infty := \max_j \{ \sum_k |\langle g_j, g_k \rangle| \} \tag{2.14}$$

since $\|\mathbf{G}\|_\infty$ is itself the ℓ^∞-induced matrix norm. Thus we have

$$\left\| \sum_k a_k g_k \right\|^2_{L^2(-\delta,\delta)} = \sum_{j,k} \langle g_j, g_k \rangle \overline{a_j} a_k \leq \|\mathbf{G}\|_\infty \|\mathbf{a}\|^2$$

for (finite) vectors $\mathbf{a} = (a_k) \in \ell^2$. Hence, using (2.13),

$$\begin{aligned} \|\mathbf{c}\|^2 &= \sum_k |c_k|^2 = \sum_k \langle f, g_k \rangle c_k = \langle f, \sum_k c_k g_k \rangle \\ &\leq \|f\| \| \sum_k c_k g_k \| \leq \|f\| \left(\|\mathbf{G}\|_\infty \|\mathbf{c}\|^2 \right)^{1/2} \end{aligned}$$

so for $f \in \overset{\circ}{\mathcal{M}}$ we have the estimate

$$\|\mathbf{c}\|_{\ell^2} \leq \left(\|\mathbf{G}\|_\infty \right)^{1/2} \|f\|_{L^2(-\delta,\delta)}. \tag{2.15}$$

We now use (2.12) and (2.8) to estimate $\|\mathbf{G}\|_\infty$ from (2.14). Fixing j, we consider $k \in \mathbb{Z}$ and set $m := |k - j|$ so

$$|\langle g_j, g_k \rangle| = \frac{1}{2\pi} \langle G_j, G_k \rangle \leq 4 e^{2Q(\delta)} \left[\frac{1}{2\pi} \int_0^\infty e^{-\gamma(s)}\, ds\, e^{-\gamma(\psi_m/2)} \right].$$

Summing over $k \in \mathbb{Z}$ then gives

$$\sum_k |\langle g_j, g_k \rangle| \leq A^2 e^{2Q(\delta)}$$

for each j so $\|\mathbf{G}\|_\infty \le (Ae^{Q(\delta)})^2$. Combining this with (2.15) gives

$$\|\mathbf{C}_\delta f\| = \|\mathbf{c}\| \le Ae^{Q(\delta)}\|f\|$$

for all $f \in \overset{\circ}{\mathcal{M}}$. By the density of $\overset{\circ}{\mathcal{M}}$ in $\mathcal{M}_\delta$, this gives precisely the desired estimate (2.10). ■

3. The Mollifier

Our next object is to re-examine Theorem K-L so as to introduce the 'mollifier' $P(\cdot)$ with a reasonably explicit estimate for $Q = Q(\delta)$. To this end, given $\omega \in \Omega$ we set

$$v(s) = \frac{\omega(s)}{s^2}, \qquad dq = -s^2 dv. \tag{3.1}$$

Note that the definition (1.9) of Ω ensures that q is an unbounded increasing function of s and that $\omega(\alpha)/\alpha \to 0$ as $\alpha \to \infty$. For each α in $(0,\infty)$ we can then set

$$\delta(\alpha) := \frac{1+2\omega(\alpha)}{\alpha} + 2\int_\alpha^\infty \frac{\omega(s)}{s^2}ds = \frac{1}{\alpha} + 2\int_\alpha^\infty \frac{dq}{s}\ . \tag{3.2}$$

LEMMA 3.1: *Fix $\omega \in \Omega$ and let $\delta(\cdot)$ be defined by (3.2). Then $\delta(\alpha)$ is nonincreasing on $(0,\infty)$ and $\delta(\alpha) \to 0$ as $\alpha \to \infty$ so for each $\delta > 0$ there exists an $\alpha := \alpha(\delta)$ such that $\delta(\alpha) \le \delta$. Further, fixing $\delta > 0$, there is a sequence (a_j) such that*

$$\sum_0^\infty a_j \ \le\ \delta(\alpha) \le \delta \tag{3.3}$$

$$\sum_{a_j|s|\le 1} [a_j]^2 \ \ge\ \frac{2\omega(|s|)-1}{s^2} \quad \text{for}\, |s| > \alpha. \tag{3.4}$$

Proof: Deferred to the Appendix. ■

We can now state our revised form of Theorem K-L, including the estimate of $Q(\delta)$.

THEOREM 3.2: *For any $\delta > 0$, define $P(z)$ by*

$$P(z) := \prod_{j=1}^\infty \cos(a_j z) \quad (z \in \mathbb{C}), \tag{3.5}$$

using the sequence (a_j) of Lemma 3.1. Then $P(\cdot)$ is an even entire function of exponential type δ with $P(0) = 1$. Further, one has

$$|P(s)| \le e^{Q(\delta)}e^{-\omega(|s|)} \tag{3.6}$$

for all $s \in \mathbb{R}$, where Lemma 3.1 is used to define

$$Q(\delta) := 1/2 + \omega(\alpha(\delta)). \tag{3.7}$$

Proof: We know that $\cos(\cdot)$ is even and entire of exponential type 1. By (3.3), it follows that P is a well-defined even, entire function of exponential type δ. Observing that

$$|\cos s| \le \exp\left[-\frac{s^2}{2}\right] \quad \text{for } |s| \le 1,$$

it follows from (3.4) that for $|s| > \alpha$ we have

$$\begin{aligned} |P(s)| &\le \prod\{|\cos(a_j s)| : a_j|s| \le 1\} \\ &\le \exp\left[-\frac{s^2}{2}\sum_{a_j|s|\le 1} a_j^2\right] \le e^{1/2}e^{-\omega(|s|)}. \end{aligned}$$

Since $|P(s)| \le 1 \le e^{\omega(\alpha)}e^{-\omega(|s|)}$ for any s, we have (3.6) for all $s \in \mathbb{R}$. ■

4. Examples

We now specialize our work to treat some particular cases more explicitly. In each case, we take $\omega = (1+\varepsilon)\Psi$, i.e., $\gamma := \varepsilon\Psi$. A principal point, here, is that the asymptotics of $Q(\delta)$ as $\delta \to 0$ are (almost) determined by the asymptotics of ψ_m as $m \to \infty$. In the first two examples, we also note the convenience of taking $\psi_m = \psi(m)$ for a suitable function $\psi(\cdot)$, giving an integral version of (2.1) for the asymptotically correct determination of $\Psi(\cdot)$.

EXAMPLE 1: We first suppose $\psi(x) = ax^p$ $(a > 0, p > 1)$; when $p = 2$ this is the case considered in [4]. It is easily seen that $\boldsymbol{\psi} := \{\psi_m\}$ satisfies (1.7). To simplify the explicit computation of various quantities, we deal with the integral version of (2.1), namely,

$$\begin{aligned} \Psi(s) &:= 2\int_0^\infty \log\left(1 + \frac{s^2}{\psi(x)^2}\right) dx \\ &= 2\int_0^\infty \log\left(1 + \frac{s^2}{a^2x^{2p}}\right) dx \\ &= 2s^{1/p}\left[\frac{1}{a^{1/p}}\int_0^\infty \log\left(1 + \frac{1}{u^{2p}}\right) du\right] =: \beta(p)\, s^{1/p} \end{aligned}$$

where (cf., e.g., [1] p.114) $\beta(p) = \frac{2\pi}{a^{1/p}\sin\frac{\pi}{2p}}$. Now, let $\varepsilon > 0$ and let $\omega(s) := (1+\varepsilon)\Psi(s)$. From (3.2),

$$\begin{aligned} \delta(\alpha) &= \frac{1 + 2\omega(\alpha)}{\alpha} + 2\int_\alpha^\infty \frac{\omega(s)}{s^2}\, ds \\ &= \frac{1 + 2(1+\varepsilon)\beta(p)\alpha^{1/p}}{\alpha} + \int_\alpha^\infty \frac{2(1+\varepsilon)\beta(p)s^{1/p}}{s^2}\, ds \\ &= \frac{1}{\alpha} + \frac{\vartheta}{\alpha^{1/q}} \end{aligned}$$

where $\vartheta := \vartheta(\varepsilon, p) = 2(1+q)\,(1+\varepsilon)\,\beta(p)$ with $pq = p + q$. Since $\delta(\alpha) \leq (1+\varepsilon)\vartheta/\alpha^{1/q}$ for large α, we see that $\alpha(\delta) \leq [(1+\varepsilon)\vartheta/\delta]^q$ for large α, i.e., for small δ. By (3.7),

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leq \tfrac{1}{2} + (1+\varepsilon)\beta(p)\left(\tfrac{(1+\epsilon)\vartheta}{\delta}\right)^{q/p}.$$

Thus, (2.10) becomes

(4.1) $$\|\mathbf{C}_\delta\| \leq Ae^{1/2}\exp\left[B\,(1/\delta)^{q/p}\right]$$

where, with a corresponding constant A, $B > B_0$ is arbitrary with

$$B_0 := \beta(p)\vartheta^{q/p} = 2^{2q-1}\left(\frac{1+q}{a}\right)^{q/p}\left(\frac{\pi}{\sin\pi/2p}\right)^q.$$

Since $\frac{q}{p} = \frac{1}{p-1}$, we have the promised estimate (1.10).

EXAMPLE 2: We next consider sequences which are even more sparse: $\psi(x) = ce^{\beta x}$ with $c, \beta > 0$, indicating how various quantities can be computed. We now have

$$\begin{aligned}
\Psi(s) &= 2\int_0^\infty \log\left(1 + \frac{s^2}{\psi(x)^2}\right)dx = \tfrac{2}{\beta}\int_0^\infty \log\left(1+\frac{s^2}{c^2e^{2\beta x}}\right)\beta\,dx \\
&= \tfrac{1}{\beta}\int_{-\sigma}^\infty \log(1+e^{-r})\,dr \qquad\qquad (\text{where } \tfrac{s^2}{c^2} = e^\sigma,\ r = 2\beta x - \sigma) \\
&= \tfrac{1}{\beta}\int_0^\infty \log(1+e^{-r})\,dr + \tfrac{2}{\beta}\int_0^\sigma \log(1+e^r)\,dr \\
&\sim \tfrac{2}{\beta}[\log s]^2 \quad \text{as} \quad s \to \infty.
\end{aligned}$$

Asymptotically, $\omega(s) \sim (1+\varepsilon)\frac{2}{\beta}[\log s]^2$, so one has $\delta(\alpha) \sim \frac{8(1+\varepsilon)}{\beta\alpha}[\log\alpha]^2$ by a simple computation and $\delta(\alpha) \leq (1/\alpha)^{1/p}$ for arbitrary $p > 1$ and large α. Hence $\alpha(\delta) \leq 1/\delta^p$ for small δ. Thus,

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leq \tfrac{1}{2} + \tfrac{2}{\beta}(1+\varepsilon)p^2[\log 1/\delta]^2$$

and one has, therefore,

(4.2) $$\|C_\delta\| \leq A\exp\left[B(\log 1/\delta)^2\right]$$

for any $B > B_0 := 2/\beta$ and a suitable constant A.

EXAMPLE 3: In this example, we consider the ultimate asymptotic sparsity: a finite sequence $\{\lambda_j\}$ of $L+1$ distinct real numbers. Taking these in increasing order and setting $c := \min\{|\lambda_k - \lambda_j| : j \neq k\} > 0$, we then automatically have the condition (1.4) with $\psi_m := mc$ for $m = 0, \ldots, L$ and (formally) $\psi_m := \infty$ for $m > L$, giving

$$\Psi(s) := 2\sum_{m=1}^{L}\log\left(1+\frac{s^2}{c^2m^2}\right) = 4L\log s + \mathcal{O}(1).$$

We now have $\omega(s) \sim 4L(1+\varepsilon)\log s$ so $\delta(\alpha) \sim 16L(1+\varepsilon)\frac{1}{\alpha}\log\alpha$ for large α. Hence, as in the previous example, $\alpha(\delta) \leq 1/\delta^p$ for arbitrary $p > 1$ and small δ so

$$Q(\delta) := \tfrac{1}{2} + \omega(\alpha) \leq \tfrac{1}{2} + 4p(1+\varepsilon)L\log 1/\delta.$$

One therefore has algebraic growth in $1/\delta$ for the norm in this case:

$$\|C_\delta\| \leq A\delta^{-\nu} \tag{4.3}$$

for any $\nu > \nu_0 := 4L$ with a corresponding constant A.

5. Appendix

LEMMA A.1: *Let $\boldsymbol{\psi}$ be any increasing positive sequence satisfying (1.7). Then (2.1) defines a function Ψ on $\mathbb{R}^+$ such that*

$$\begin{array}{lll} & (i) & \Psi \text{ is continuous and unbounded on } [0,\infty) \text{ with } \Psi(0)=0,\\ & (ii) & \Psi \text{ is } C^1 \text{ and (strictly) increasing on } \mathbb{R}^+,\\ (5.1) & (iii) & \Psi(s)/s^2 \text{ is decreasing on } (0,\infty),\\ & (iv) & \Psi(s)/s^2 \text{ is integrable at } \infty,\\ & (v) & e^{-\Psi} \text{ is integrable on } \mathbb{R}^+. \end{array}$$

Proof: Since $\psi = \psi_k \to \infty$ as $k \to \infty$ so $\log(1+1/\psi) \sim 1/\psi$, we see from (1.7) that the sum in (2.1) is well defined and finite for each $s \geq 0$. Further, each term in that sum is (strictly) increasing in s and continuous. By the Weierstrass M-test, the series converges uniformly on any closed and bounded interval in $\mathbb{R}^+$ so $\Psi(s)$ is continuous. Similarly, $\Psi' = 4\sum_1^\infty \frac{s}{(s^2+\psi_m^2)}$ which is finite by (1.7) and positive on $\mathbb{R}^+$. Thus we have (5.1-*i*,*ii*). To see (*iii*), we observe that

$$\Psi(s)/s^2 = 2\sum_1^\infty \frac{\rho([s/\psi_m]^2)}{\psi_m^2}$$

with $\rho(u) := \frac{\log(1+u)}{u}$ and note that ρ is strictly decreasing for $u > 0$.

To get (*iv*), we observe that $\Psi(s)/s^2$ will be integrable at ∞ if and only if the series $\{\int_1^\infty (1/s^2)\log(1+s^2/\psi_k^2)\,ds\}$ is summable. From the identity

$$\int \frac{\log(1+u^2)}{u^2} = 2\tan^{-1}u - \frac{\log(1+u^2)}{u},$$

we get

$$\int_1^\infty (1/s^2)\log\left(1+\frac{s^2}{\psi^2}\right)\,ds = \frac{\pi}{\psi} - 2\frac{1}{\psi}\tan^{-1}\frac{1}{\psi} + \log\left(1+\frac{1}{\psi^2}\right).$$

Using (1.7) and (i), we get (iv). Statement (v) is obvious. ■

Finally, we provide the promised proof of Lemma 3.1.

Proof [of Lemma 3.1]: As already noted, the definition (1.9) of Ω ensures that q is increasing, so the right hand side of (3.2) is (strictly) decreasing to 0 as $\alpha \to \infty$ by the integrability of ω/s^2. Thus, $\delta(\cdot)$ is invertible with $\alpha(\delta)$ defined for (small) $\delta > 0$.

Now, fixing $\delta > 0$ and so $\alpha = \alpha(\delta)$, we use[4] (3.1) to define

$$a_j := 1/q^{-1}(z_j) \text{ with } z_j := q(\alpha) + j/2 \tag{5.2}$$

for $j = 0, 1, \ldots$. An integral comparison, noting that the function $1/q^{-1}(\cdot)$ is decreasing and that $z_{j+1} - z_j \equiv 1/2$, gives

$$\sum_0^\infty a_j = \frac{1}{\alpha} + \sum_1^\infty \frac{1}{q^{-1}(z_j)} \le \frac{1}{\alpha} + 2\int_{q(\alpha)}^\infty \frac{dz}{q^{-1}(z)}$$

which precisely gives (3.3) on using (3.2) for $z = q(s)$.

For $|s| > \alpha$, we now note that $j_* \ge 1$ where $j_* = j_*(s,\alpha)$ is the smallest j for which $a_j|s| \le 1$; hence, $0 \le z_{j_*} - q(|s|) < \frac{1}{2}$. An argument similar to the above then gives

$$\begin{aligned}
\sum_{a_j|s|\le 1} [a_j]^2 &= \sum_{j_*}^\infty \frac{1}{[q^{-1}(z_j)]^2} \ge 2\int_{z_{j_*}}^\infty \frac{dz}{[q^{-1}(z)]^2} \\
&\ge 2\int_{q(|s|)}^\infty \frac{dz}{[q^{-1}(z)]^2} - \frac{1}{s^2} \\
&\qquad\qquad \text{(since } q^{-1}(z) \ge |s| \text{ for } s \le z \le z_{j_*}\text{)} \\
&= 2v(|s|) - 1/s^2
\end{aligned}$$

which is just (3.4). ■

References

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[4] Although (3.1) only defines $q(\cdot)$ to within an additive constant, the formula (5.2) suffices to specify (a_j).

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